ON A QUESTION OF PETER SARNAK

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Consider the following pair of integer matrices (taken from [CDGP, p. 43]):

$$A = \begin{pmatrix} -9 & -3 & 5 & 3 \\ 0 & 1 & 0 & -1 \\ -20 & -5 & 11 & 5 \\ -15 & 5 & 8 & -4 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

They preserve the skew-symmetric form with matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ and so they belong to the integral symplectic group $\mathrm{Sp}(4,\mathbb{Z})$. In his 2012 MSRI "Notes on thin groups" [Sar], Peter Sarnak asks if the index of the subgroup $\langle A, T\rangle$ in $\mathrm{Sp}(4,\mathbb{Z})$ is finite or infinite. In this note we prove the following result:

Proposition. *The subgroup* $\langle A, T^3\rangle$ *is of infinite index in* $\mathrm{Sp}(4,\mathbb{Z})$*.*

The proposition will follow from the proof of Corollary 2 below.

Consider a real symplectic vector space $\mathbb{R}^{2m}$ with the standard basis $(e_1, \dots, e_{2m})$ endowed with the skew-symmetric bilinear form with matrix $J = \left(\begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix}\right)$. Recall that for any $a \in \mathbb{R}^{2m}$ the *symplectic transvection* along a, $T_a \colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is defined as follows:

$$T_a(v) = v + J(a, v)a.$$

Note that A has order 5 and T is the matrix of transvection T_{e_2} along vector e_2. Therefore the group $\langle A, T\rangle$ is the semidirect product of

$$H = \langle T, ATA^{-1}, A^2TA^{-2}, A^3TA^{-3}, A^4TA^{-4}\rangle \quad \text{and} \quad \langle A\rangle \cong \mathbb{Z}/5\mathbb{Z},$$

and A^iTA^{-i} is the matrix of the transvection $T_{A^ie_2}$ along A^ie_2.

If we could prove that H is a free group, then H would have infinite index in $\mathrm{Sp}(4,\mathbb{Z})$, as the latter is not virtually free (see Corollary 2 below). This leads us to consider a question:

When does a collection of symplectic transvections along vectors $a_1, \dots, a_n$ from $\mathbb{R}^{2m}$ generate a free group of rank n?

In a very similar setting, H. Hamidi-Tehrani [HT, Section 7] gave some sufficient conditions for an arbitrary collection of Dehn twists along simple closed curves on an oriented surface to generate a free group of finite rank. We will try to axiomatize his result and apply it for the family of symplectic transvections.

Let X be a nonempty set. Suppose that there is a pairing $(.,.) : X \times X \to \mathbb{R}_{\geq 0}$ and for all c in X there exists a bijective transformation $T_c \colon X \to X$ satisfying the following conditions:

(1) $(a, a) = 0$ for all $a \in X$;
(2) $(a, b) = (b, a)$ for all $a, b \in X$;
(3) $T_a(a) = a$ for all $a \in X$;
(4) $(T_c(a), T_c(b)) = (a, b)$ for all $a, b, c \in X$;

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(5) $(T_a^k(x), b) \le |k|(a,b)(x,a) + (x,b)$ for all $a, b \in X$, $k \in \mathbb{Z}$;
(6) $(T_a^k(x), b) \ge |k|(a,b)(x,a) - (x,b)$ for all $a, b \in X$, $k \in \mathbb{Z}$.

Suppose $a_1, \ldots, a_n$ is a collection of $n \ge 3$ elements of X such that $(a_i, a_j) > 0$ for all $i \ne j$. For arbitrary distinct i, j, k from $\{1, \ldots, n\}$ fix real numbers $\lambda_{ijk} > 1$ and $\mu_{ij} > 0$, such that $\mu_{ji} = \mu_{ij}^{-1}$. Hamidi-Tehrani introduces the following "attracting" sets for the ping-pong process, depending on λ_{ijk} and μ_{ij}:

$$N_{a_i} = \{x \in X \mid (x, a_i) < \mu_{ij}(x, a_j), \quad \frac{(x, a_k)}{(x, a_j)} < \lambda_{ijk} \frac{(a_i, a_k)}{(a_i, a_j)}, \text{ for all } j \ne i,\ k \ne i,\ j \ne k\}.$$

Obviously $a_i \in N_{a_i}$. Also all N_{a_i} are mutually disjoint: indeed, if some $x \in X$ belongs to both N_{a_i} and N_{a_j} then $(x, a_i) < \mu_{ij}(x, a_j)$ and $(x, a_j) < \mu_{ji}(x, a_i)$, so $(x, a_i) < \mu_{ij}\mu_{ji}(x, a_i) = (x, a_i)$, a contradiction.

