QUESTIONS AND COMMENTS ON THE PAPER "DEHN FUNCTIONS OF SUBGROUPS OF RIGHT-ANGLED ARTIN GROUPS" BY NOEL BRADY AND IGNAT SOROKO [BS19]

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Our paper [BS19] touches upon several themes involving some intriguing open questions, which we are going to discuss here.

1. Free-by-cyclic groups

If a group embeds into a right-angled Artin group, it has certain nice properties, such as linearity over $\mathbb{Z}$, for example. There is a specifically remarkable class of subgroups of right-angled Artin groups, the so-called special subgroups in the terminology of Haglund and Wise [HW08]. They are the fundamental groups of some non-positively curved cubical complexes and they are undistorted (quasi-convex) in the respective RAAGs. For many purposes, such as linearity, for example, only embeddability of finite index subgroups is important. So one may ask if a group has this property virtually. In regard of free-by-cyclic groups we have the following well-known question:

Question 1.1. *Which free-by-cyclic groups virtually embed in right-angled Artin groups?*

More specifically,

Question 1.2. *Characterize free-by-cyclic groups which are virtually special.*

In [Ger94], Gersten defines an $F_3 \rtimes \mathbb{Z}$ group: $\langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = ba, tct^{-1} = ca^2\rangle$, which does not embed into a RAAG, and we prove in the Appendix to [BS19] that no finite index subgroup of Gersten's group admits an embedding into a RAAG. Another such example was given by Woodhouse [Woo16, Ex. 5.1] (in view of [Woo18, Th. 5.10]). The groups considered by Gersten and Woodhouse are not CAT(0) groups, and this prompts the following question:

Question 1.3 (Question 1 of [BS19])**.** *Does every* CAT(0) *free-by-cyclic group virtually embed into a RAAG?*

A stronger statement would be:

Question 1.4. *Is every* CAT(0) *free-by-cyclic group virtually special?*

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Another way to understand the phenomenon exhibited by Gersten's group is to observe that the automorphism $\phi\colon F_3 \to F_3, a \mapsto a, b \mapsto ba, c \mapsto ca^2$, of Gersten's group has linear growth, whereas the maximum possible polynomial growth rate of an automorphism of F_3 is quadratic. In general, for a free group F_n of rank $n \geqslant 3$ one can model the behavior of Gersten's group automorphism on a subset of a basis of F_n, and get the same phenomenon for F_n. However this approach will generate a monodromy automorphism whose order of growth is strictly smaller than $n-1$, the maximal possible order of growth for F_n. (This is the case with Woodhouse's group mentioned above, which is isomorphic to $F_5 \rtimes_\phi \mathbb{Z}$ with ϕ growing linearly.) This leads to the following question that Martin Bridson asked the second-named author:

Question 1.5. *Let F_k be a rank k free group and $\phi \in \mathrm{Aut}(F_k)$ an automorphism of maximal polynomial growth, i.e. whose monodromy grows as $\simeq n^{k-1}$. Is the free-by-cyclic group $F_k \rtimes_\phi \mathbb{Z}$ virtually special?*

A natural example to test this and the previous questions are the famous Hydra groups studied by Dison and Riley [DR13]: $H_n = F_n \rtimes_{\phi_n} \mathbb{Z} = \langle a_1, \dots a_n, t \mid ta_1t^{-1} = a_1, ta_it^{-1} = a_ia_{i-1}, i = 2, \dots, n\rangle$.

Question 1.6. *Are Hydra groups virtually special?*

The second-named author has established that the answer to this question is affirmative for $n \leqslant 4$ and it is his current work in progress to prove this result for arbitrary $n \geqslant 5$.

One of the main results of our article [BS19] is the construction of analogues of the Hydra groups (where the base $\mathbb{Z}^2$ subgroup is replaced by a more complicated RAAG) which are CAT(0) free-by-cyclic groups with polynomially growing monodromy automorphisms of arbitrary degree and which are virtually special.

2. Growth of automorphisms

Recall that we defined the growth of an automorphism ψ of a group G with the generating set $\mathcal{A}$ as $\mathrm{gr}_{\psi,\mathcal{A}}(n) := \max_{a\in\mathcal{A}} \|\psi^n(a)\|_{\mathcal{A}}$, where $\|g\|_{\mathcal{A}}$ is equal to $d_{\mathcal{A}}(1, g)$ for $g \in G$, i.e. the distance in the Cayley graph for G with respect to $\mathcal{A}$ from the identity to the element g. We consider growth functions up to the equivalence relation $\sim$, which is defined as follows: two functions $f, g\colon [0,\infty) \to [0,\infty)$ are said to be $\sim$ equivalent if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means that there exist constants $A > 0$ and $B \geq 0$ such that $f(n) \leqslant Ag(n) + B$ for all $n \geqslant 0$. We proved in Proposition 2.4 of [BS19] that when G is a free group then up to the equivalence relation $\sim$ the growth function of an automorphism ψ does not depend on the generating set $\mathcal{A}$ and is the same for all ψ-invariant subgroups of finite index $H \leqslant G$. This suggests a general question:

Question 1.7. *For an arbitrary group G, an arbitrary automorphism $\alpha \in \mathrm{Aut}(G)$ and an α–invariant subgroup $H \leqslant G$ of finite index, what is the range for possible gaps between $\mathrm{gr}_\alpha(n)$, the growth rate of an automorphism α, and $\mathrm{gr}_{\alpha|_H}(n)$, the growth of its restriction to H, $\alpha|_H$?*

Yves Cornulier informed us that the infinite dihedral group gives an example of the gap of one polynomial degree. Namely, let $G = \langle a, b \mid aba^{-1} = b^{-1}, a^2 = 1\rangle$ be the infinite dihedral group. Then the inner automorphism $i_b\colon x \mapsto bxb^{-1}$ has linear growth, but its restriction to the index 2 subgroup $\langle b\rangle$ is trivial. To see that, observe that $aba^{-1} = b^{-1}$ implies $ab = b^{-1}a$ and hence $ba = ab^{-1}$. Thus $bab^{-1} = ab^{-2}$ and $i_b^n(a) = b^nab^{-n} = ab^{-2n}$. Looking at the Cayley graph of G shows that the element $g = ab^{-2n}$ is at distance $2n + 1$ from 1, so that ab^{-2n} is a word of minimal length representing element g, and i_b indeed grows linearly on G.

