[9] T. Ibukiyama, On symplectic Euler factors of genus two, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30** (1984) and Proc. Japan Acad., **57** Ser. A no. 5 (1981), 271–275.
[10] ——, On automorphic forms of Sp (2, R) and its compact forms *Sp* (2), Sémi. Delange-Pisou-Poitou 1982–83, Birkhäuser Boston Inc. (1984), 125–134.
[11] ——, On the graded rings of Siegel modular forms of genus two belonging to certain level two congruence subgroups, preprint.
[12] J. Igusa, On Siegel modular forms of genus two, Amer. J. Math., **84** (1962), 175–200, (II) ibid., **86** (1964), 392–412.
[13] Y. Ihara, On certain arithmetical Dirichlet series, J. Math. Soc. Japan, **16** (1964), 214–225.
[14] H. Jacquet and R. P. Langlands, Automorphic forms on GL(2), Lecture Notes in Math., **260**, Springer (1972).
[15] R. P. Langlands, Problems in the theory of automorphic forms, Lecture Notes in Math., **170**, Springer (1970), 18–61.
[16] ——, Stable conjugacy: Definitions and Lemmas, Canad. J. Math., **31** (1979), 700–725.
[17] H. Shimizu, On zeta functions of quaternion algebras, Ann. of Math., **81** (1965), 166–193.
[18] J. Tits, Reductive groups over local fields, Proc. Symp. Pure Math., XXXIII part 1 (1979), 29–69.

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Advanced Studies in Pure Mathematics 7, 1985
Automorphic Forms and Number Theory
pp. 31–102

On Relations of Dimensions of Automorphic Forms of $Sp(2, \boldsymbol{R})$ and Its Compact Twist $Sp(2)$ (II)

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In this paper, we show some good global dimensional relations between automorphic forms of $Sp(2, \boldsymbol{R})$ (matrix size four) and its compact twist $Sp(2)$. One of the authors has already shown such relations when the p-adic completions (for a fixed prime p) of the discrete subgroups in question are maximal compact (See [24]). In this paper, we treat discrete subgroups whose p-adic completions are minimal parahoric. Our aim is a generalization of Eichler-Jacquet-Langlands correspondence between SL_2 and $SU(2)$ to the symplectic case of higher degree. Such correspondence should be proved by comparison of the traces of all the Hecke operators. Our results mean that there exist relations of traces at least for $T(1)$ for some explicitly defined discrete subgroups of $Sp(2, \boldsymbol{R})$ and $Sp(2)$ (§ 2 Main Theorem I). Besides, they give meaningful examples for Langlands philosophy on stable conjugacy classes (§ 2 Main Theorem II). Roughly speaking, such comparison is divided into character relations at infinite places (which are more or less known) and arithmetics at finite places. Our point is to execute the comparison of the *arithmetical* part explicitly. It seems that our Theorems are the first global results on such relations except for GL_n (cf. also [24]). In Section 1, after a brief introduction, we give a precise formulation on our problems between $Sp(n, \boldsymbol{R})$ and $Sp(n)$ for general n, e.g. on how to choose discrete subgroups explicitly. For automorphic forms with respect to these explicitly chosen discrete subgroups, we propose there two conjectures (which were first given in [21], [23]): coincidence of dimensions and existence of an isomorphism between new forms as Hecke algebra modules. For $n=1$, these are nothing but the *theorems* by Eichler [10], [11], and the above conjectures are a natural generalization of his results. Langlands [34] has given a quite general philosophy on correspondence of automorphic forms of any reductive algebraic groups, but we understand that his philosophy does not give very detailed formulation at present for such typical and explicit cases as

Received March 5, 1984.
*) The authors were partially supported by SFB 40, Univ. Bonn and Max-Planck-Institut für Mathematik.

treated in this paper, and we believe that the above conjectures have its own interest. In Section 2, we state our Main Theorems, which assert that the first conjecture is true also for $n=2$. The proof consists of explicit calculation of dimensions. Such explicit dimension formulae, given in Section 3, Theorems 3.2, 3.3, 3,4, 3.5, have their own value. Our Main Theorems are corollary to the results in Section 3 and [16], [19], [24]. The proofs of the formulae in Section 3 start from Section 4, where the computation of the dimensions of our spaces of automorphic forms are reduced to the detailed study of conjugacy classes in the arithmetic subgroups. We shall give in Section 4 an expository review on the results in [15], [16], and [19], for the convenience of the readers. There we shall also give some new remarks: (i) a formula for the number of semi-simple conjugacy classes in the arithmetic subgroup, and (ii) a relation of orbital integrals for semi-simple elements of G and G'. They will play an important role in the studies to extend the results of this paper for higher rank groups. In Section 5 and Section 6, we list up explicitly all data that we need for the calculation of the dimensions. In Section 7, we shall give a brief survey on some related topics.

