L-functions of $S_3(\Gamma(4, 8))$

Takeo Okazaki

Department of Mathematics, Graduate School of Science, Osaka University,
Machikaneyama 1-16, Toyonaka, Osaka 560-0043, Japan

Received 6 February 2006; revised 28 August 2006
Available online 9 February 2007
Communicated by D. Zagier

Abstract

We prove most of B. van Geemen and D. van Straten’s conjectures on the explicit description of Andrianov L-functions of Siegel cuspforms of degree 2 of weight 3 for the group $\Gamma(4, 8)$, which are contained in [B. van Geemen, D. van Straten, The cuspform of weight 3 on $\Gamma_2(2, 4, 8)$, Math. Comp. 61 (204) (1993) 849–872]. These L-functions are related to the Galois representations on the Siegel modular threefold $\Gamma(4, 8)/\mathfrak{S}_2$ as determined by B. van Geemen and N. Nygaard [B. van Geemen, N.O. Nygaard, On the geometry and arithmetic of some Siegel modular threefolds, J. Number Theory 53 (1995) 45–87].

© 2006 Elsevier Inc. All rights reserved.

Keywords: Siegel modular form and modular threefold; Andrianov L-function; Yoshida lift; Jacquet–Langlands theory

1. Introduction and main idea

As a next step of the Eichler–Shimura theory, B. van Geemen and N. Nygaard [3] compare L-functions related to Galois representations on Siegel modular threefolds $\Gamma\backslash \mathfrak{S}_2$ and Andrianov L-functions of cuspforms in $S_3(\Gamma)$. Here, the $\Gamma$’s are congruence subgroups larger than

$$\Gamma(4, 8) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(4) \mid \text{diag}(B) \equiv \text{diag}(C) \equiv 0 \pmod{8} \right\}.$$ 

They determined the Galois representations on $H_3^\Gamma$ of the modular threefolds, and give a conjecture relating these to Andrianov L-function of certain cuspforms.

E-mail address: okazaki@gaia.math.wani.osaka-u.ac.jp.

0022-314X/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jnt.2006.11.005
Further, B. van Geemen and D. Straten [4] analyzed $S_3(\Gamma(4, 8))$ and determined all the Hecke eigenforms belonging to $S_3(\Gamma(4, 8))$ as follows. Using a theta embedding $\Theta : \Gamma(2, 4) \setminus \mathbb{H}_2 \to \mathbb{P}^1$, and regarding $M_3(\Gamma(2, 4, 8))$ as a quotient space of homogeneous polynomials of degree 6 with respect to the theta constants in $\Theta$, they showed that $S_3(\Gamma(4, 8))$ is spanned by certain six-fold products of theta constants. Considering the action of $Sp_2(\mathbb{Z})$ on these products due to the transformation formula, they showed $S_3(\Gamma(4, 8))$ is divided into direct sums of seven irreducible $Sp_2(\mathbb{Z})$-modules. The seven modules contain the elements in Table 1.

Here, we set the Igusa theta constant associated to a characteristic $m = (m_1, m_2, m_3, m_4)$, with $m_i \in \{0, 1\}$ by

$$\theta_m(Z) = \sum_{a, b \in \mathbb{Z}} e\left(\left(Z\left[\frac{m_1 + (a/2)}{m_2 + (b/2)}\right] + m_3(m_1 + 2a)/2 + m_4(m_2 + 2b)/2\right)/2\right),$$

where we denote $e(x) = \exp(2\pi \sqrt{-1}x), x \in \mathbb{C}$, and $Z[v] = i\nu Z v, Z \in S_2$.

For a six-fold product $\theta$, a character $\chi_\theta$ on $\Gamma(2)$ is determined by $\chi_\theta(\nu) = \frac{\vartheta(\nu)}{\vartheta(1)}$ and satisfies $\chi_\theta^2 = 1$. They showed that $\chi_\theta$ is characterized by a unique $\theta$. When $\chi_\theta^2 = 1$, the Hecke algebra $\mathcal{H}(\mathbb{Z}) = \bigotimes_{p \neq 2} \mathcal{H}_v(GSp_2(\mathbb{Q}_p), GSp_2(\mathbb{Z}_p))$ outside of 2 acts on the one-dimensional space $\mathbb{C}\theta$, and thus $\theta$ is a Hecke eigenform. When $\chi_\theta$ is not real-valued, $\mathcal{H}(\mathbb{Z})$ acts on the two-dimensional space spanned by $\theta$ and $\theta'$ which has the complex conjugate character of $\chi_\theta$, so an appropriate linear combination of $\theta$ and $\theta'$ is a Hecke eigenform (cf. [4, Proposition 7.4]).

Computing some Hecke operators for the eigenforms obtained as above, they conjectured that their Andrianov $L$-functions are as in Table 2.

Here $\omega_d$ denotes the quadratic character associated to the extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ and $\otimes$ denotes the convolution product. The symbols $\theta_\mu, \rho_1, \psi_1$ denote some elliptic eigenforms belonging to the spaces (see Table 3).

### Table 1

<table>
<thead>
<tr>
<th>Space</th>
<th>dim</th>
<th>Theta series</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_3(\Gamma(4))$</td>
<td>15</td>
<td>$\theta_1 = \theta(1,0,0,0)\theta(0,1,0,0)\theta(1,1,0,0)\theta(0,1,1,0)\theta(1,1,1,1)(Z)$</td>
</tr>
<tr>
<td>$S_3(\Gamma(4,8))$</td>
<td>90</td>
<td>$\theta_2 = \theta(0,0,0,1)\theta(0,0,0,0)\theta(1,0,0,0)\theta(0,1,0,0)\theta(0,0,1,0)\theta(0,0,0,1)\theta(0,0,0,1)(Z)$</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>$\theta_3 = \theta(1,0,0,0)\theta(1,0,1,0)\theta(0,1,0,0)\theta(1,0,0,0)\theta(0,0,1,0)\theta(0,0,0,1)\theta(0,0,0,1)(Z)$</td>
</tr>
<tr>
<td></td>
<td>360</td>
<td>$\theta_4 = \theta(0,0,1,0)\theta(1,0,1,0)\theta(0,1,0,0)\theta(0,0,1,1)\theta(0,0,0,1)\theta(0,1,1,1)(Z)$</td>
</tr>
<tr>
<td></td>
<td>180</td>
<td>$\theta_5 = \theta(0,0,0,0)\theta(0,0,1,0)\theta(0,1,0,0)\theta(0,0,1,0)\theta(0,0,0,1)\theta(0,1,1,1)(Z)$</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>$\theta_6 = \theta(0,0,0,0)\theta(0,0,0,0)\theta(0,0,1,0)\theta(0,0,0,1)\theta(1,0,0,0)\theta(0,1,1,1)(Z)$</td>
</tr>
<tr>
<td></td>
<td>360</td>
<td>$\theta_7 = \theta(0,0,0,0)\theta(0,0,0,0)\theta(1,0,0,0)\theta(0,1,0,0)\theta(0,0,1,1)\theta(0,0,1,1)(Z)$</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>Label</th>
<th>Eigenform</th>
<th>Conjectured Andrianov $L$-function outside of 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_6^-; 0; 2$</td>
<td>$F_1 = \theta_1$</td>
<td>$\zeta(s-1)\zeta(s-2)L(s, \rho_1)$</td>
</tr>
<tr>
<td>$R_4^-; (0; \text{2})$</td>
<td>$F_2 = \theta_2 - 40\theta_2'$</td>
<td>$\zeta(s-1)\zeta(s-2)L(s, \rho_1)$</td>
</tr>
<tr>
<td>$R_4^-; (1; \text{1}; 0)$</td>
<td>$F_3 = \theta_3 + 16\theta_3'$</td>
<td>$\zeta(s-1)\zeta(s-2)L(s, \rho_1 \otimes \omega_{-1})$</td>
</tr>
<tr>
<td>$R_4^-; (1; \text{1})$</td>
<td>$F_4 = \theta_4 + 40\theta_4'$</td>
<td>$L(s-1, \theta_\mu \otimes \omega_{-2})L(s, \rho_3 \otimes \omega_{-2})$</td>
</tr>
<tr>
<td>$R_6^-$</td>
<td>$F_5 = \theta_5$</td>
<td>$L(s-1, \theta_\mu)L(s, \rho_2)$</td>
</tr>
<tr>
<td>$R_6^-; (0; \text{2})$</td>
<td>$F_6 = \theta_6$</td>
<td>$L(s-1, \theta_\mu \otimes \omega_{-2})L(s, \rho_2 \otimes \omega_{-2})$</td>
</tr>
<tr>
<td>$R_6^-; (1; \text{0})$</td>
<td>$F_7 = \theta_7$</td>
<td>$L(s, \theta_\mu \otimes \psi_1)$</td>
</tr>
</tbody>
</table>
Table 3

