Divergence in Coxeter groups

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Joint work with

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BG–UToledo joint Geometry and Topology seminar
September 8, 2022
Geometry of groups

Let $G$ be a finitely presented group: $G = \langle A \mid R \rangle$

$$1 \rightarrow \langle\langle R\rangle\rangle \rightarrow F(A) \rightarrow G \rightarrow 1$$
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Geometric models:

- Cayley graph: $\text{Cay}^1_A(G)$:
  - \{vertices\} $\longleftrightarrow$ $G$
  - \{directed edges\} $\longleftrightarrow$ $G \times A$

- Cayley 2-complex: $\text{Cay}^2_{\langle A \mid R \rangle}(G)$
  - attach 2-cells to the Cayley graph $\text{Cay}^1_A(G)$ equivariantly
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$G$ acts on $\text{Cay}_A^1(G)$ and $\text{Cay}_{\langle A\mid R \rangle}^2(G)$ by isometries.
Examples:

\[ \mathbb{Z} \times \mathbb{Z} = \langle a, b \mid [a, b] \rangle : \]

\[ \text{Cay}^2_{\langle A, IR \rangle} = \mathbb{R}^2 \]
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\[ \mathbb{Z}_2 \star \mathbb{Z}_3 = \langle a \mid a^2 \rangle \star \langle b \mid b^3 \rangle : \]

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Divergence

Let

- $X$ be a 1-ended geodesic metric space
- $e$ basepoint,
- $S(e, r)$ sphere of radius $r$ around $e$

The **divergence** of $X$ is:

\[
div_X(r) = \sup_{x,y\in S(e,r)} \inf \text{(lengths of } r\text{-avoidant paths from } x \text{ to } y)\]

Examples

- $\mathbb{R}^2$: $div_{\mathbb{R}^2}(r) = \pi r$, linear
- $\mathbb{H}^2$: $div_{\mathbb{H}^2}(r) = \pi \sinh(r) \xrightarrow{r} \pi e^{r/2}$, exponential
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Up to equivalence $\sim$ on functions, $\text{div}_G$ does not depend on the choice of $A$. 
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Q: Given your favorite class of groups, what spectrum of divergence functions does it have?
**Gromov (1991):** Same dichotomy should be true for more general non-positively curved spaces, such as CAT(0) spaces.

**CAT(0):**

- **$X$:**
  - $p$
  - $q$

- **$\mathbb{R}^2$:**
  - $\overline{p}$
  - $\overline{q}$

- $d_X(p, q) \leq d_{\mathbb{R}^2}(\overline{p}, \overline{q})$

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- **Gersten (1994):** $\text{div}_G \sim r^2$

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Non-CAT(0) groups exhibit even wilder behavior:

- Brady–Tran (2021): \( \text{div}_G \sim r^\alpha \) for \( \alpha \) dense in \([2, \infty)\)
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Coxeter groups

A Coxeter group $W$ is given by:

- finite set $S$
- symmetric matrix $(m_{st})_{s,t \in S}$ such that:
  \[ m_{ss} = 1, \quad m_{st} = m_{ts} \in \{2, 3, 4, \ldots, \infty\} \]

$(W, S)$ is given by presentation:

\[ W = \langle S \mid (st)^{m_{st}} = 1, \text{ for all } s, t \in S \rangle \]

$m_{st} = \infty$ means that $st$ has infinite order.
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Encoded by a Coxeter graph (a.k.a. Dynkin graph) with edges labeled $m_{st}$:
Spherical Coxeter groups = finite

Here’s the list of irreducible ones (Coxeter, 1935):

\[ A_n, (n \geq 1): \]

\[ B_n, (n \geq 2): \]

\[ D_n, (n \geq 4): \]

\[ F_4: \]

\[ H_4: \]

\[ H_3: \]

\[ E_6: \]

\[ E_7: \]

\[ E_8: \]

\[ I_2(m), \] (\( m \geq 5, m \neq \infty \)): \[ m \]
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Groups generated by reflections in the faces of a simplex in $S^n$