The authors would like to express their hearty thanks to professors I. Satake and Y. Morita who gave them an opportunity to write a paper for this volume, in spite of the fact that they did not attend this Symposium.

Contents

§ 1. Conjectures

Let G and G' be two different reductive algebraic groups over algebraic number fields. For some good choice of G and G', sometimes

we know that there exists a correspondence between automorphic representations $\pi=\otimes_v \pi_v$ of G_A and $\pi'=\otimes_v \pi'_v$ of G'_A which preserves L-functions, where G_A or G'_A is the adelization of G or G'. Langlands [34] has given a general philosophy on such problems: he defined so called L-groups ${}^LG$ or ${}^LG'$, and he conjectured that, if G is quasi split, and if there exists an L-homomorphism u: ${}^LG'\rightarrow {}^LG$, then there should exist a "good" correspondence of automorphic representations. (As for more precise contents of this conjecture, see Langlands [34], or Borel [3].) For example, if G' is an inner twist of G, then ${}^LG\cong {}^LG'$, and we can expect a good correspondence. One of the reason of this conjecture seems to be the fact that there exists a good character relations between π_∞ and π'_∞ (cf. (4.50)).

The basic example is $GL(2)$. The first typical results on the relation between $GL(2)$ and division quaternion algebras were due to Eichler [10], [11], and later completed by many mathematicians, notably, Shimizu [43], Jacquet-Langlands [29]. One obvious direction of generalizations of the $GL(2)$-case is $GL(n)$, which has been studied also by various mathematicians.

Now, another direction is the symplectic groups, because we can regard $GL(2)$ as the symplectic group of size two with similitudes. Let $Sp(n, \boldsymbol{R})$ be the symplectic group of size $2n$, and $Sp(n)$ be its compact twist: $Sp(n)=\{g \in M_n(\boldsymbol{H}); g\,{}^t\bar{g}=1_n\}$, where $\boldsymbol{H}$ is the division quaternion algebra over R and $^-$ is the canonical involution. When $n=2$, for pairs of $\boldsymbol{Q}$-forms of $Sp(2, \boldsymbol{R})$ and $Sp(2)$, Ihara [28] raised a conjectural problem on an existence of correspondence of automorphic forms (independent from and older than Langlands [34]). He clarified, among others, what should be the correspondence of weights (i.e. representations at infinity) of those forms by showing some character relations (unpublished). (As for some other works by him, see [28] or § 7.) Later, Hina and Masumoto [20] gave character relations between some admissible representations of $GSp(2, F)$ (size four) and its inner twist, when F is a non archimedean local field. But, there was no global result, and any global example had not been known before [21]. We would like to have some global (and rather classical) approach to this problem, and aim a generalization of the typical results of Eichler. Even if we restrict ourselves to such typical cases, the precise formulation had not been known before [21], [23]. Besides, such typical cases have their own fruitful structures. Our aim of this paper is to give good global dimensional relations in such cases. This can be regarded as the first step to the proof of such correspondence of automorphic forms.

Now, we explain our problem more precisely. Put

$$G=GSp(n, \boldsymbol{Q})=\{g \in M_{2n}(\boldsymbol{Q}); gJ\,{}^tg=n(g)J,\ n(g) \in \boldsymbol{Q}^\times\},$$

where $J=\begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ and 1_n is the unit matrix of size n. On the other hand, let D be the definite quaternion algebra over $\boldsymbol{Q}$ with prime discriminant p. (We fix a prime number p.) Put

$$G'=\{g \in M_n(D);\ g\,{}^t\bar{g}=n(g)1_n,\ n(g) \in \boldsymbol{Q}^\times\}.$$

Then, G' is an inner twist of G. Let G_A (resp. G'_A) be the adelization of G (resp. G'), and for any place v of $\boldsymbol{Q}$, let G_v (resp. G'_v) be the v-adic component of G_A (resp. G'_A). We have $G_\infty=GSp(n, \boldsymbol{R})$ and $G'_\infty=GSp(n)$ (i.e. the group of symplectic similitudes). We note that

$$G'_v \cong G_v=GSp(n, \boldsymbol{Q}_v), \quad \text{if } v\neq p, \infty.$$

We consider subgroups U_A (resp. U'_A) of G_A (resp. G'_A) of the following forms:

(1.1) $$U_A=G_\infty P \prod_{q\neq p} GSp(n, \boldsymbol{Z}_q) \quad (\text{resp.}$$

(1.2) $$U'_A=G'_\infty P' \prod_{q\neq p} GSp(n, \boldsymbol{Z}_q)\,),$$

where P (resp. P') is an open compact subgroup of G_p (resp. G'_p), and, for any prime q,

$$GSp(n, \boldsymbol{Z}_q)=\{g \in GSp(n, \boldsymbol{Q}_q);\ g, g^{-1} \in M_{2n}(\boldsymbol{Z}_q)\}.$$

We define automorphic forms and Hecke operators. Let $\mathfrak{H}_n$ be the Siegel upper half space of degree n:

$$\mathfrak{H}_n=\{X+iY;\ X, Y \in M_n(\boldsymbol{R}),\ {}^tX=X,\ {}^tY=Y,\ Y>0, \text{ i.e. } Y \text{ is positive definite}\}.$$

An element $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp(n, \boldsymbol{R})^+$ acts on $\mathfrak{H}_n$ by:

$$Z \longmapsto (AZ+B)(CZ+D)^{-1}.$$

Put

$$GSp(n, \boldsymbol{Q})^+=\{g \in G;\ n(g)>0\} \quad \text{and} \quad U=U_A \cap GSp(n, \boldsymbol{Q})^+.$$

Then, U acts on H_n discontinuously and $\mathrm{vol}\,(U\backslash\mathfrak{H}_n)$ is finite. The space $S_k(U)$ of cusp forms of weight k with respect to U is defined by:

$S_k(U)=\Big\{$holomorphic functions f on $\mathfrak{H}_n$ such that

(1) $f(\gamma Z)=f(Z) \det (CZ+D)^k$ for all $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U$,

(2) $f(Z)(\det Y)^{k/2}$ is bounded on $\mathfrak{H}_n\Big\}$.

For any natural integer m ($p\nmid m$), the action of the Hecke operator $T(m)$ on $S_k(U)$ is defined as follows: Put $T(m)=\bigcup_g UgU$, where g runs through elements of

$$\Delta_m=\{g \in G \cap M_{2n}(\boldsymbol{Z});\ gJ\,{}^tg=mJ\}.$$

We take a coset decomposition $T(m)=\coprod_{i=1}^d Ug_i$ (disjoint). For any $f \in S_k(U)$, we define $f|T(m)$ by:

$$(f|T(m))(Z)=m^{nk-n(n+1)/2}\sum_{i=1}^d f(g_iZ) \det (C_iZ+D_i)^{-k},$$

where $g_i=\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$.

On the other hand, let (ρ, V) be an irreducible representation of G'_∞. We regard ρ as a representation of G_A by composing it with projection: ρ: $G'_A \to G'_\infty \to GL(V)$. The space $\mathfrak{M}_\rho(U'_A)$ of automorphic forms on G'_A of weight ρ with respect to U'_A is defined by:

$$\mathfrak{M}_\rho(U'_A)=\{f\colon G'_A \to V;\ f(uga)=\rho(u)f(g) \text{ for all } a \in G',\ u \in U'_A, \text{ and } g \in G'_A\}.$$

As well known, we can realize V in a space of some spherical functions. The strong approximation theorem does not hold for G' and the 'class number' of U'_A is not one in general. A 'classical' interpretation of $\mathfrak{M}_\rho(U'_A)$ is given as follows: Take a double coset decomposition $G'_A=\coprod_{i=1}^h U'_A g_i G'$ (disjoint), and put

(1.3) $$\Gamma_i=g_i^{-1}U'_A g_i \cap G'.$$

Put

$$V^{\Gamma_i}=\{v \in V;\ \rho(\gamma)v=v \text{ for all } \gamma \in \Gamma_i\}.$$

Then, we have

(1.4) $$\mathfrak{M}_\rho(U'_A)=\bigoplus_{i=1}^h V^{\Gamma_i},$$

where the isomorphism is given by $f \to (\rho(g_i^{-1})f(g_i))_{i=1\ldots h}$. Let ρ_ν be the representation of $Sp(n)$ which corresponds with the Young diagram

1	···	ν
·	···	·
·	···	·
·	···	ν

$\Big\}\, n$. We extend it by putting $\rho_\nu(a1_n)=a^{n\nu}$ for $a \in \boldsymbol{R}^\times$, $a>0$.

