Stable commutator length in right-angled Artin groups

Ignat Soroko

University of Oklahoma

ignat.soroko@ou.edu

Joint work with

Max Forester and **Jing Tao**, University of Oklahoma

52nd STDSC, Auburn University, March 14–17, 2018

Commutator length: algebraic definition

Given group G, $g \in [G, G]$,

$$cl(g) = \min\{N \mid g = \prod_{i=1}^{N} [x_i, y_i],\ x_i, y_i \in G\}$$

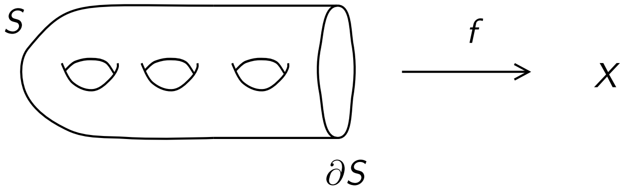
Commutator length: algebraic definition

Given group G, $g \in [G, G]$,

$$cl(g) = \min\{N \mid g = \prod_{i=1}^{N}[x_i, y_i],\ x_i, y_i \in G\}$$

Commutator length: topological definition

Let X be a top. space with $\pi_1(X) \cong G$. Let S be a cpt, oriented surface with 1 boundary component and a continuous map $f\colon S \to X$, such that $(f|_{\partial S})_* = g$:



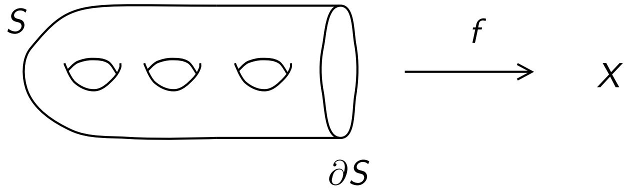
Commutator length: algebraic definition

Given group G, $g \in [G, G]$,

$$cl(g) = \min\{N \mid g = \prod_{i=1}^{N} [x_i, y_i],\ x_i, y_i \in G\}$$

Commutator length: topological definition

Let X be a top. space with $\pi_1(X) \cong G$. Let S be a cpt, oriented surface with 1 boundary component and a continuous map $f: S \to X$, such that $(f|_{\partial S})_* = g$:



$$cl(g) = \min\{genus(S) \mid \exists f: S \to X, (f|_{\partial S})_* = g\}$$

Example: Let $G = F(x, y)$, the free group of rank 2. Take $g = [x, y] = xyx^{-1}y^{-1}$. Consider its powers:

$$g^2 = [x, y] \cdot [x, y]$$

Example: Let $G = F(x, y)$, the free group of rank 2. Take $g = [x, y] = xyx^{-1}y^{-1}$. Consider its powers:

$$g^2 = [x, y] \cdot [x, y]$$

$$g^3 = [x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2]$$

Example: Let $G = F(x, y)$, the free group of rank 2. Take $g = [x, y] = xyx^{-1}y^{-1}$. Consider its powers:

$$g^2 = [x, y] \cdot [x, y]$$

$$g^3 = [x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2]$$

A striking fact: $cl([x, y]^3)$ is only 2!

Example: Let $G = F(x, y)$, the free group of rank 2. Take $g = [x, y] = xyx^{-1}y^{-1}$. Consider its powers:

$$g^2 = [x, y] \cdot [x, y]$$

$$g^3 = [x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2]$$

A striking fact: $cl([x, y]^3)$ is only 2!

$$g^4 = [x, y]^4 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2] \cdot [x, y]$$

Example: Let $G = F(x, y)$, the free group of rank 2. Take $g = [x, y] = xyx^{-1}y^{-1}$. Consider its powers:

$$g^2 = [x, y] \cdot [x, y]$$

$$g^3 = [x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2]$$

A striking fact: $cl([x, y]^3)$ is only 2!

$$g^4 = [x, y]^4 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2] \cdot [x, y]$$

$$g^5 = [x, y]^5 = [w_1, w_2] \cdot [w_3, w_4] \cdot [w_5, w_6]$$

so that $cl([x, y]^5)$ is, again, only 3.