<table>
<thead>
<tr>
<th>Elliptic cuspform</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_\mu )</td>
<td>( S_2(F_0(32)) )</td>
</tr>
<tr>
<td>( \psi_1 )</td>
<td>( S_3(F_0(32), \omega_{-1}) )</td>
</tr>
<tr>
<td>( \rho_1 )</td>
<td>( S_4(F_0(8)) )</td>
</tr>
<tr>
<td>( \rho_2 = \theta_{\mu,3} )</td>
<td>( S_4(F_0(32)) )</td>
</tr>
<tr>
<td>( \rho_3 )</td>
<td>( S_4(F_0(32)) )</td>
</tr>
</tbody>
</table>

In particular, \( \theta_\mu \) is obtained by the Größen-character \( \mu \) related to the elliptic curve \( y^2 = x^3 - x \) with complex multiplication:

\[
\theta_\mu(z) \sum_{a} \mu(a)e(N(a)z), \quad z \in \mathfrak{H},
\]

where \( a \) runs through all integral ideals of \( \mathbb{Z}[i] \) prime to 2. For these conjectures, our main result is

**Main Theorem.** The conjectures for \( F_i \), \( 1 \leq i \leq 6 \), are true.

Our proof is using the Yoshida lift as follows. The conjectured \( L(s, F_i) \) for \( 1 \leq i \leq 6 \) are products of \( L \)-functions of elliptic modular forms, and the Yoshida lift [14] can provide a Siegel modular form having such a type of \( L \)-function. Indeed, in the \( Sp_2(\mathbb{Z}) \) module generated by \( F_i \), due to the Yoshida lift, we construct an eigenform having the conjectured \( L(s, F_i) \). At this moment, since \( L(s, F) = L(s, F|\gamma) \) with \( F|\gamma \) translated for \( \gamma \in Sp_2(\mathbb{Z}) \) (see Proposition 2.2 for a more rigorous discussion), we see that \( L(s, F_i) \) is just the conjectured one.

Although we believe that the conjecture for \( F_7 \) is true, it seems to need more preparations. By base change, \( \psi_1 \) is lifted to an automorphic form on \( SL_2(\mathbb{Q}(\sqrt{-1})) \). But, the theta lift from \( SO(3, 1) \simeq SL_2(\mathbb{C}) \) to \( Sp_2(\mathbb{R}) \) as in [6] cannot provide a Siegel modular form of weight 3. Further, we are interested in the Galois representation related to \( \psi_1 \) and that related to the modular threefold \( \ker(\chi_{\theta_7}) \setminus \mathfrak{H}_2 \).

This paper is organized as follows. In Section 2, we review the definition of Andrianov \( L \)-function by Evdokimov [2] for adélic forms. In Section 3, we give a short review of the Yoshida lift. In Section 4, we prove the conjectures.

**Notation.** For a ring \( A \) with norms, the group of units of \( A \) is denoted by \( A^\times \) and by \( A^1 \) the group of elements of norm 1. We denote by \( M_k^p(\Gamma, \chi) \) and \( S_k^p(\Gamma, \chi) \) the space of Siegel modular forms and that of cusps of degree \( n \), of weight \( k \), with a character \( \chi \) on a congruence subgroup \( \Gamma \subset Sp_n(\mathbb{Z}) \).

2. Andrianov \( L \)-function for adélic forms

We review the definition of the Andrianov \( L \)-function by Evdokimov [2] for adélic forms, and see how the \( L \)-function changes w.r.t. translations of forms by \( \gamma \in Sp_2(\mathbb{Z}) \) (Proposition 2.2). Further, using this occasion, we recall the definition of the spinor \( L \)-function, and clarify the difference between Andrianov and spinor \( L \)-functions. These \( L \)-functions are likely to be regarded as the same thing, but they are different things, strictly. Indeed, the spinor \( L \)-function is invariant w.r.t. translations by elements of \( Sp_2(\mathbb{Z}) \).
In [2] originally, the Andrianov $L$-function is defined for classical Siegel modular forms, using his Hecke operators. The spinor $L$-function is defined for adélic forms on $GSp_2(\mathbb{A})$, canonically. Then, the Andrianov $L$-function of $F$ coincides with the spinor $L$-function of $F^\flat$. However, when we do not extend $F$ canonically, there may be differences between the $L$-functions. It is caused by the difference of Hecke operators by which the $L$-functions are defined.

The Hecke operators of the former act on forms globally, but those of the latter act locally.

Now, we treat the Andrianov $L$-function. Let $\Gamma(N)$ be the principal congruence subgroup of level $N$. For Dirichlet characters $\eta, \psi$ defined modulo $N$, let $M_k(N, \eta, \psi) \subset M_k(\Gamma(N))$ denote the space of all Siegel modular forms $F$ satisfying

$$ F|_k \gamma(a, b) = \eta(a)\psi(b)F, $$

for every $\gamma(a, b) \equiv \text{diag}[a, ab, a^{-1}, (ab)^{-1}]$ (mod $N$) in $Sp_2(\mathbb{Z})$. Here, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GSp_2(\mathbb{R})$ and $F$ we set

$$ F|_k g(z) = \det(cz + d)^{-k} F((az + b)(cz + d)^{-1}). $$

Every $F \in M_k(\Gamma(N))$ can be decomposed as $F = \sum_{\eta, \psi} F_{\eta, \psi}, F_{\eta, \psi} \in M_k(N, \eta, \psi)$. We set, for $t \in \mathbb{Q}$,

$$ \delta(t) = \text{diag}[1, 1, t, t], \quad \varepsilon(t) = \text{diag}[1, t, t^2, t]. $$