By expressing the condition

$$T_{a_i}^k(N_{a_j}) \subseteq N_{a_i} \quad \text{for all} \quad 1 \le i \ne j \le n$$

in terms of numbers k, λ_{ijk} and μ_{ij} and using only properties (1)–(6) above, he obtains a rather complicated system of nonlinear inequalities involving k, λ_{ijk}, μ_{ij}, see [HT, Lemma 7.1]. He also provides simple sufficient conditions on numbers (a_i, a_j) guaranteeing that there exist values of λ_{ijk} and μ_{ij} such that these inequalities are satisfied for all $|k| \ge 1$ [HT, Th. 7.2]:

Theorem (Hamidi-Tehrani Ping Pong)**.** *Let $a_1, \ldots, a_n$ be $n \ge 3$ elements from X such that $(a_i, a_j) > 0$ for all $i \ne j$. Denote $M = \max\{(a_i, a_j)\}_{i \ne j}$ and $m = \min\{(a_i, a_j)\}_{i \ne j}$. If*

$$\frac{M}{m^2} \le \frac{1}{6},$$

then the group generated by transformations $\langle T_{a_1}, \ldots, T_{a_n} \rangle$ is free of rank n. □

In his article, he works in the situation where:

X is the set of all simple closed curves on the surface;
(a, b) denotes the geometric intersection number of curves a, b;
T_c denotes the right Dehn twist about the curve c.

In our situation, we can set these ingredients to be:

$X = \mathbb{R}^{2m} \setminus 0$;
$(a, b) := |J(a, b)|$, the absolute value of the symplectic form J on a, b;
$T_c =$ the symplectic transvection along vector c.

We just need to prove

Lemma. *The X, $(.,.)$ and T_c defined above satisfy conditions (1)–(6).*

Proof. The properties (1)–(3) are obvious. Property (4) holds since a symplectic transvection preserves the symplectic form J. Let's show (5) and (6).

If $a, b \in \mathbb{R}^{2m}$ then

$$T_a^k(x) = x + k \cdot J(a, x)a$$

and so

$$J(T_a^k(x), b) = J(x, b) + k \cdot J(a, x)J(a, b), \text{ so}$$
$$J(T_a^k(x), b) - J(x, b) = k \cdot J(a, x)J(a, b),$$

and therefore

$$|k \cdot J(a, x)J(a, b)| = |J(T_a^k(x), b) - J(x, b)|.$$

Thus,

$$|k \cdot J(a,x)J(a,b)| \le |J(T_a^k(x),b)| + |J(x,b)|,$$

or

$$|J(T_a^k(x),b)| \ge |k| \cdot |J(a,x)| \cdot |J(a,b)| - |J(x,b)|,$$

which gives us (6). Also,

$$|J(T_a^k(x),b)| = |(J(T_a^k(x),b) - J(x,b)) + J(x,b)| \le |J(T_a^k(x),b) - J(x,b)| + |J(x,b)| = \\ |k \cdot J(a,x)J(a,b)| + |J(x,b)| = |k| \cdot |J(a,x)| \cdot |J(a,b)| + |J(x,b)|,$$

which gives us (5). □

Let's now figure out what are the numbers (a_i, a_j) for the group

$$H = \langle T, ATA^{-1}, A^2TA^{-2}, A^3TA^{-3}, A^4TA^{-4}\rangle = \langle T_{e_2}, T_{Ae_2}, T_{A^2e_2}, T_{A^3e_2}, T_{A^4e_2}\rangle.$$

In this case,

$$a_1 = e_2, \quad a_2 = Ae_2, \quad a_3 = A^2e_2, \quad a_4 = A^3e_2, \quad a_5 = A^4e_2.$$

Here are the powers of matrix A:

$$A = \begin{pmatrix} -9 & -3 & 5 & 3 \\ 0 & 1 & 0 & -1 \\ -20 & -5 & 11 & 5 \\ -15 & \boxed{5} & 8 & -4 \end{pmatrix}, \quad A^2 = \begin{pmatrix} -64 & 14 & 34 & -11 \\ 15 & -4 & -8 & 3 \\ -115 & 25 & 61 & -20 \\ 35 & \boxed{-10} & -19 & 6 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 61 & -19 & -34 & 8 \\ -20 & 6 & 11 & -3 \\ 115 & -35 & -64 & 15 \\ -25 & \boxed{10} & 14 & -4 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 11 & 8 & -5 & 0 \\ 5 & -4 & -3 & 1 \\ 20 & 15 & -9 & 0 \\ 5 & \boxed{-5} & -3 & 1 \end{pmatrix}.$$

We have:

$$J(a_i, a_j) = J(A^ie_2, A^je_2) = J(e_2, A^{j-i}e_2),$$

so we need only to compute $J(e_2, A^me_2)$ for $m = 1, 2, 3, 4$.