One can modify the above example by omitting the relation $a^2 = 1$, with the same effect.

It must be noted that there exist a different notion of the growth of automorphisms, the so-called cyclic growth, which is invariant under the composition with inner automorphisms, and hence is better suited for studying the growth of outer automorphisms. For details see [PR19, Section 7.2].

3. Subgroups of RAAGs and their Dehn functions

Another main result of [BS19] is a theorem that stipulates existence of subgroups having arbitrary polynomial Dehn functions inside RAAGs. Examples known previously had linear, quadratic, cubic and quartic Dehn functions [Bra07], and also an example due to Bridson with the exponential Dehn function [Bri13]. A natural question is:

Question 1.8 (Question 2 of [BS19])**.** *Do there exist finitely presented subgroups of right-angled Artin groups whose Dehn functions are either super-exponential or sub-exponential but not polynomial?*

Our construction of subgroups with polynomial Dehn functions places them inside RAAGs with 3–dimensional Salvetti complex [BS19, Th. B].

Question 1.9 (Question 3 of [BS19])**.** *Do there exist subgroups with polynomial Dehn functions of arbitrary degrees inside* 2*–dimensional RAAGs?*

In Remark 7.1 of [BS19] we noticed that the groups Γ having polynomial Dehn functions of arbitrary degree can be found in the Bestvina–Brady kernel (call it BBK) of the corresponding RAAG $A(\Delta) \times F_2$. Dison proved in [Dis08] that Bestvina–Brady kernels have Dehn functions bounded above by n^4. Also, the RAAG itself (being a CAT(0) group) has at most quadratic Dehn function. This tells us that in the series of inclusions

$$\Gamma \leqslant BBK \leqslant \mathrm{RAAG}$$

we observe a 'distortion of areas' phenomenon. On the other hand, it can be proven using technique from [A$^+$13] that the subgroup distortion (i.e. the 'distortion of lengths') of BBK in RAAG is at most quadratic.

Question 1.10. *What is the subgroup distortion of* Γ *in* BBK*?*

4. A difficult question

If we replace in the main result of [BS19] RAAGs with Gromov hyperbolic groups, we get the following question, which is very interesting but probably extremely difficult:

Question 1.11. *Do there exist subgroups of Gromov hyperbolic groups having polynomial Dehn functions of arbitrary degree?*

It is known that finitely presented subgroups of hyperbolic groups may not be themselves hyperbolic [Bra99, Lod18]. This implies that the Dehn function of such subgroups will be at least quadratic. However the exact order of the Dehn functions of these groups is not known.

Question 1.12. *Determine the Dehn functions of the non-hyperbolic subgroups of hyperbolic groups constructed in* [Bra99, Lod18]*. Are they quadratic?*

References

[A$^+$13] Abrams, A., Brady, N., Dani, P., Duchin, M., Young, R., Pushing fillings in right-angled Artin groups. *J. Lond. Math. Soc.* (2) 87 (2013), no. 3, 663–688.

[Bra99] Brady, N., Branched coverings of cubical complexes and subgroups of hyperbolic groups. *J. London Math. Soc.* (2) 60 (1999), no. 2, 461–480.

[Bra07] Brady, N., Dehn functions and non-positive curvature, in *The geometry of the word problem for finitely generated groups.* (Birkhäuser, Basel, 2007) 1–79.

[BS19] Brady, N., Soroko, I., Dehn functions of subgroups of right-angled Artin groups. *Geom. Dedicata* 200 (2019), 197–239.

[Bri13] Bridson, M. R., On the subgroups of right-angled Artin groups and mapping class groups. *Math. Res. Lett.*, 20 (2013), no. 2, 203–212.

[Dis08] Dison, W., An isoperimetric function for Bestvina–Brady groups. *Bull. Lond. Math. Soc.* 40 (2008), no. 3, 384–394.

[DR13] Dison, W., Riley, T. R., Hydra groups. *Comment. Math. Helv.* 88 (2013), no. 3, 507–540.

[Ger94] Gersten, S. M., The automorphism group of a free group is not a CAT(0) group. *Proc. Amer. Math. Soc.* 121 (1994), no. 4, 999–1002.

[HW08] Haglund, F., Wise, D. T., Special cube complexes. *Geom. Funct. Anal.* 17 (2008), no. 5, 1551–1620.

[Lod18] Lodha, Y., A hyperbolic group with a finitely presented subgroup that is not of type FP_3. *Geometric and cohomological group theory*, 67–81, *London Math. Soc. Lecture Note Ser.*, 444, Cambridge Univ. Press, Cambridge, 2018.

[PR19] Pueschel, K., Riley, T., Dehn functions of mapping tori of right-angled Artin groups. arXiv:1906.09368 (2019).

[Woo16] Woodhouse, D. J., Classifying finite dimensional cubulations of tubular groups. *Michigan Math. J.* 65 (2016), no. 3, 511–532.

[Woo18] Woodhouse, D. J., Classifying virtually special tubular groups. *Groups Geom. Dyn.* 12 (2018), no. 2, 679–702.

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