We write $\mathfrak{M}_{\rho_\nu}(U'_A)=\mathfrak{M}_\nu(U'_A)$. If $-1 \in U'_A$, then $\mathfrak{M}_\nu(U'_A)=0$, unless

$(-1)^{n\nu}=1$. We put $T'(m)=\bigcup_g U'_A g U'_A$ $(p\nmid m)$, where g runs through elements of

$$\Delta'_m=\{g=(g_v)\in G'_A;\ g_v\in M_{2n}(Z_v) \text{ and } n(g_v)\in mZ_v^\times \text{ for all finite } v\neq p,\ g_p\in P\}.$$

Take a coset decomposition

$$T'(m)=\coprod_{i=1}^{d'} g'_i U'_A \quad \text{(disjoint)}.$$

For any $f\in M_\nu(U'_A)$, $f\mid T'(m)$ is defined by:

$$(f\mid T'(m))(g)=\sum_{i=1}^{d'}\rho_\nu(g'_i)f(g'^{-1}_i g),\quad g\in G'_A.$$

The (abstract) Hecke algebra spanned by $T(m)$ $(p\nmid m)$ is isomorphic to the one spanned by $T'(m)$ $(p\nmid m)$. We sometimes denote $T'(m)$ by $T(m)$. For a common eigen form $f\in S_k(U)$ or $\mathfrak{M}_\nu(U'_A)$ of all the Hecke operators $T(m)$ $(p\nmid m)$, the L-function of f is defined (up to the p-Euler factors) by:

$$L(s,F)=\text{the denominator of } \sum_{p\nmid m}\lambda(m)m^{-s}, \text{ where } T(m)f=\lambda(m)f.$$

Now, we review a typical case of Eichler's results on $GL(2)$. Let O be a maximal order of D and O_p be its p-adic completion. Put $P'=O_p^\times$ in (1.2). On the other hand, put

$$P=\left\{g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL(2,Z_p);\ c\equiv 0 \bmod p\right\}.$$

In the usual notation, $U=\Gamma_0(p)$ in this case. We write $U'_A=O_A^\times$ in this case.

Theorem 1.5 (Eichler [10], [11]). *If we denote by* $S_k^0(\Gamma_0(p))$ *the space of new forms of* $S_k(\Gamma_0(p))$, *then for* $k\geqq 2$, *we have*: $\mathfrak{M}_{k-2}(O_A^\times)=S_k^0(\Gamma_0(p))$ $(\oplus C$, *if* $k=2)$, *as modules over Hecke algebras* (i.e. *this isomorphism preserves L-functions*).

The new forms $S_k^0(\Gamma_0(p))$ are actually defined as the orthogonal complement of $S_k(SL_2(Z))\oplus S_k(\rho SL_2(Z)\rho^{-1})$ in $S_k(\Gamma_0(p))$ with respect to the Petersson inner product, where $\rho=\begin{pmatrix} 0 & -1\\ p & 0\end{pmatrix}$. So, we get

Corollary 1.6 (Eichler, loc. cit.). *For* $k\geqq 2$, *we have* $\dim\mathfrak{M}_{k-2}(O_A)=\dim S_k(\Gamma_0(p))-2\dim S_k(SL_2(Z))+\delta$, *where* $\delta=1$, *if* $k=2$, *and* $\delta=0$, *otherwise.*

This Corollary 1.6 will be extended for $n=2$ in this paper. But, before we state our Main Theorem, we would like to propose general formulations and conjectures. If we want to generalize such simple and beautiful typical results, several natural questions arise:

(1) What are the corresponding weights in the general case?

(2) What kind of U or U'_A should be taken instead of $\Gamma_0(p)$ and O_A?

(3) What are new forms?