Example: Let $G = F(x, y)$, the free group of rank 2. Take $g = [x, y] = xyx^{-1}y^{-1}$. Consider its powers:

$$g^2 = [x, y] \cdot [x, y]$$
$$g^3 = [x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2]$$

A striking fact: $cl([x, y]^3)$ is only 2!

$$g^4 = [x, y]^4 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2] \cdot [x, y]$$
$$g^5 = [x, y]^5 = [w_1, w_2] \cdot [w_3, w_4] \cdot [w_5, w_6]$$

so that $cl([x, y]^5)$ is, again, only 3.

In general,

$$cl([x, y]^n) = \lfloor n/2 \rfloor + 1 \quad \text{(Culler)}.$$

Example: Let $G = F(x, y)$, the free group of rank 2. Take $g = [x, y] = xyx^{-1}y^{-1}$. Consider its powers:

$$g^2 = [x, y] \cdot [x, y]$$

$$g^3 = [x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2]$$

A striking fact: $cl([x, y]^3)$ is only 2!

$$g^4 = [x, y]^4 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2] \cdot [x, y]$$

$$g^5 = [x, y]^5 = [w_1, w_2] \cdot [w_3, w_4] \cdot [w_5, w_6]$$

so that $cl([x, y]^5)$ is, again, only 3.
In general,

$$cl([x, y]^n) = \lfloor n/2 \rfloor + 1 \quad \text{(Culler)}.$$

Q: What is the "compression ratio" of $cl(g^n)$ when the power n of g^n grows unboundedly?

Example: Let $G = F(x, y)$, the free group of rank 2. Take $g = [x, y] = xyx^{-1}y^{-1}$. Consider its powers:

$$g^2 = [x, y] \cdot [x, y]$$

$$g^3 = [x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2]$$

A striking fact: $cl([x, y]^3)$ is only 2!

$$g^4 = [x, y]^4 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2] \cdot [x, y]$$

$$g^5 = [x, y]^5 = [w_1, w_2] \cdot [w_3, w_4] \cdot [w_5, w_6]$$

so that $cl([x, y]^5)$ is, again, only 3.

In general,

$$cl([x, y]^n) = \lfloor n/2 \rfloor + 1 \quad \text{(Culler)}.$$

Q: What is the "compression ratio" of $cl(g^n)$ when the power n of g^n grows unboundedly?

A: The **stable commutator length**, or scl:

$$scl(g) := \lim_{n \to \infty} \frac{cl(g^n)}{n}.$$

Example: Let $G = F(x, y)$, the free group of rank 2. Take $g = [x, y] = xyx^{-1}y^{-1}$. Consider its powers:

$$g^2 = [x, y] \cdot [x, y]$$

$$g^3 = [x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2]$$

A striking fact: $cl([x, y]^3)$ is only 2!

$$g^4 = [x, y]^4 = [xyx^{-1}, y^{-1}xyx^{-2}] \cdot [y^{-1}xy, y^2] \cdot [x, y]$$

$$g^5 = [x, y]^5 = [w_1, w_2] \cdot [w_3, w_4] \cdot [w_5, w_6]$$

so that $cl([x, y]^5)$ is, again, only 3.

In general,

$$cl([x, y]^n) = \lfloor n/2 \rfloor + 1 \quad \text{(Culler)}.$$

Q: What is the "compression ratio" of $cl(g^n)$ when the power n of g^n grows unboundedly?

A: The **stable commutator length**, or scl:

$$scl(g) := \lim_{n \to \infty} \frac{cl(g^n)}{n}.$$

For $g = [x, y]$ we have: $scl([x, y]) = 1/2$.

Q: Why is scl interesting?

Q: Why is scl interesting?
A:

Q: Why is scl interesting?
A:

- *scl* $\approx$ Gromov–Thurston norm on $H_2(X)$;
- Bounded cohomology;
- Bavard duality;
- Dynamics, etc. — see book D.Calegari “scl”

Q: Why is scl interesting?
A:

- *scl* $\approx$ Gromov–Thurston norm on $H_2(X)$;
- Bounded cohomology;
- Bavard duality;
- Dynamics, etc. — see book D.Calegari "scl"

For many important groups **scl has gap above zero**:

- Free groups: $\forall g$, $scl(g) \geq 1/6$ (Culler'81); $scl(g) \geq 1/2$ (Duncan–Howie'91, Chen'16);
- Baumslag–Solitar groups: $scl(g) \geq 1/12$ (Clay–Forester–Louwsma'12);
- Hyperbolic groups: $scl(g) \geq f(\delta)$, where δ is the hyperbolicity constant (Calegari–Fujiwara'10);
- Right-angled Artin groups: $scl(g) \geq 1/24$ (Fernós–Forester–Tao'16).