Then, Evdokimov defined for a prime $p \nmid N$ Hecke operators on $M_k(N, \eta, \psi)$ by

$$ T(1, 1, p, p)F = T(\delta(p))F = p^{k-3} \sum_j F|_k H_j, $$

$$ T(1, p, p^2, p)F = T(\varepsilon(p))F = p^{2k-6} \sum_j F|_k L_j, $$

$$ T(p, p, p, p)F = p^{2k-6} \eta(p)F, $$

where the $H_j, L_j$ satisfy $\Gamma \delta(p) \Gamma = \bigsqcup_j \Gamma H_j$, and $\Gamma \varepsilon(p) \Gamma = \bigsqcup_j \Gamma L_j$, $H_j \equiv \delta(p), L_j \equiv \varepsilon(p) \pmod{N}$ with $\Gamma = \Gamma(N)$. Of course, these definitions are independent from the choice of $H_j, L_j$. For an eigenform $F \in M_k(N, \eta, \psi)$ at $p$ with eigenvalues $\lambda(\delta(p)), \lambda(\varepsilon(p))$ for the above Hecke operators, Evdokimov defined the Andrianov $L$-function attached to $F$ by

$$ L^{ae}(s, F)_p = 1 - \lambda(\delta(p)) p^{-s} + (p\lambda(\varepsilon(p)) + p^{2k-5}(p^2 - 1)\eta(p)) p^{-2s} - \eta(p)\lambda(\delta(p)) p^{2k-3-3s} + \eta(p)^2 p^{4k-6-4s}. $$

Next, we recall the definition of the spinor $L$-function. For an automorphic form $f$ on $GSp_2(\mathbb{A})$ which is right $GSp_2(\mathbb{Z}_p)$-invariant, the Hecke operators $T_p(\delta(p)), T_p(\varepsilon(p))$ are defined by

$$ T_p(\delta(p)) f(g) = \sum_j f(g(H_j) p^{-1}), \quad T_p(\varepsilon(p)) f(g) = \sum_j f(g(L_j) p^{-1}). $$
with \((H_j)_p, (L_j)_p\) being the images of \(H_j\) respectively \(L_j\) under the embedding \(GSp_2(\mathbb{Q}) \to GSp_2(\mathbb{Q}_p)\). Using the eigenvalues \(\lambda^2(\delta(p))\) and \(\lambda^2(\epsilon(p))\), local spinor \(L\)-function of \(f\) is defined by

\[
L^p(s, f)_p = 1 - \lambda^2(\delta(p))p^{-s} + (p\lambda^2(\epsilon(p)) + p(p^2 + 1)\eta(p))p^{-2s}
- \eta(p)\lambda^2(\delta(p))p^{3-3s} + \eta(p)^2p^{6-4s}.
\]

For a classical \(F \in M_k(N, \eta, \psi)\), we extend \(F\) to a function \(F^\natural\) on \(GSp_2(\mathbb{A})\) as follows. By the strong approximation theorem for \(Sp_2(\mathbb{A})\), any element \(g \in GSp_2(\mathbb{A})\) can be decomposed as

\[g = \gamma g_\infty kt_\infty \times \prod_p \delta(tp).\]

Here \(\gamma \in Sp_2(\mathbb{Q}), g_\infty \in Sp_2(\mathbb{R}), k \in \prod_p \Gamma(N)_p, t_\infty \in \mathbb{R}^\times, t_p \in \mathbb{Z}_p^\times\). We set

\[F^\natural(g) = F(g_\infty(t)) \det(ct + d)^{-k}, \quad g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad t = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}.\]

We call \(F^\natural\) the ‘canonical extension of \(F\).’

**Proposition 2.1.** Let \(F \in M_k(\Gamma(N), \eta, \psi)\) be a classical form on \(S_2\) and \(F^\natural\) the canonical extension of \(F\) on \(GSp_2(\mathbb{A})\). Suppose that \(F\) is an eigenform at \(p\). Then, we have

\[L^{ae}(s, F)_p = L^p(s - k + 3, F^\natural)_p.\]

**Proof.** It suffices to see \(\lambda(\delta(p)) = p^{k-3}\lambda^2(\delta(p))\) and \(\lambda(\epsilon(p)) = p^{2k-6}\lambda^2(\epsilon(p))\). This is clear by observing that \(H_j\delta(p)^{-1} \equiv L_j\epsilon(p)^{-1} \equiv 1 \pmod{N}\), and the way \(F^\natural\) is defined.

We now consider the case where \(F \in S_3(\Gamma(4, 8))\) has a character \(\chi\) on \(\Gamma(2)\), for our proof of the conjectures.

**Proposition 2.2.** Suppose that \(F \in S_3(\Gamma(4, 8))\) is a Hecke eigenform with

\[L^{ae}(s, F)_p = 1 - \lambda(\delta(p))a_p p^{-s} + \lambda(\epsilon(p))a_p p^{3-3s} + p^{6-4s}\]

and has a character \(\chi\) on \(\Gamma(2)\). Then \(F|_\chi\) is also a Hecke eigenform with

\[L^{ae}(s, F|_\chi) = 1 - \xi(p)a_p p^{-s} + \lambda(\epsilon(p))a_p p^{3-3s} + p^{6-4s},\]

for a certain function \(\xi\) on \(\mathbb{Z}_2^\times\) defined modulo 8.

**Proof.** Put \(\Gamma = \Gamma(8)\) and take an odd prime \(p\). Then we compute the Hecke operator \(T(\delta(p))\) for \(F|_k\chi\):

\[T(\delta(p))(F|_k\chi) = \sum_j F|_k\chi H_j, \quad H_j \equiv H_1 \pmod{8}, \quad H_1 = \delta(p)\]
with \( \Gamma \delta(p) \Gamma = \bigsqcup_j \Gamma H_j \). Instead of this computation for \( F \), we consider that for \( F^\natural \):

\[
\sum_j F^\natural(\gamma_\infty H_j, \gamma_\infty g_\infty) = \sum_j F^\natural(H_j^{-1} \gamma_\infty^{-1} \gamma_\infty H_j, \gamma_\infty g_\infty)
\]

\[
= \sum_j F^\natural(g_\infty H_j^{-1} \gamma_2^{-1} H_j, g_\infty p), \quad (2.2)
\]

where \( g_\infty \) is an element of \( \text{Sp}_2(\mathbb{A}) \) whose finite components are all 1 and \( \gamma_{v,H}, \gamma_{v,H} \) denote the images by the embedding \( \text{GSp}_2(\mathbb{Q}) \rightarrow \text{GSp}_2(\mathbb{Q}_v) \). Here we use the left \( \text{GSp}_2(\mathbb{Q}) \)-invariance and right product \( \prod_{v \neq 2} \text{GSp}_2(\mathbb{Z}_v) \)-invariance of \( F^\natural \). This computation is continued to

\[
\sum_j F^\natural(g_\infty H_j^{-1} \gamma_2^{-1} H_j p) = \sum_j F^\natural(g_\infty \gamma_2^{-1} \gamma_2 H_j^{-1} \gamma_2^{-1} H_j, g_\infty H_j^{-1} \gamma_2^{-1} H_j, g_\infty p)
\]

\[
= \sum_j \chi_2([\gamma_2, \delta(p)_2^{-1}]) F^\natural(g_\infty \gamma_2^{-1} H_j, g_\infty p)
\]

\[
= \sum_j \lambda^\natural_2(\delta(p)) \chi_2([\gamma_2, \epsilon(p)_2^{-1}]) F^\natural(g_\infty g_\infty g_\infty),
\]

where \([a, b] = aba^{-1}b^{-1}\) for \( a, b \in \text{GSp}_2(\mathbb{Q}_2) \) and \( \chi_2 \) denotes the 2-component of the extended \( \chi \), which is characterized by

\[
\chi_2(k) = \chi(\alpha)^{-1}
\]

for \( k \in \Gamma(2)_2, \alpha \in \Gamma(2), \alpha \equiv k \pmod{8} \). The computation for \( T(\epsilon(p)) \) is also given by

\[
T(\epsilon(p))(F|\gamma) = \chi_2([\gamma_2, \epsilon(p)_2^{-1}]) \lambda(\epsilon(p))(F|\gamma).
\]