The answer to the question (1) for $n=2$ was given by Ihara. The general case seems more or less known: If we take Siegel cusp forms of degree n with weight $k\geqq n+1$, the corresponding weight of automorphic forms on G'_A should be ρ_{k-n-1} (cf. (4.50)). To questions (2) and (3), a hypothetical answer has been given in Ibukiyama [21], [23], [24]: First of all, as far as we take U_A or U'_A as in (1.1) or (1.2), this question is a local problem how to choose P or P'. Secondly, it is known that every reductive algebraic group over a non archimedean local field has the unique minimal parahoric subgroup up to conjugation (Tits [46]). Roughly speaking, the minimal parahoric subgroup is a group such that its reduction mod p is the Borel subgroup. For example, P or P' chosen in Theorem 1.5 is minimal parahoric. So, it is natural to choose the minimal parahoric subgroup B of G_p or B' of G'_p as the first candidate for P or P', respectively. (As for another kinds of candidates, see [23], [24].) To obtain new forms, we should subtract automorphic forms belonging to U_A or U'_A with $P\supsetneqq B$ or $P'\supsetneqq B'$. To explain more precisely, we review briefly the Bruhat-Tits theory. The extended Dynkin diagram of G_p is the Coxeter graph of the affine Weyl group W of G_p, and the set S of vertices of this graph can be regarded as a set of generators of W as a Coxeter system. We fix a minimal parahoric subgroup B of G_p. By the Bruhat decomposition, there is a one to one correspondence between the set of all subsets θ of S and the set of all subgroups of G_p containing B. More precisely, for each $w\in W$, there is a good representative of w in G_p, which we denote also by w. For a subset θ of S, put

$$P_\theta=\{\text{the group generated by all double cosets } BwB \text{ such that } w\in\theta\}.$$

Such groups are called standard parahoric subgroups. Then, we have $P_\theta\supset B$, and $P_\theta=P_{\theta'}$, if and only if $\theta=\theta'$. Besides, every group P which contains B is obtained in this way. For example, $P_\phi=B$ and $P_S=G_p$. For $\theta\subset S$, we put

$$U_\theta(p)_A=G_\infty P_\theta\prod_{q\neq p}GSp(n,Z_q),\quad\text{and}$$
$$U_\theta(p)=GSp(n,Q)^+\cap U_\theta(p)_A. \tag{1.7}$$

The above theory is completely the same also for G'_p, and we denote by S' the set of generators of the affine Weyl group of G'_p. We denote by P' the standard parahoric subgroup defined by $\theta'\subset S'$. We put

$$U'_{\theta'}(p)_A = G'_\infty P'_{\theta'} \prod_{q\neq p} GSp(n, Z_q). \tag{1.8}$$

We often omit the suffix A in this case, because we do not treat 'global' discrete subgroups. We put $U_\phi(p)=B(p)$ and $U'_\phi(p)=B'(p)$. The second author gave the following conjectures ([21], [23]):

Conjecture 1.9. *For any integer n, $\nu\geqq 1$, we should have:*

$$\sum_{\substack{\theta\subset S\\ \theta\neq S}} (-1)^{\#(\theta)} \dim S_{\nu+n+1}(U_\theta(p)) = \sum_{\substack{\theta'\subset S'\\ \theta'\neq S'}} (-1)^{\#(\theta)} \dim \mathfrak{M}_\nu(U'_{\theta'}(p)). \tag{1.10}$$

If $\nu=0$, we should add one to the right hand side.

We define the space $S^0_k(B(p))$ of new forms of $S_k(B(p))$ as the orthogonal complement of $\sum_{\substack{\#(\theta)=1\\ \theta\subset S}} S_k(U_\theta(p))$ (summation as C-vector spaces) in $S_k(B(p))$ with respect to the Petersson inner product. We define $\mathfrak{M}^0_\nu(B'(p))$ completely in the same way. These definitions mean that the p-adic admissible representation attached to a new form is the special representation (cf. [4])*).

Conjecture 1.11. *For any integer $n\geqq 1$ and $\nu\geqq 1$, we have:*

$$S^0_{\nu+n+1}(B(p)) = \mathfrak{M}^0_\nu(B'(p)), \tag{1.12}$$

as modules over the Hecke algebra spanned by $T(m)$ ($p\nmid m$).

For $n=1$, these conjectures are nothing but Theorem 1.5 and Corollary 1.6 by Eichler. For $n=2$, Conjecture 1.9 is true (at least) for $\nu\geqq 2$ and $p\neq 3$. This is our Main Theorem. For $n=2$ and $p=2$, there have been given some explicit examples $f\in S^0_{\nu+3}(B(p))$ and $f'\in\mathfrak{M}^0_\nu(B'(p))$ in Ibukiyama [21] such that Euler 3-factors of $L(s,f)$ and $L(s,f')$ coincide with each other and satisfy the Ramanujan Conjecture at 3 (i.e. these cannot be obtained as 'liftings' of one dimensional automorphic forms). For general n, the both sides of (1.10) are expressed as a sum of contributions of conjugacy classes of elements of G or G'. For some conjugacy classes, we can show the equality of contributions. For example, the main terms

*) For $\nu\geqq 1$, $\dim S^0_{\nu+n+1}(B(p))$ (resp. $\dim \mathfrak{M}^0_\nu(B'(P))$) is equal to the left (resp. right) hand side of (1.10).
Prof. W. Casselman proved this fact, answering to the question by one of the authors.