Since $J(e_2, e_4) = 1$, and $J(e_2, e_i) = 0$ for $i \ne 4$, we see that for any $v \in \mathbb{R}^{2m}$,

$$J(e_2, v) = \text{coefficient at } e_4 \text{ of } v.$$

This gives us the following values (see the boxed entries in matrices A^i):

$$|J(e_2, Ae_2)| = 5, \quad |J(e_2, A^2e_2)| = 10, \quad |J(e_2, A^3e_2)| = 10, \quad |J(e_2, A^4e_2)| = 5.$$

So we see that $M = \max\{(a_i, a_j)\}_{i\ne j} = 10, \quad m = \min\{(a_i, a_j)\}_{i \ne j} = 5$ and

$$\frac{M}{m^2} = \frac{2}{5} \not\le \frac{1}{6}$$

so we cannot apply Hamidi-Tehrani Ping Pong theorem for the group H.

Scaling trick. Observe that

$$T_{\alpha a}(v) = v + J(\alpha a, v)\alpha a = v + \alpha^2 J(a, v)a = T_a^{\alpha^2}(v),$$

if α^2 is an integer. Thus we may try to scale our vectors a_i's by a suitable constant α so that we will get $\frac{M}{m^2} \le \frac{1}{6}$. (And this is the reason why we are working with the real symplectic space $\mathbb{R}^{2m}$ instead of the integer symplectic module $\mathbb{Z}^{2m}$.) By doing so, we need to ensure that α^2 is an integer so that we obtain the freeness of the group generated by the *integer* powers of transvections $\langle T_{a_1}^{\alpha^2}, \dots, T_{a_n}^{\alpha^2} \rangle$.

Let's try $\alpha = \sqrt{2}$. Then M and m will be scaled by $\alpha^2 = 2$, and we will have:

$$\frac{M}{m^2} = \frac{20}{10^2} = \frac{1}{5} \not\leq \frac{1}{6},$$

so $\alpha = \sqrt{2}$ doesn't work.

But if we take $\alpha = \sqrt{3}$ then

$$\frac{M}{m^2} = \frac{30}{15^2} = \frac{2}{15} = \frac{4}{30} \leq \frac{5}{30} = \frac{1}{6},$$

and the Hamidi-Tehrani Ping Pong theorem tells us that the group generated by transvections

$$\langle T_{\sqrt{3}a_i}, i = 1 \dots, 5 \rangle$$

is free of rank 5. Since $T_{\sqrt{3}a_i} = T_{a_i}^3 = T_{A^{i-1}e_2}^3 = A^{i-1}T^3A^{-(i-1)}$, we get the following

Corollary 1. *The group* $H_3 := \langle T^3, AT^3A^{-1}, A^2T^3A^{-2}, A^3T^3A^{-3}, A^4T^3A^{-4} \rangle$ *is free of rank* 5.

Corollary 2. *Subgroups* H_3 *and* $\langle A, T^3 \rangle = H_3 \rtimes \langle A \rangle$ *have infinite index in* $\mathrm{Sp}(4, \mathbb{Z})$.

Proof. Suppose H_3 has finite index in $\mathrm{Sp}(4, \mathbb{Z})$. Then H_3 contains a normal (in $\mathrm{Sp}(4, \mathbb{Z})$) subgroup N of finite index. Since H_3 is a free group by Corollary 1, N will be a free group as well. By the solution of the congruence subgroup problem for $\mathrm{Sp}(4, \mathbb{Z})$ [BMS], subgroup N must contain some congruence subgroup

$$C = \mathrm{Sp}(4, \mathbb{Z}, m) = \{A \in \mathrm{Sp}(4, \mathbb{Z}) : A \equiv I \mod m\}.$$

Being a subgroup of a free group N, C itself must be free. However, this is not possible since the abelianization $C^{\mathrm{ab}} = C/[C, C]$ consists entirely of torsion elements, see [Sat, Cor. 10.2]. ☐

Remark. One might expect that the application of Lemma 7.1 of [HT] itself may lead to a stronger result than the application of its consequence, the Hamidi-Tehrani Ping Pong theorem. However, we were able to show through lengthy computations that for the case of vectors $(e_2, Ae_2, A^2e_2, A^3e_2, A^4e_2)$ the system of inequalities from Lemma 7.1 of [HT] has no solutions with $|k| = 1$, so it does not allow to establish the freeness of the group $H = \langle A, T \rangle$ either.

References

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