We observe that both of the maps

\[
\mathbb{Z}_2^\times \ni t \rightarrow \chi_2([\gamma_2, \delta(t)_2^{-1}]) \in \mathbb{C}^\times, \\
\mathbb{Z}_2^\times \ni t \rightarrow \chi_2([\gamma_2, \epsilon(t)_2^{-1}]) \in \mathbb{C}^\times
\]

are defined modulo 8, and that the latter is always 1 since

\[
[\epsilon(p), \text{Sp}_2(\mathbb{Z}_2)] \subset \Gamma(4, 8) \subset \ker(\chi_2),
\]

reminding that the commutator subgroup of \( \Gamma(2) \) is \( \Gamma(4, 8) \). This proves the assertion. \( \square \)

**Remark 2.3.** Indeed, an example with a nontrivial \( \xi \) is given in [3].
In contrast, for a general automorphic form $f$ on $\text{GSp}_2(\mathbb{A})$, the spinor $L$-function is stable under $\text{Sp}_2(\mathbb{Z})$-translations:

$$L^{sp}(s, f(\gamma_{\infty}g)) = L^{sp}(s, f(g))$$

for every $g \in \text{GSp}_2(\mathbb{A})$ and $\gamma_{\infty} \in \text{Sp}_2(\mathbb{Z}) \subset \text{Sp}_2(\mathbb{R})$. This is clear from the definition. We note that $(F|\gamma)^{\flat}(g) = F^{\flat}(\gamma_{\infty}g)$ does not necessarily hold.

3. Review of the Yoshida lift

The Yoshida lift is a theta lift from a pair of automorphic forms on a definite quaternion algebra $D_{\mathbb{Q}}$ defined over $\mathbb{Q}$ to a Siegel modular form whose spinor $L$-function is the product of the $L$-functions of the pair. Jacquet–Langlands theory [7] associates cuspidal automorphic forms on $D_{\mathbb{A}} \times \mathbb{A}$ to elliptic cuspforms. For every cuspidal automorphic form on $D_{\mathbb{A}} \times \mathbb{A}$, there exists an elliptic cuspform having the same $L$-function. So, we can construct a Siegel modular form whose $L$-function is a product of that of a pair of elliptic modular forms.

We start with a short review of the Yoshida lift. Let $D_{\mathbb{Q}}$ be a definite quaternion algebra over $\mathbb{Q}$ attached to $a, b \in \mathbb{Q} > 0$:

$$D_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}I + \mathbb{Q}J + \mathbb{Q}IJ, \quad I^2 = -a, \quad J^2 = -b, \quad IJ = -JI,$$

with the canonical involution $* : a + bI + cJ + dIJ \rightarrow a - bI - cJ - dIJ$. We denote by $N(x) = x \cdot x^*$ and $\text{Tr}(x) = x + x^*$ the reduced norm and trace of $x \in D_{\mathbb{Q}}$. We put $W_1 = \mathbb{R}I + \mathbb{R}J + \mathbb{R}IJ \subset D_{\infty}$. Considering the action $\tau$ of $D_{\mathbb{A}} \times \mathbb{A}$ on $W_1$ such as $\tau(d)w = d^{-1}wd, d \in D_{\infty}^\times, w \in W_1$, we obtain a representation $\sigma$ of $D_{\infty}^\times / \mathbb{R}^\times$. We denote by $\sigma_{2n} = \text{Sym}^n(\sigma)$ the tensor $n$-tuple product representation on the space $W_n = \text{Sym}^n(W_1)$.

**Definition 3.1 (Automorphic form of type $(\sigma_{2n}, R, \chi)$).** Let $R$ be an order in $D_{\mathbb{Q}}$ and $\chi = \bigotimes_p \chi_p$ be a product of character $\chi_p$ on $R_p^\times$ ($\chi_p$ is trivial at almost all $p$). We define an automorphic form on $D_{\mathbb{A}}^\times$ of type $(\sigma_{2n}, R, \chi)$ to be a $W_n$-valued function $f$ on $D_{\mathbb{A}}^\times$ which satisfies the following conditions (1)–(3):

1. For any $\gamma \in D_{\mathbb{Q}}^\times$ and $x \in D_{\mathbb{A}}^\times$, $f(\gamma x) = f(x)$.
2. For any $h \in D_{\infty}^\times$, $f(xh_v) = \sigma_{2n}(h)f(x)$.
3. For any $k_p \in R_p^\times$, $f(xk_p) = \chi_p(k_p)f(x)$.

We denote by $\mathcal{A}(\sigma_{2n}, R, \chi)$ the space of automorphic forms on $D_{\mathbb{A}}^\times$ of type $(\sigma_{2n}, R, \chi)$. If $\chi$ is trivial, we abbreviate it to $\mathcal{A}(\sigma_{2n}, R)$.

**Remark 3.2.** See [7] for the general definition of automorphic forms. Only the above types of automorphic forms are needed for our use in the Yoshida lift.

We only describe the Yoshida lift from a pair of eigenforms $f_1 \in \mathcal{A}(\sigma_0, R, \chi)$ and $f_2 \in \mathcal{A}(\sigma_2, R, \chi)$ as follows. Associated to the pair, we take a certain $W_1$-valued Schwartz function (i.e., theta kernel or test function) $\Phi = \prod_v \Phi_v$ on $D_{\mathbb{A}}^2$ satisfying (i)–(iii):
(i) \( \Phi_\infty(x_1, x_2) = P(x_1^*x_2) \exp(-2\pi(N(x_1) + N(x_2))) \) for \( x_i \in D_\infty \), where \( P(x) = P(a + bI + cJ + dIJ) = bI + cJ + dIJ \).

(ii) If \( \chi_p \) on \( R_p^\times \) is trivial, \( \Phi_p \) is the characteristic function of \( R_p^2 \).

(iii) If \( \chi_p \) is nontrivial, \( \Phi_p \) has the property such as

\[
\Phi_p(k^{-1}x_1k_2, k^{-1}x_2k_2) = \chi_p(k^{-1}k_2) \Phi_p(x_1, x_2), \quad k_i \in R_p^\times, \quad x_j \in D_p.
\]  

(3.1)

Then, by the Weil representation of \( Sp_2(\mathbb{A}) \) in [14], we obtain a Siegel modular form on \( Sp_2(\mathbb{A}) \).