(i.e. the contribution of the unit elements) of both sides of (1.10) coincide with each other and given by:

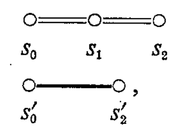
$$\frac{2(\nu+1)(\nu+2)\cdots(\nu+n)}{n!}\prod_{1\leq i\leq j\leq n}\frac{2\nu+i+j}{i+j}\,\frac{\prod_{i=1}^n\zeta(2i)}{(2\pi)^{n(n+1)}}$$
$$\times(p-1)(p^3-1)\cdots(p^{2n-1}-1).$$

As for this kind of relations for other algebraic groups which are not symplectic, see Ibukiyama [23]. We have some results also for some kind of unipotents elements of G or G'.

§ 2. Main Theorem

In this section, we explain our Main Theorem more in detail. For $n=2$, the extended Dynkin diagrams of G_p and G'_p are given as follows:

G_p G'_p



where $\{s_0, s_1, s_2\}$ or $\{s'_0, s'_1\}$ is the set of generators of the affine Weyl group W or W' of G_p or G'_p, respectively. We can take the minimal parahoric subgroup B of G_p as follows:

$$B=GSp(2, Z_p)\cap\begin{pmatrix} * & * & * & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & p* & * & * \end{pmatrix},$$

where *'s run through all the p-adic integers. We can fix representatives of s_i ($i=0, 1, 2$) in G_p as follows:

$$s_0=\begin{pmatrix} 0 & 0 & -p^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\quad s_1=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\quad s_2=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Put

$$\rho=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

Then, it is easy to show that

$$P_{\{s_1\}}=B\cup Bs_1B=\left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in GSp(2,Z_p)\colon C\equiv 0 \bmod p\right\},$$

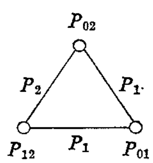
$$P_{\{s_2\}}=B\cup Bs_2B=GSp(2,Z_p)\cap\begin{pmatrix} * & * & * & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & * & * & *\end{pmatrix},$$

$$P_{\{s_0\}}=B\cup Bs_0B=\rho P_{\{s_2\}}\rho^{-1}=GSp(2,Q_p)\cap\begin{pmatrix} * & * & p^{-1}* & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & p* & * & *\end{pmatrix},$$

$$P_{\{s_0,s_2\}}=GSp(2,Q_p)\cap\begin{pmatrix} * & * & p^{-1}* & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & * & * & *\end{pmatrix},$$

$$P_{\{s_1,s_2\}}=GSp(2,Z_p),\ \text{and}\ P_{\{s_0,s_1\}}=\rho GSp(2,Z_p)\rho^{-1},$$

where $*$ runs through all the p-adic integers. For the sake of simplicity, we write $P_{\{s_0\}}=P_0$, $P_{\{s_0,s_2\}}=P_{02}$, etc. The relations of the standard parahoric subgroups of G_p are illustrated as follows:



which means that every face is the intersection of its boundaries and every vertex is spanned by simplexes containing it. For example, $P_2=P_{02}\cap P_{12}$ and P_{12} is generated by P_1 and P_2 etc. To explain the parahoric subgroups of G'_p, it is convenient to take another model. Put $D_p=D\otimes_Q Q_p$ and

$$G^*_p=\left\{g\in M_2(D_p);\ g\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}{}^t\bar{g}=n(g)\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},\ n(g)\in Q_p^\times\right\}.$$

Then, $G^*_p\cong G_p$. We fix such an isomorphism once and for all. Let π be a prime element of $O_p=O\otimes_Z Z_p$. We can take a minimal parahoric subgroup B' of G^*_p as follows:

$$B'=\begin{pmatrix} O_p & O_p \\ \pi O_p & O_p\end{pmatrix}^\times\cap G^*_p.$$

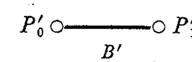
We can put

$$s'_0=\begin{pmatrix} 0 & \pi^{-1} \\ \pi & 0\end{pmatrix},\quad\text{and}\quad s'_1=\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