(m $\geq I_2(m)$, $m \neq \infty$): $m$
Affine Coxeter groups

\[ \tilde{A}_n, (n \geq 2): \]

\[ \tilde{B}_n, (n \geq 4): \]

\[ \tilde{C}_n, (n \geq 3): \]

\[ \tilde{D}_n, (n \geq 5): \]

\[ \tilde{E}_6: \]

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Groups generated by reflections in the faces of a simplex in \( \mathbb{R}^n \)
Lannér’s hyperbolic Coxeter groups

with \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \),
Lannér’s hyperbolic Coxeter groups

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Groups generated by reflections in the faces of a simplex in $\mathbb{H}^n$
Our results, part I

Theorem 1
Let \((W, S)\) be a 1-ended Coxeter system. If \((W, S)\) is irreducible and non-affine, then the divergence of \(W\) is at least quadratic.
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As a corollary we get a complete characterization of linear divergence:

Corollary 2

Let \((W, S)\) be a 1-ended Coxeter system. Then \(W\) has linear divergence if and only if \((W, S) = (W_1, S_1) \times (W_2, S_2)\) where either

1. both \(W_1\) and \(W_2\) are infinite, or
2. \(W_1\) is finite (possibly trivial) and \(W_2\) is irreducible affine of rank \(\geq 3\).
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Corollary 3
If a 1-ended Coxeter group has a super-linear divergence, then its divergence is at least quadratic.
I.e. there is a gap between \(r\) and \(r^2\).
Our results, part II

Ivan Levcovitz introduced what he called a hypergraph index for RACGs, which is an integer $\geq 0$ or $\infty$, computable directly from the Coxeter graph. We generalize it for general Coxeter groups.

Theorem 4

1. $h = 0 \iff W$ has linear divergence.
2. $h = 1 \implies W$ has quadratic divergence.
3. $h$ is finite $\implies$ the divergence of $W$ is bounded above by a polynomial of degree $h + 1$.
4. $h = \infty \iff$ the divergence of $W$ is exponential.
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**Conjecture**

$h$ is finite $\iff$ the divergence of $(W, S)$ is polynomial of degree $h + 1$. 

Levcovitz (2020): true for right-angled Coxeter groups ($m_s, t \in \{2, 1\}$). We proved it for certain series of non-right-angled Coxeter groups.
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**Theorem 5**

Let \((W, S)\) be a Coxeter system with the Coxeter graph \(\Delta = \Delta(W, S)\) and hypergraph index \(h = h(W, S)\). If \(h\) is finite then \(h \leq b_1(\Delta) + 1\), where \(b_1(\Delta)\) is the 1-st Betti number of \(\Delta\).
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\[ b_1(\Delta) = e - v + k, \quad v = \#\text{vertices}, \quad e = \#\text{edges}, \quad k = \#\text{components}. \]
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\[ b_1(\Delta) = e - v + k, \quad v = \#\text{vertices}, \quad e = \#\text{edges}, \quad k = \#\text{components}. \]

Corollary 6
If a Coxeter group \(W\) is not relatively hyperbolic, then the divergence of \(W\) is bounded above by a polynomial of degree \(b_1(\Delta) + 2\).

Corollary 7
If the Coxeter graph of \((W, S)\) is a tree and \(W\) is 1-ended, then \(W\) has divergence linear, quadratic or exponential only. Moreover, each of these possibilities is realized.
Key idea

Behrstock–Caprace–Hagen–Sisto (2017): A Coxeter group $W$ is either:

- relatively hyperbolic $\iff$ $\text{div} \simeq \text{exponential}$
- thick $\iff$ $\text{div}$ is $\preceq$ a polynomial
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Goal: Determine the exact upper bound.
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Goal: Determine the exact upper bound.