The classical form of the Yoshida lift \( \Theta \Phi, f_1 \times f_2(Z) \) from \( f_1 \times f_2 \) for a Schwartz function \( \prod_{v \leq \infty} \Phi_v \) is

\[
\sum_{i,j=1}^h (n_in_j)^{-1} \sum_{x_1, x_2 \in D_\mathbb{Q}} \Phi_0(y_j^{-1}x_1y_j, y_j^{-1}x_2y_j) P_j(x_1^*x_2)f_1(y_1)e[x_1, x_2, Z].
\]  

(3.2)

The meanings of the symbols are as follows. We decompose

\[
D^\times = \bigsqcup_{1 \leq i \leq h} D_\mathbb{Q}^\times y_i R^\times_{\mathbb{A}}
\]  

(3.3)

with \( (y_i)_{\infty} = 1 \) and denote \( n_i = \sharp(D_\mathbb{Q} \cap y_i R^1_{\mathbb{A}} y_i^{-1}) \). \( \Phi_0 = \prod_{p<\infty} \Phi_p \), \( P_j \) means

\[
P_j(a + bI + cJ + dIJ) = \text{Tr}(f_2(y_j)(bI + cJ + dIJ)),
\]

where we remark that \( P_j \) plays the role of the contribution of the \( \Phi_\infty \). \( e[x_1, x_2, Z] = e(N(x_1)z_{11} + \text{Tr}(x_1^*x_2)z_{12} + N(x_2)z_{22}), \quad Z = (z_{ij}) \in \mathfrak{f}_2 \). Using this classical form, we can calculate the Fourier coefficients.

It is known that \( \Theta \Phi, f_1 \times f_2 \) is a cuspform of weight 3 and Hecke eigenform at almost all places. Its Andrianov \( L \)-function is described as follows. Suppose that \( \chi_p \) is trivial and \( R_p \) is isomorphic to \( M_2(\mathbb{Z}_p) \). By the computation in [9] which is a modification of Yoshida’s original one, the Andrianov \( L \)-function of \( \Phi f_1 \times f_2 \) is given by

\[
L^{ae}(s, \Theta \Phi, f_1 \times f_2)_p = L(s - 1, f_1)_p L(s, f_2)_p,
\]

where the theta kernels are not fixed to be the characteristic functions of \( R^2 \). We note that, if the central character of \( f_1 \) is trivial, the same computation is used in [1] to describe the standard \( L \)-function as

\[
Z(s, \Theta \Phi, f_1 \times f_2)_p = \zeta(s)_p L(s - 2, f_1 \otimes f_2)_p.
\]

4. Proofs

In order to prove the conjectures, we need to check two things.

(1) To show the existence of eigenforms having the conjectured Andrianov \( L \)-functions in the irreducible \( Sp_2(\mathbb{Z}) \) module generated by \( F_i \).

(2) To check eigenvalues of the eigenforms at 3, 5, and 7 (cf. Proposition 2.2).
For (1), we will construct eigenforms in $S_3(\Gamma(4, 8))$ by the Yoshida lift and show the existence of such eigenforms in $Sp_2(\mathbb{Z}) : F_i$. For (2), we will consult the table of [4]. We first fix some notations. In the remainder of this paper, we consider the definite quaternion algebra

$$D_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}I + \mathbb{Q}J + \mathbb{Q}IJ, \quad I^2 = J^2 = -1, \quad IJ = -JI,$$

which is split at every odd prime. We will use the orders

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}I + \mathbb{Z}J + \mathbb{Z}(1 + I + J + IJ)/2,$$

$$\mathcal{O}(l) = \mathbb{Z} + \omega^l \mathcal{O}, \quad N(\omega) = 2, \quad l \in \mathbb{Z}_{\geq 0},$$

$$R = \mathbb{Z} + 2\mathbb{Z}I + 2\mathbb{Z}J + 2\mathbb{Z}IJ.$$

Note that $\mathcal{O}_p \simeq \mathcal{O}(l)_p \simeq R_p \simeq M_2(\mathbb{Z}_p)$ at odd prime $p$ and $\mathcal{O}(l)^\times$ is a normal subgroup of $D_{\mathbb{Q}}^\times$. With respect to $\mathcal{O}$ or $R$, we have decompositions of $D_{\mathbb{Q}}$ as

$$D_{\mathbb{Q}}^\times = D_{\mathbb{Q}}^\times \mathcal{O}_{\mathbb{A}} = D_{\mathbb{Q}}^\times y_1 R_{\mathbb{A}} \sqcup D_{\mathbb{Q}}^\times y_2 R_{\mathbb{A}},$$

for $y_1 = 1$ and $(y_2)_2 = I + J + IJ$, $(y_2)_v = 1, v \neq 2$. Here $\mathcal{O}_{\mathbb{A}} = D_{\infty}^\times \times \prod_{p < \infty} \mathcal{O}_p^\times$ and so on.

4.1. Proof for $F_2$

Now, we start to prove the conjecture for $F_2$. We need first a pair of automorphic forms $f_1, f_2$ such that $L(s, f_1) = \zeta(s)\zeta(s - 1), L(s, f_2) = L(s, \rho_1)$. We can construct them in $A(\sigma_0, R)$ and $A(\sigma_2, R)$ as follows. By direct calculation, we have

$${\rm dim}_{\mathbb{C}} A(\sigma_0, R) = 2, \quad \text{and} \quad {\rm dim}_{\mathbb{C}} A(\sigma_2, R) = 6.$$

Now define $f_1 \in A(\sigma_0, R, 1)$ and $f_2 \in A(\sigma_2, R, 1)$ by

$$f_1(y_1) = f_1(y_2) = 1,$$

$$f_2(y_1) = 2bI - cJ + 2dIJ, \quad f_2(y_2) = -3cJ.$$

Proposition 4.1. The above $f_1$ and $f_2$ are Hecke eigenforms with

$$L(s, f_1) = \zeta(s)\zeta(s - 1), \quad L(s, f_2) = L(s, \rho_1),$$

up to the Euler factor at 2.

Proof. The assertion for $f_1$ is clear. We give the proof for $f_2$. Since $\mathcal{O}(3) \subset R$, Lemma 4.2 yields $\theta_{f_2} \in S_4(\Gamma_0(16))$ having the same $L$-function up to the Euler factor at 2.

The unique cusform $\rho_1(z) \in S_4(\Gamma_0(8))$ yields two oldforms of level 16 (namely $\rho_1(z)$ and $\rho_1(2z)$). They give the same eigenvalue (namely, $-4$) of the Hecke operator $T_3$, by Stein’s table in [13]. The newform of $S_4(\Gamma_0(16))$ has eigenvalue $+4$ for $T_3$. So $f_2$, corresponding to eigenvalue $-4$, comes from an oldform. □
Lemma 4.2. For every Hecke eigenform \( f \in A(\sigma_2, \mathcal{O}(l)) \), there exists a Hecke eigenform \( \theta_f \in S_4(\Gamma_0(2^{l+1})) \) having the same \( L \)-function, up to the Euler factor at 2.

Proof. Let \( V = \sum C f_i \) be the subspace of \( A(\sigma_2, \mathcal{O}(l)) \) spanned by Hecke eigenforms \( f_i \) having the same \( L \)-function as \( f \), outside of 2. We see that \( V \) is stable with respect to the right translation \( \rho \) of \( D_2^\times \): \( \rho(g)f'(x) = f'(xg) \), \( f' \in V \), since

\[
\rho(g)f'(xk) = f'(xkg) = f'(xg^{-1}kg) = \rho(g)f'(x)
\]

for every \( k \in \mathcal{O}(l)^\times \) and \( g \in D_2^\times \) (note that \( \mathcal{O}(l)^\times \) is a normal subgroup of \( D_2^\times \)).

We take an irreducible component \( \Omega \) taking values on \( V_\Omega \subset V \). From a certain automorphic form in \( V_\Omega \), we take a function \( f_\Omega \), which is an automorphic form in the sense of [7, p. 330].