Then,

$$P'_0=P_{\{s'_0\}}=\begin{pmatrix} O_p & \pi^{-1}O_p \\ \pi O_p & O_p\end{pmatrix}\cap G^*_p,\quad\text{and}\quad P'_1=P_{\{s'_1\}}=GL_2(O_p)\cap G^*_p.$$

We can illustrate these groups as follows:



We define $U_\theta(p)$ or $U'_\theta(p)$ as in (1.1), and put $U_0(p)=U_{\{s_0\}}(p)$, etc. In the notation of [21], [23], [24], $U_1(p)=\Gamma_0(p)$, $U_2(p)=\Gamma'_0(p)$, $U_0(p)=\Gamma''_0(p)$, and $U_{02}(p)=K(p)$. You can get an expression of $U_\theta(p)$ or $B(p)$, if you replace p-adic integers or numbers by rational integers or numbers in the expression of P or B, and take $n(g)$ to be one. We have no standard global expression of $U'_\theta(p)$ or $B'(p)$, partly because the 'class number' of G' is not one. You can find some explicit examples of Γ_i defined in (1.3) in [21], [26].

Main Theorem I. *For* $n=2$, *any integer* $k\geqq 5$, *and any prime* $p\neq 3$, *the conjecture* 1.9 *is true*, i.e., *we have the following equality*:

(2.1)

$$\begin{aligned}&\dim S_k(B(p))-\dim S_k(U_0(p))-\dim S_k(U_1(p))-\dim S_k(U_2(p))\\&\quad+2\dim S_k(Sp(2,Z))+\dim S_k(U_{02}(p))\\&\qquad=\dim \mathfrak{M}_{k-3}(B'(p))-\dim \mathfrak{M}_{k-3}(U'_1(p))-\dim \mathfrak{M}_{k-3}(U'_0(p)).\end{aligned}$$

As we shall explain in Section 4, dimension formulae are expressed as summations over the contributions of conjugacy classes of elements (with $n(g)=1$) of G or G' of various types. Any elements of G' (with $n(g)=1$) are semi-simple, because they are embedded into the compact group $Sp(2)$. We have $G\otimes C=G'\otimes C=GSp(2,C)$. Let C be a conjugacy class of some semi-simple elements of $GSp(2,C)$. It is well known that C is determined only by the principal polynomial $f(x)$ of all elements of C. Let $T(f)$ (resp. $H(f)$) be the set of all G-(resp. G'-) conjugacy classes contained in C.

Main Theorem II. *The contribution of non-elliptic* (i.e. *non torsion*) *conjugacy classes to the left hand side of* (2.1) *is zero. For any polynomial* $f(x)$ *which is the principal polynomial of some elements of* G *or* G' *of finite order, the contribution of* $T(f)$ *to the left hand side of* (2.1) *is equal to the contribution of* $H(f)$ *to the right hand side.*

Remark 2.2. This Main Theorem II can be regarded as an evidence for the philosophy by Langlands [36] on stable conjugacy classes.

Remark 2.3. The proof of the Main Theorem consists of an explicit calculation of each dimension. Some of the above mentioned dimensions have been already known: dim $S_k(Sp(2, \mathbf{Z}))$ by Igusa [27] (cf. also [16]), dim $S_k(U_1(p))$ by Hashimoto [16], dim $S_k(U_{02}(p))$ by Ibukiyama [24], and dim $\mathfrak{M}_{k-3}(U_i'(p))$ $(i=0, 1)$ by Hashimoto and Ibukiyama [19]. So, we shall calculate dim $S_k(B(p))$, dim $S_k(U_2(p))$ $(=\dim S_k(U_0(p)))$, and dim $\mathfrak{M}_\nu(B'(p))$ explicitly in the following sections. We note that (2.1) has been known for $p=2$ with $k\geqq 3$, although we must add one to the right hand side, if $k=3$ (cf. [24]). For general p, we need the trace formula, but it does not work well at present, unless $k\geqq 5$. We assumed $p\neq 3$, because it sometimes makes computation easier. By virtue of the above mentioned results, we can also assume that $p\neq 2$ in the following considerations.