A relatively hyperbolic group $H$ has a family of peripheral subgroups $P_i$:

1. Each $\mathbb{Z} \times \mathbb{Z}$ subgroup of $H$ must be contained in some of $P_i$
2. Groups $P_i$ and all their conjugates must intersect in finite subgroups
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Plan: Build candidates for peripheral subgroups $P_i \subseteq W$ forced by (1) and (2). Start with obvious subgroups containing $\mathbb{Z} \times \mathbb{Z}$ and take their joins if they intersect infinitely. Once the process stops:

- if no subgroup $P_i$ equal $W$: we get an honest peripheral structure and $W$ is relatively hyperbolic $\implies$ div $\simeq$ exponential
- if some $P_i = W$, the Coxeter group $W$ is thick, and the number of steps before stabilization is our hypergraph index $h \implies \text{div} \leq r^{h+1}$. 
More formally:

**Wide subsets:** \( \Omega(S) = \text{maximal sets of the form } A \times B \) where
- \( A, B \) both nonspherical, or
- \( A \) irreducible affine of \( \text{rk} \geq 3 \), \( B \) spherical (or empty)

**Slab subsets:** \( \Psi(S) = \text{maximal sets of the form: } A \times K \), such that
- \( A \) is minimal nonspherical
- \( K \) is maximal nonempty spherical, commuting with \( A \)
- there does not exist \( T \in \Omega(S) \) such that \( A \times K \subseteq T \).

**Define:**

\[ \Lambda_0(S) = \Omega(S) \cup \Psi(S), \]
\[ \Lambda_{i+1}(S) = \text{set of all unions of elements in } \equiv_i \text{ equivalence class on } \Lambda_i(S), \]

generated by the condition “\( T \cap T' \) is nonspherical”

Then the **hypergraph index** \( h \) is:

- if \( S \in \Lambda_h(S) \setminus \Lambda_{h-1}(S) \) and \( \Omega(S) \neq \emptyset \): \( h \in \mathbb{N} \)
- otherwise \( h = \infty \)
\[ \Lambda_0 = \{ T_1, T_2, T_3 \} \]

\[ T_i \cap T_j = C_2 \times C_2, \text{ spherical} \]

\[ \Lambda_1 = \Lambda_0, \text{ relatively hyperbolic with peripheral subgroups } \{ W_{T_1}, W_{T_2}, W_{T_3} \} \]

\[ h = \infty \]
(b) \( h = 1 \)

\[
T_1 = \{ s_2, s_3, s_9, s_8 \} \times \{ s_4, s_5, s_6 \} = S \setminus \{ s_3, s_7 \}
\]

\[
T_2 = \{ s_3, s_2, s_1, s_9 \} \times \{ s_5, s_6, s_7 \} = S \setminus \{ s_4, s_8 \}
\]

\[
T_1 \cap T_2 = \{ s_1, s_2, s_3 \} \times \{ s_5, s_6 \} = \tilde{C}_2 \times B_2
\]

\[
T_1 \cup T_2 = \text{all of } S \quad h = 1
\]

(c) \( h = 2 \)

\[
T_1 = S \setminus \{ s_1, s_5 \} = \{ s_2, s_3, s_4 \} \times \{ s_6, s_7, s_8, s_9 \}
\]

\[
T_2 = S \setminus \{ s_1, s_6 \} = \{ s_2, s_3, s_4 \} \times \{ s_5, s_7, s_8, s_9 \}
\]

\[
T_1 \cap T_2 = \{ s_2, s_3, s_4 \} \times \{ s_5, s_6, s_7, s_8 \} = \tilde{C}_2 \times \tilde{C}_2, \text{ non spherical}
\]

\[
T_1 \cup T_2 = S \setminus \{ s_1 \}
\]

\[
T_3 = S \setminus \{ s_2, s_9 \} = \{ s_3, s_4, s_5, s_6 \} \times \{ s_1, s_7, s_8 \} = \tilde{C}_5 \times A_1
\]

\[
T_3 \cap T_4 = \tilde{C}_5, \text{ non spherical}, \quad h = 2
\]

\[
T_5 = T_3 \cup T_4 = S
\]