The right translation by \( D_2^\times \) of \( f_\Omega \) determines the irreducible admissible representation \( \pi' = \Omega \times \bigotimes_{v \neq 2} \pi'_v \) of \( D_2^\times \).

At \( \infty \), the Weil representation in [7] associates \( \sigma_2 \) to a discrete series representation of \( \mathcal{H}_\mathbb{R} \). By [5, p. 142], the discrete series are in the space of right \( \mathcal{O}_2(\mathbb{R}) \)-finite functions on \( GL_2(\mathbb{R}) \) such that

\[
\phi \left( \begin{pmatrix} t_1 & \ast \\ 0 & t_2 \end{pmatrix} g \right) = \mu_1(t_1)\mu_2(t_2)|t_1/t_2|^{1/2}\phi(g),
\]

for the character \( \mu_1(a) = |a|^{5/2}, \mu_2(a) = |a|^{1/2}, a \in \mathbb{R}^\times \).

At 2, \( \Omega \) is associated to an irreducible admissible representation \( \pi_2(\Omega) \) of \( GL_2(\mathbb{Q}_2) \) by the Weil representation. We define a Schwartz function \( \phi \in S(D_2) \otimes V_\Omega \) by

\[
\phi(k) := \Omega(k)v, \quad k \in D_2^\times,
\]

for a nonzero \( v \in V_\Omega \), and zero if \( k \notin \mathcal{O}_2^\times \). Noting \( \Omega|\mathcal{O}(l)^\times \) is trivial, we see \( \phi \) is fixed by the action of \( \Gamma_0(2^{l+1}) \). Thus, the conductor of \( \pi_2(\Omega) \) divides \( 2^{l+1} \).

At the other places, by Theorem 4.4, \( \pi'_p \) are mapped to unramified \( \pi_p \) of \( GL_2(\mathbb{Q}_p) \). \( \pi_p \) is related to a cuspform, so is infinite-dimensional, due to Deligne’s theorem on Ramanujan’s conjecture.

Summing up Theorem 14.4 of [7], Theorem 5.19 of [5] and the above discussions, we get the assertion. \( \Box \)

For the case of \( A(\sigma_0, \mathcal{O}(l)) \), a similar result to the previous lemma is obtained in almost the same way. So, we omit the proof.

Lemma 4.3. Suppose that \( f \in A(\sigma_0, \mathcal{O}(l)) \) is an eigenform such that

\[
\int_{\mathcal{D}_Q^1 \setminus \mathcal{D}_h^1} f(h) \, dh = 0. \tag{4.1}
\]

Then, there exists a Hecke eigenform \( \theta_f \in S_2(\Gamma_0(2^{l+1})) \) having the same \( L \)-function, up to the Euler factor at 2.
Theorem 4.4. [10] Suppose that a definite quaternion algebra $B_{\mathbb{Q}}$ ramifies at only one prime $q$ and at $\infty$, and that an order $\mathcal{O}' \subset B_{\mathbb{Q}}$ is isomorphic to $M_2(\mathbb{Z}_p)$ at every $p \neq q$.

Then, the theta lifting from $\mathcal{A}(\sigma_{2n}, R')$ to elliptic modular forms is not vanishing. If $n > 0$, or if $n = 0$ and $f$ satisfies (4.1), the image is in $S_{2n+2}(\Gamma(qN))$ for some $N \in \mathbb{N}$.

Remark 4.5. As mentioned after Theorem 14.4 of [7], we also think that every eigenform $f$ is mapped to an eigen cuspform, except the case of $f(x) = \psi \circ N(x), \ x \in D_{\mathbb{A}}$, for a certain character $\psi$ on $\mathbb{Q}_q^\times$. But, we do not know references showing it.

Next, we will compute the Yoshida lift from $f_1$ and $f_2$. We define a theta kernel $\Phi_2 \in S(D_2^2)$ satisfying the condition (3.1) by

$$\Phi_2(x_1, x_2) = \begin{cases} e((a_1 + b_2)/4) & \text{if } x_1 = a_1 + b_1 I + c_1 J + d_1 IJ \equiv 1, \\ 0 & \text{and } x_2 = a_2 + b_2 I + c_2 J + d_2 IJ \equiv 1 \pmod{2}, \text{otherwise.} \end{cases}$$

We check the Fourier coefficient of $\Theta_{\Phi, f_1 \times f_2}(Z)$ at $(0, 1) \neq 0$.

Theorem 4.6. The Andrianov $L$-function of $F_2$ is equal to $\zeta(s-1) \zeta(s-2)L(s, \rho_1)$, up to the Euler factor at 2. The conjecture for $F_2$ is true.

Proof. Put $\Theta(Z) = \Theta_{\Phi, f_1 \times f_2}(8^{-1}Z)$. We can see easily $\Theta \in S_3(\Gamma(4), 8)$ by the properties of the Weil representation at 2 in [14] and the definition of $\Phi_2$. We observe that $N(x_1), N(x_2) \in \mathbb{Z}_2^\times$ whenever $\Phi_2(x_1, x_2) \neq 0$, and from the action of $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$ on $\Theta_{\Phi, f_1 \times f_2}(Z)$ for $S = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 \\ 1 & 1/2 \end{pmatrix}$, we find $\Theta \notin S_3(\Gamma(4))$.

Hence, one of the seven irreducible $Sp_3(\mathbb{Z})$ modules (excluded that of $F_1 \in S_3(\Gamma(4))$) must contain a Hecke eigenform whose Andrianov $L$-function is equal to $\zeta(s-1) \zeta(s-2)L(s, \rho_1)$. Consulting the table of eigenvalues of $F_i$ in [4], we see $\Theta$ is not orthogonal to the $Sp_2(\mathbb{Z})$ module of $F_2$.

Thus, observing the eigenvalues at 3, 5, 7 of $F_2$, from Proposition 2.2, we find the precise Andrianov $L$-function of $F_2$ is equal to the conjectured one. \qed

4.2. Proof for $F_3$

We define $f_2^{(-1)} \in \mathcal{A}(\sigma_2, R)$ by

$$f_2^{(-1)}(y_1) = -2bI - 2cJ + dIJ, \quad f_2^{(-1)}(y_2) = 3dIJ.$$

Similar to the proof of Proposition 4.1, we see $L(s, f_2^{(-1)}) = L(s, f_2 \otimes \omega_{-1})$, where $\omega_l$ denotes the quadratic character associated to $\mathbb{Q}(\sqrt{-l})/\mathbb{Q}$.

We check $\Theta_{\Phi, f_1 \times f_2^{(-1)}}$ has nonzero Fourier coefficient at $(1, 0)$, and thus get the next theorem analogous to Theorem 4.6.