Thm 1. (Forester–S.–Tao'17)

Let $G = A(\Gamma)$ be a right-angled Artin group whose defining graph Γ has chromatic number k. Then every nontrivial element $g \in G$ satisfies: $scl(g) \geq 1/(6k)$.

Thm 1. (Forester–S.–Tao'17)

Let $G = A(\Gamma)$ be a right-angled Artin group whose defining graph Γ has chromatic number k. Then every nontrivial element $g \in G$ satisfies: $scl(g) \geq 1/(6k)$.

Thm 2. (Forester–S.–Tao'17)

Let $G = A(\Gamma)$ be a right-angled Artin group whose defining graph Γ does not contain triangles. Then every nontrivial element $g \in G$ satisfies $scl(g) \geq 1/20$.

Thm 1. (Forester–S.–Tao'17)

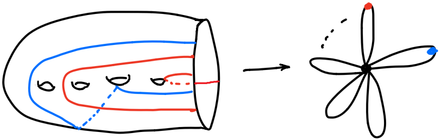
Let $G = A(\Gamma)$ be a right-angled Artin group whose defining graph Γ has chromatic number k. Then every nontrivial element $g \in G$ satisfies: $scl(g) \geq 1/(6k)$.

Thm 2. (Forester–S.–Tao'17)

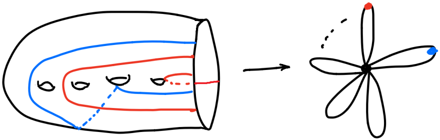
Let $G = A(\Gamma)$ be a right-angled Artin group whose defining graph Γ does not contain triangles. Then every nontrivial element $g \in G$ satisfies $scl(g) \geq 1/20$.

Note: Thm 2 is not a consequence of Thm 1, since there exist triangle-free graphs with large chromatic number.

Culler's Proof: G free group, $scl(g) \geq 1/6$ for all $g \in G$.

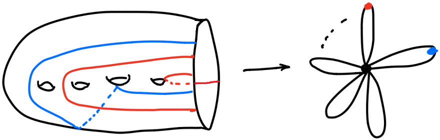


Culler's Proof: G free group, $scl(g) \geq 1/6$ for all $g \in G$.



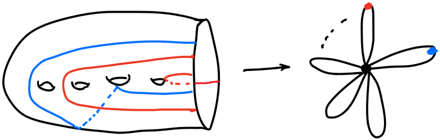
Transversality $\Longrightarrow$ system of nontrivial pairwise disjoint arcs labeled by the generators.

Culler's Proof: G free group, $scl(g) \geq 1/6$ for all $g \in G$.



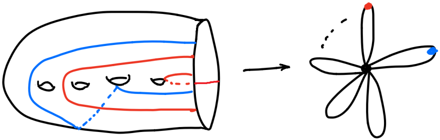
Transversality $\Longrightarrow$ system of nontrivial pairwise disjoint arcs labeled by the generators. **Band** $=$ maximal collection of parallel arcs.

Culler's Proof: G free group, $scl(g) \geq 1/6$ for all $g \in G$.



Transversality $\Longrightarrow$ system of nontrivial pairwise disjoint arcs labeled by the generators. **Band** = maximal collection of parallel arcs.
$A(S)$ = maximal number of arcs on S that are pairwise non-parallel and disjoint = number of arcs in an ideal triangulation of $S/\partial S$.

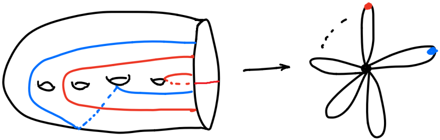
Culler's Proof: G free group, $scl(g) \geq 1/6$ for all $g \in G$.



Transversality $\implies$ system of nontrivial pairwise disjoint arcs labeled by the generators. **Band** = maximal collection of parallel arcs.
$A(S)$ = maximal number of arcs on S that are pairwise non-parallel and disjoint = number of arcs in an ideal triangulation of $S/\partial S$.

$$A(S) = 6genus(S) - 3$$

Culler's Proof: G free group, $scl(g) \geq 1/6$ for all $g \in G$.

