QUESTIONS AND COMMENTS ON THE PAPER "GENUS BOUNDS IN RIGHT-ANGLED ARTIN GROUPS" BY MAX FORESTER, IGNAT SOROKO, AND JING TAO [FST20]

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1. Right-angled Artin groups

Nicolaus Heuer proved the sharp bound $\mathrm{scl}(w) \geq \frac{1}{2}$ for all w in all RAAGs [Heu19], thus extending the known result for free groups to the general right-angled Artin groups. Other possible generalizations of results known for free groups to the case of RAAGs are:

Question 2.1. *Is* scl *computable for RAAGs? (Devise an algorithm to compute* scl *in RAAGs.)*

Question 2.2. *Is* scl *rational for RAAGs?*

I expect that the answer is 'yes' for both questions.

2. Free groups

Even for free groups there are some intriguing questions for the behavior of scl. It is known that for a free group F, and any nontrivial $w \in F'$, the $\mathrm{scl}(w)$ is rational and moreover that $\mathrm{scl}(w) \geq \frac{1}{2}$, i.e. there is a gap $(0, \frac{1}{2})$. Lvzhou Chen has brought to my attention the following conjectures based on the experimental data from the article [CW11]:

Question 2.3. *For a free group F:*

1. *is the second* scl *gap equal to $(\frac{1}{2}, \frac{7}{12})$?*
2. *is the first accumulation point for the values of* scl *equal $\frac{3}{4}$?*
3. *are the values of* scl *dense after $\frac{3}{4}$?*

Obviously elements $w \in F$ with $\mathrm{scl}(w) = \frac{1}{2}$ are interesting since they minimize scl. We call them *commutator-like*. (The motivation for this comes from the fact that $\mathrm{scl}([x, y]) = \frac{1}{2}$ in $F(x, y)$.)

Question 2.4. *Describe all commutator-like words in a free group.*

Question 2.5. *Describe all commutator-like words in an arbitrary RAAG.*

Lvzhou Chen gave a relatively simple independent proof of the fact that $\mathrm{scl}(w) \geq \frac{1}{2}$ in free groups [Che18]. One may try to distill from his technique a tool to describe all commutator-like words. Of course, there are proofs for RAAGs, which can be adapted to the case of free groups, like the one by Heuer [Heu19], mentioned above, and a subsequent topological proof of Chen and Heuer [CH20, Th. 7.4], as well as the original proof of Duncan and Howie [DH91] for free products of locally indicable groups, and its recent generalization by Ivanov and Klyachko [IK18].

The next question was suggested to the author by Jonah Gaster:

Question 2.6. *What is the growth of* $\mathrm{scl}(w)$ *as a function of the word length of w? Consider this question for a free group and for a RAAG.*

Clearly, scl is not monotonic, so one needs to specify what kind of growth to consider. A possible candidate is:

$$\mathrm{gr}_{\mathrm{scl}}(n) = \max\{\, \mathrm{scl}(w) \mid \text{word length}(w) \leq n \,\}.$$

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References

[CW11] Danny Calegari, Alden Walker, Isometric endomorphisms of free groups. *New York J. Math.* 17 (2011), 713–743.

[Che18] Lvzhou Chen, Spectral gap of scl in free products, *Proc. Amer. Math. Soc.* 146 (2018), no.7, 3143–3151.

[CH20] Lvzhou Chen, Nicolaus Heuer, Spectral gap of scl in graphs of groups and 3-manifolds. Preprint arXiv:1910.14146

[DH91] Andrew J. Duncan, James Howie, The genus problem for one-relator products of locally indicable groups. *Math. Z.* 208(2):225–237, 1991.

[FST20] Max Forester, Ignat Soroko, Jing Tao, Genus bounds in right-angled Artin groups, *Publicacions Matemàtiques* 64 (2020), 1, pp. 233–253.

[Heu19] Nicolaus Heuer, Gaps in scl for amalgamated free products and RAAGs, *Geometric and Functional Analysis* 29 (2019), 1, pp. 198–237.

[IK18] Sergei V. Ivanov, Anton A. Klyachko, Quasiperiodic and mixed commutator factorizations in free products of groups. *Bull. Lond. Math. Soc.* 50 (2018), no. 5, 832–844.

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