Theorem 4.7. The Andrianov $L$-function of $F_3$ is, up to the Euler factor at 2, equal to $\zeta(s-1) \zeta(s-2)L(s, \rho_1 \otimes \omega_{-1})$. The conjecture for $F_3$ is true.
4.3. Proof for $F_4$

We define the character $\chi_4 = (\chi_4)_2 \times \prod_{v \neq 2} 1_v$ on $R_\mathbb{A}^\times$ with

$$(\chi_4)_2(1 + 2a + 2bI + 2cJ + 2dIJ) = (-1)^d,$$

for $k = 1 + 2a + 2bI + 2cJ + 2dIJ \in R_2^\times$ and calculate

$$\dim \mathcal{A}(\sigma_0, R, \chi_4) = 2, \quad \dim \mathcal{A}(\sigma_2, R, \chi_4) = 6.$$ 

We define $f_1 \in \mathcal{A}(\sigma_0, R, \chi_4)$ and $f_2 \in \mathcal{A}(\sigma_2, R, \chi_4)$ by

$$f_1(y_1) = 1, \quad f_1(y_2) = 0,$$$$
$$f_2(y_1) = 2bI + cJ, \quad f_2(y_2) = bI + 2dIJ.$$ 

Proposition 4.8. The $f_1$ and $f_2$ are Hecke eigenforms and

$$L(s, f_1) = L(s, \theta_\mu \otimes \omega_{-2}), \quad L(s, f_2) = L(s, \rho_3 \otimes \omega_{-2}),$$

up to the Euler factor at 2.

Proof. We give only a proof for $f_2$, since that for $f_1$ is similar. Since $(\chi_4)_2$ is trivial on $\mathcal{O}(5)^\times$, we have $\mathcal{A}(\sigma_2, R, \chi_4) \subset \mathcal{A}(\sigma_2, \mathcal{O}(5))$. The same discussion as in the proof of Proposition 4.1 tells that the Jacquet–Langlands correspondence maps $\mathcal{A}(\sigma_2, R, \chi_4)$ to $S_4(\Gamma_0(64))$. We calculate the Brandt matrices (representing matrix of the Hecke algebra on $\mathcal{A}(\sigma_2, R, \chi_4)$) and obtain

<table>
<thead>
<tr>
<th>Space $\mathcal{A}(\sigma_2, R, \chi_4)$</th>
<th>Eigenvalues at 3</th>
<th>Eigenvalues at 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>${\pm 8, 0}$</td>
<td>${-22, 10}$</td>
</tr>
</tbody>
</table>

We see $f_2$ has eigenvalues 10 at 5 and 8 at 3.

On the other hand, Stein’s table tells that

<table>
<thead>
<tr>
<th>Space $\subset S_4(\Gamma_0(32))$</th>
<th>Eigenvalues at 3</th>
<th>Eigenvalues at 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\subset \rho_3 \subset S_4(\Gamma_0(32))$</td>
<td>8</td>
<td>$-10$</td>
</tr>
<tr>
<td>$S_4(\Gamma_0(32))$</td>
<td>${\pm 8, \pm 4, 0}$</td>
<td>${22, -10, 2}$</td>
</tr>
<tr>
<td>$S_4(\Gamma_0(64))$</td>
<td>${\pm 8, \pm 4, 0}$</td>
<td>${\pm 22, \pm 10, \pm 2}$</td>
</tr>
</tbody>
</table>

Thus, by Proposition 3.64 of [12], we find that $\rho_3 \otimes \omega_{-2}$ belongs to $S_4(\Gamma_0(64))$. Stein’s table tells that only $\rho_3 \otimes \omega_{-2}$ has eigenvalue 8 at 3 and 10 at 5.

Taking into account that $\mathcal{A}(\sigma_2, R, \chi_4)$ is spanned by eigenforms, we can easily conclude $f_2$ is an eigenform outside of 2 with $L$-function $L(s, \rho_3 \otimes \omega_{-2})$. □
We define a theta kernel $\Phi_2$ associated to the pair of $f_1$ and $f_2$ by

$$\Phi_2(x_1, x_2) = \begin{cases} e((a_1 + d_1 + b_2)/4) & \text{if } x_1 = a_1 + b_1 I + c_1 J + d_1 IJ \equiv 1, \\
0 & \text{and } x_2 = a_2 + b_2 I + c_2 J + d_2 IJ \equiv I \pmod{2},\\
& \text{otherwise.}
\end{cases}$$

Then the Fourier coefficient of $\Theta_{\Phi, f_1 \times f_2} (Z)$ at $\left( \begin{smallmatrix} 1 \\ 0 \\ 0 \\ 1 \end{smallmatrix} \right)$ is not zero, from which we obtain the next theorem analogous to Theorem 4.6.

**Theorem 4.9.** The Andrianov $L$-function of $F_4$ is, up to the Euler factor at 2, equal to $L(s - 1, \theta_\mu \otimes \omega_{-2})L(s, \rho_3 \otimes \omega_{-2})$. The conjecture for $F_4$ is true.

### 4.4. Proofs for $F_5$ and $F_6$

Suppose that the conjectures for $F_5$ and $F_6$ are true. By Proposition 2.2, we notice that there exist eigenforms with the same Andrianov $L$-function in the different modules $Sp_2(\mathbb{Z}) \cdot F_5$ and $Sp_2(\mathbb{Z}) \cdot F_6$. It is not sufficient to construct eigenforms in $S_3(\Gamma(4, 8))$ having the Andrianov $L$-functions, different form the previous cases. The eigenforms obtained by the Yoshida lift may be in the same $Sp_2(\mathbb{Z})$ module. So, after the constructions, we will see that they are belonging to different $Sp_2(\mathbb{Z})$ modules.

We will first prove the conjecture for $F_5$. Define a character $\chi_5 = (\chi_5)_2 \times \prod_{v \neq 2} 1_v$ on $R_\mathbb{A}^\times$ with

$$(\chi_5)_2(1 + 2a + 2bI + 2cJ + 2dIJ) = (-1)^{b+c}.$$ 

We define $f_1 \in \mathcal{A}(\sigma_0, R, \chi_5)$ and $f_2 \in \mathcal{A}(\sigma_2, R, \chi_5)$ by

$$f_1(y_1) = 1, \quad f_1(y_2) = 0,$$

$$f_2(y_1) = 0, \quad f_2(y_2) = 2bI + 2cJ - dIJ.$$ 

The next proposition is analogous to Proposition 4.8.

**Proposition 4.10.** The above $f_1$ and $f_2$ are Hecke eigenforms outside of 2 with

$L(s, f_1) = L(s, \theta_\mu), \quad L(s, f_2) = L(s, \theta_\mu^3),$ 

up to the Euler factor at 2.

Associated to the pair $f_1$ and $f_2$, we define

$$\Phi_2(x_1, x_2) = \begin{cases} e(d_2/4) & \text{if } x_1 = a_1 + b_1 I + c_1 J + d_1 IJ \equiv 1 + J + IJ, \\
0 & \text{and } x_2 = a_2 + b_2 I + c_2 J + d_2 IJ \equiv I + J \pmod{2},\\
& \text{otherwise.}
\end{cases}$$

This theta kernel is the four-fold product of the Igusa theta constants (see Introduction and Main idea of [9])

$$\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(1,1,0,0)}\theta_{(1,0,0,1)}(Z),$$
which is the complement of $F_5$ of ten-fold product of all even theta constants. Using Proposition 6.2 of [4] and Lemma 2.2 of [11], we see the ten-fold product belongs to $S_5(\Gamma(2))$. Hence the four-fold product has the same character $\chi_{F_5}$ on $\Gamma(2)$ (note that $\chi_{F_5}$ is $[\pm 1]$-valued). Of course, $\Theta_{\Phi, f_1 \times f_2}$ has the same character $\chi_{F_5}$. Thus, we conclude that $\Theta_{\Phi, f_1 \times f_2}$ is in the $Sp_2(\mathbb{Z})$-orbit of $F_5$, consulting the lengths of the orbits in Theorem 6.4 of [4] which is the classification of characters on $\Gamma(2)$. The Fourier coefficient of $\Theta_{\Phi, f_1 \times f_2}(Z)$ at $\left(\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right)$ is not zero. Hence, consulting eigenvalues at 3, 5, 7, we have

**Theorem 4.11.** Up to the Euler factor at 2, the Andrianov $L$-function of $F_5$ is equal to $L(s - 1, \theta_{\mu})L(s, \theta_{\mu})$. The conjecture for $F_5$ is true.