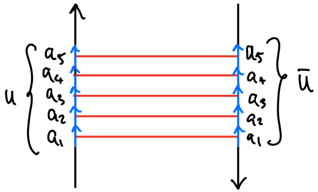


Transversality $\Longrightarrow$ system of nontrivial pairwise disjoint arcs labeled by the generators. **Band** = maximal collection of parallel arcs.
$A(S)$ = maximal number of arcs on S that are pairwise non-parallel and disjoint = number of arcs in an ideal triangulation of $S/\partial S$.

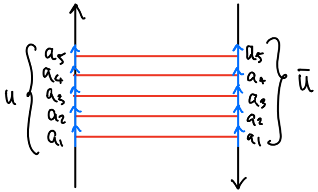
$$A(S) = 6genus(S) - 3$$

$$genus(S) = \frac{A(S)}{6} + \frac{1}{2} \geq \frac{\#\text{Bands}}{6} + \frac{1}{2}$$

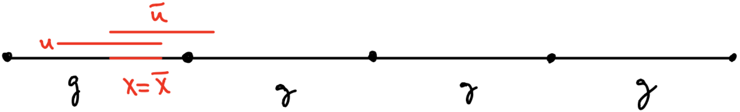
Counting bands:



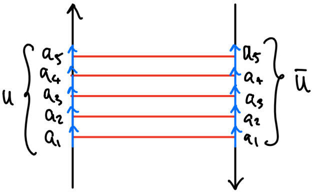
Counting bands:



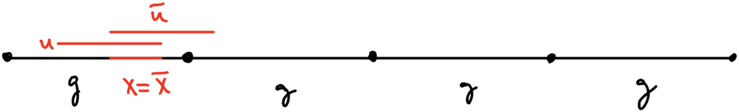
Non-overlapping property: If both u and u^{-1} are subwords of a word g in a free group, then $|u| \leq |g|/2$.



Counting bands:



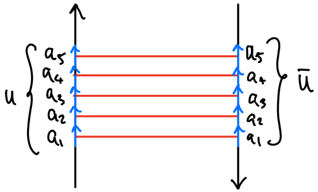
Non-overlapping property: If both u and u^{-1} are subwords of a word g in a free group, then $|u| \leq |g|/2$.



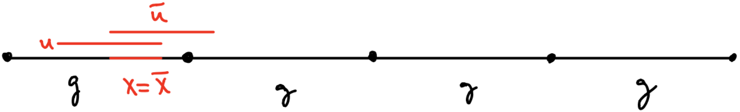
Corollary: Number of bands in S with $\partial S = g^n$ is at least n.

$$cl(g^n) = genus(S) \geq \frac{\#\text{Bands}}{6} + \frac{1}{2} \geq \frac{n}{6} + \frac{1}{2}$$

Counting bands:



Non-overlapping property: If both u and u^{-1} are subwords of a word g in a free group, then $|u| \leq |g|/2$.

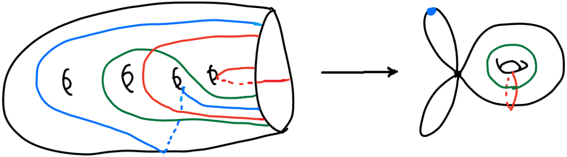


Corollary: Number of bands in S with $\partial S = g^n$ is at least n.

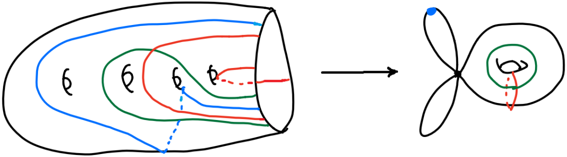
$$cl(g^n) = genus(S) \geq \frac{\#\text{Bands}}{6} + \frac{1}{2} \geq \frac{n}{6} + \frac{1}{2}$$

$$scl(g^n) = \lim_{n\to\infty} \frac{cl(g^n)}{n} \geq \frac{1}{6}. \quad \square$$

Back to RAAGs: $G = A_\Gamma$, $\quad X =$ Salvetti complex for A_Γ:

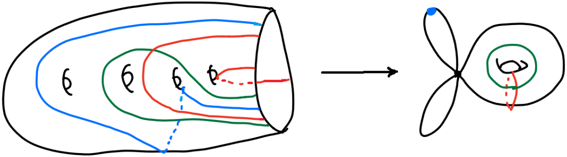


Back to RAAGs: $G = A_\Gamma$, $X =$ Salvetti complex for A_Γ:



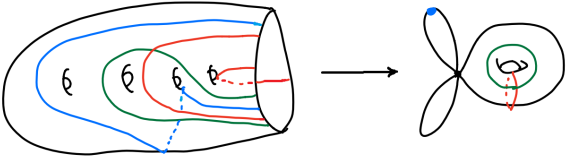
Now arcs may cross, but when they do, their labels must commute.
$k =$ chromatic number of Γ, $V(\Gamma) = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k$.
$a, b \in A_i$ means a, b do not commute. Hence, $A(S) \geq \#\text{Bands}/k$.

Back to RAAGs: $G = A_\Gamma$, X = Salvetti complex for A_Γ:



Now arcs may cross, but when they do, their labels must commute.
k = chromatic number of Γ, $V(\Gamma) = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k$.
$a, b \in A_i$ means a, b do not commute. Hence, $A(S) \geq \#\text{Bands}/k$.
All we need to finish the proof is the inequality $\#\text{Bands} \geq n$.

Back to RAAGs: $G = A_\Gamma$, $\quad X =$ Salvetti complex for A_Γ:

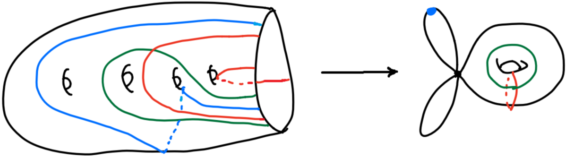


Now arcs may cross, but when they do, their labels must commute.
$k =$ chromatic number of Γ, $V(\Gamma) = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k$.
$a, b \in A_i$ means a, b do not commute. Hence, $A(S) \geq \#\text{Bands}/k$.
All we need to finish the proof is the inequality $\#\text{Bands} \geq n$.

Non-overlapping property for RAAGs:

Suppose w is a word representing the conjugacy class of g^n, for $g \in A_\Gamma$. If both u and u^{-1} are subwords of w, then $|u| \leq |w|/(2n)$.

Back to RAAGs: $G = A_\Gamma$, X = Salvetti complex for A_Γ:



Now arcs may cross, but when they do, their labels must commute.
k = chromatic number of Γ, $V(\Gamma) = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k$.
$a, b \in A_i$ means a, b do not commute. Hence, $A(S) \geq \#\text{Bands}/k$.
All we need to finish the proof is the inequality $\#\text{Bands} \geq n$.

Non-overlapping property for RAAGs:

Suppose w is a word representing the conjugacy class of g^n, for $g \in A_\Gamma$. If both u and u^{-1} are subwords of w, then $|u| \leq |w|/(2n)$.

The proof uses geometry of half-spaces in the CAT(0) cover of the Salvetti complex and all four of axioms of Haglund and Wise for hyperplane pathologies in special cube complexes, recast in terms of actions.

Open Questions

Open Questions

Open Question 1: What is the best possible lower bound for *scl* in RAAGs?

Open Questions

Open Question 1: What is the best possible lower bound for *scl* in RAAGs?

Breaking News: In February 2018, Nicolaus Heuer has proved using quasimorphisms that $scl(g) \geq 1/2$ uniformly for all RAAGs.

Open Questions

Open Question 1: What is the best possible lower bound for *scl* in RAAGs?

Breaking News: In February 2018, Nicolaus Heuer has proved using quasimorphisms that $scl(g) \geq 1/2$ uniformly for all RAAGs.

Open Question 2: Is *scl* computable for RAAGs?

Open Questions

Open Question 1: What is the best possible lower bound for *scl* in RAAGs?

Breaking News: In February 2018, Nicolaus Heuer has proved using quasimorphisms that $scl(g) \geq 1/2$ uniformly for all RAAGs.

Open Question 2: Is *scl* computable for RAAGs?

Open Question 3: Is *scl* rational for RAAGs?

Open Questions

Open Question 1: What is the best possible lower bound for *scl* in RAAGs?
Breaking News: In February 2018, Nicolaus Heuer has proved using quasimorphisms that $scl(g) \geq 1/2$ uniformly for all RAAGs.

Open Question 2: Is *scl* computable for RAAGs?

Open Question 3: Is *scl* rational for RAAGs?

Thank you!