We are going to prove the conjecture for $F_6$. Define a character $\chi_6 = (\chi_6)_2 \times \prod_{v \neq 2} 1_v$ on $R^\times_k$ with

$$(\chi_6)_2(1 + 2a + 2bI + 2cJ + 2dIJ) = (-1)^c.$$ 

We define $f'_1 \in \mathcal{A}(\sigma_0, R, \chi_6)$ and $f'_2 \in \mathcal{A}(\sigma_2, R, \chi_6)$ by

$$f'_1(y_1) = 0, \quad f'_1(y_2) = 1,$$

$$f'_2(y_1)(bI + cJ + dIJ) = 0, \quad f'_2(y_2)(bI + cJ + dIJ) = 2b - c + 2d.$$ 

The next proposition is analogous to Proposition 4.8.

**Proposition 4.12.** The above $f'_1$ and $f'_2$ are Hecke eigenforms outside of 2 and $L(s, f'_1) = L(s, \theta_{\mu} \otimes \omega_{-2})$, $L(s, f'_2) = L(s, \theta_{\mu, 3} \otimes \omega_{-2})$, up to the Euler factor at 2.

Associated to the pair $f'_1$ and $f'_2$, we define

$$\Phi'_2(x_1, x_2) = \begin{cases} e((a_1 + c_1 + b_2)/4) & \text{if } x_1 = a_1 + b_1I + c_1J + d_1IJ \equiv 1, \\ 0 & \text{if } x_2 = a_2 + b_2I + c_2J + d_2IJ \equiv 1 \pmod{2}, \text{ otherwise.} \end{cases}$$

The Fourier coefficient of $\Theta_{\Phi', f'_1 \times f'_2}(Z)$ at $\left(\begin{smallmatrix} 1/0 \\ 0/1 \end{smallmatrix}\right)$ is not zero.

**Theorem 4.13.** Up to the Euler factor at 2, the Andrianov $L$-function of $F_6$ is equal to $L(s - 1, \theta_{\mu} \otimes \omega_{-2})L(s, \theta_{\mu, 3} \otimes \omega_{-2})$. The conjecture for $F_6$ is true.

**Proof.** From the definitions we see, for $k = (1 + 2IJ)(1 + 2J)^{-1} \in R^1_2$,

$$(\chi_5)_2(k) = 1 \neq -1 = (\chi_6)_2(k).$$

Thus, using Lemma 4.14, we know that $\Theta_{\Phi', f'_1 \times f'_2}$ cannot belong to $Sp_2(\mathbb{Z}) \cdot F_5$. Consulting the table in [4] for some Euler factors of Andrianov $L$-functions of $F_i$, we find that $\Theta_{\Phi', f'_1 \times f'_2}$ belongs to $Sp_2(\mathbb{Z}) \cdot F_6$ (and not to the orbit $Sp_2(\mathbb{Z}) \cdot F_5$). Consulting the eigenvalues of $F_6$ at 3, 5, 7, we determine the precise Andrianov $L$-function of $F_6$. \[\square\]
Lemma 4.14. Let $p$ be a bad prime and $\pi_p$ be the Weil representation of $Sp_2(\mathbb{Q}_p)$. The property (3.1) of the theta kernel $\Phi_p$ is stable for translations by $Sp_2(\mathbb{Q}_p)$:

$$(\pi_p(g)\Phi_p)(k_1^{-1}x_1k_2, k_1^{-1}x_2k_2) = \chi_p(k_1^{-1}k_2)(\pi_p(g)\Phi_p)(x_1, x_2)$$

for every $g \in Sp_2(\mathbb{Q}_p)$, $k_i \in R^1_p$ and $x_j \in D_p$.

Proof. Obvious from the fact that the action of $Sp_2(\mathbb{Q}_p)$ commutes with that of $R^1_p$ on $\Phi_p$.

Remark 4.15. van Geemen and Nygaard [3] showed that the $L$-function of the Galois representation $\rho : \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL(H^3(Y', \mathbb{Q}_l)) \simeq GL_4(\mathbb{Q}_l)$ is equal to $L(s-1, \mu)L(s, \mu^3)$. Here $Y'$ is a resolution of $\ker(\chi_{F_5}) \setminus \mathfrak{S}_2$. So, we have

$L(s, \rho) = L(s, F_5)$.

4.5. Proof for $F_1$

In [3], the Andrianov $L$-function of $F_1$ was determined by using Oda lift [8] (Converse of Saito–Kurokawa lift).

Our method using Yoshida lift is also effective to $F_1$. We only write down the automorphic forms and theta kernel. We set $R' = \mathbb{Z} + 2\mathbb{Z}I + 2\mathbb{Z}J + \mathbb{Z}(I + J + IJ)$ and have $D^\times = D^\times (R'_p)^\times$. Define the character $\chi$ on $(R'_2)^\times$ by $\chi(k) = \omega_{-1}(N(k))$. The automorphic forms are

$$f_1(1) = 1, \quad f_2(1) = bI,$$

and we set the theta kernel $\Phi_2$ by

$$\Phi_2(x_1, x_2) = \begin{cases} e((a_1 + b_2)/4) & \text{if } x_1 = a_1 + b_1I + c_1J + d_1IJ \equiv 1 + I, \\
0 & \text{and } x_2 = a_2 + b_2I + c_2J + d_2IJ \equiv I + J \pmod{2}, \text{ otherwise.} \end{cases}$$

One can show easily that $\Theta_{\Phi_2, f_1 \times f_2}$ belongs to $S_3(\Gamma(4))$. Its Fourier coefficient at $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ is not zero.

Remark 4.16. In the constructions of Yoshida lifts above, we use odd Igusa theta constants. For example, for the above $F_1$, we use

$$\theta_{(1,0,1,0)}\theta_{(1,1,1,0)}\theta_{(0,1,0,0)}\theta_{(0,0,0,0)}(Z).$$

In the case of $F_5$, we use

$$\theta_{(0,0,0,1)}\theta_{(1,1,0,1)}\theta_{(0,1,0,1)}\theta_{(0,0,0,0)}(Z),$$
which is obtained from the four-fold product of even theta constants

$$\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(1,1,0,0)}\theta_{(1,0,0,1)}(Z)$$

by translating over $y_2 = I + J + IJ \in D_2^\times$. Using the polynomial $P$ described in Section 3, one verifies that they do not vanish. In contrast, if $\Phi_2$ is obtained from a four-fold product of even theta constants, then the theta kernel $\sum_{x_i \in D} P(x_1^*x_2) \times \Phi(x_1, x_2)e[x_1, x_2, Z]$ vanishes.

Acknowledgments

I express my thanks to Professor Riccardo Salvati Manni for suggesting to solve these problems, to Professor Tomoyoshi Ibukiyama for suggesting the line of research, and to the referee for very attentive reading.

References

[10] T. Okazaki, Siegel modular cuspidal and non-cuspidal having the same spinor $L$-function, preprint.