§ 3. Explicit dimension formulae

In this section, we give explicit formulae for the dimensions of $S_k(B(p))$, $S_k(U_2(p))$, and $\mathfrak{M}_\nu(B'(p))$. The proof will be found in the following sections. First, we treat $B(p)$ and $U_2(p)$. The dimensions are expressed as sums of contributions of $B(p)$- or $U_2(p)$-conjugacy classes. But, by definition, we have $B(p), U_2(p)\subset Sp(2, \mathbf{Z})$. So, it is convenient to group together those $B(p)$- or $U_2(p)$-conjugacy classes that are contained in a $Sp(2, \mathbf{Z})$-conjugacy class. Representatives of the $Sp(2, \mathbf{Z})$-conjugacy classes of elements of finite order were given by α_i $(i=0, \cdots, 22)$, β_i $(i=1, \cdots, 6)$, γ_i $(i=1, 2, 3)$, or δ_i $(i=1, 2)$, up to sign, according to the notation of [16]. The non-semi-simple $Sp(2, \mathbf{Z})$-conjugacy classes are divided into various types as in [16], and those which have a contribution to the dimension formulae are of type $\pm\hat{\beta}_i(n)$ $(i=1, \cdots, 10)$, $\hat{\hat{\delta}}_i(m, n)$ $(i=1, \cdots, 4)$, $\pm\hat{\delta}_i(n)$ $(i=1, 2)$, $\hat{\gamma}_i(n)$ $(i=1, \cdots, 4)$, $\pm\hat{\gamma}_i(n)$ $(i=5, 6, 7)$, or $\pm\varepsilon_i(S)$ $(i=1, \cdots, 4)$, according to the notation in [16]. We use above notations to denote the set of conjugacy classes of that "type". For example, α_1 (resp. $\hat{\beta}_1$) denotes the set of $Sp(2, \mathbf{Z})$-conjugacy classes which contain α_1 or $-\alpha_1$ (resp. $\hat{\beta}_1(n)$ or $-\hat{\beta}_1(n)$ for some $n\in \mathbf{Z}$, $n\neq 0$). For $U=B(p)$, or $U_2(p)$, and a set C of some $Sp(2, \mathbf{Z})$-conjugacy classes, we denote by $t(U, C, k)$ the total sum of the contributions to dim $S_k(U)$ of U-conjugacy classes contained in C. We sometimes omit U and denote it by $t(C, k)$, if no confusion is likely. The principal polynomial of the above $Sp(2, \mathbf{Z})$-conjugacy classes are given as follows:

(3.1)

$f_1(x)=(x-1)^4$, or $f_1(-x)$ for $\pm\alpha_0$, $\pm\varepsilon_i$ $(i=1, \cdots, 4)$,

$f_2(x)=(x-1)^2(x+1)^2$ for δ_i $(i=1, 2)$, $\hat{\hat{\delta}}_i$ $(i=1, \cdots, 4)$, $\pm\hat{\delta}_i$ $(i=1, 2)$,

$f_3(x)=(x-1)^2(x^2+1)$, or $f_3(-x)$ for $\pm\beta_i$ $(i=5, 6)$, $\pm\hat{\beta}_i$ $(i=7, \cdots, 10)$,

$f_4(x)=(x-1)^2(x^2+x+1)$, or $f_4(-x)$ for $\pm\beta_i$ $(i=1, 2)$, $\pm\hat{\beta}_i$ $(i=3, \cdots, 6)$,

$f_5(x)=(x-1)^2(x^2-x+1)$, or $f_5(-x)$ for $\pm\beta_i$ $(i=3, 4)$, $\pm\hat{\beta}_i$ $(i=1, 2)$,

$f_6(x)=(x^2+1)^2$ for $\pm\alpha_1$, γ_i $(i=1, 2)$, $\hat{\gamma}_i$ $(i=1, \cdots, 4)$,

$f_7(x)=(x^2+x+1)^2$, or $f_7(-x)$ for $\pm\alpha_i$ $(i=2, 3)$, $\pm\hat{\gamma}_i$ $(i=5, 6, 7)$,

$f_8(x)=(x^2+1)(x^2+x+1)$, or $f_8(-x)$ for $\pm\alpha_i$ $(i=19, 20, 21, 22)$,

$f_9(x)=(x^2+x+1)(x^2-x+1)$ for α_i $(i=7, 8)$, α_i $(i=9, 10, 11, 12)$,

$f_{10}(x)=x^4+x^3+x^2+x+1$, or $f_{10}(-x)$ for $\pm\alpha_i$ $(i=15, 16, 17, 18)$,

$f_{11}(x)=x^4+1$ for α_i $(i=4, 5)$, $\pm\alpha_6$,

$f_{12}(x)=x^4-x^2+1$ for α_i $(i=13, 14)$.

Theorem 3.2. *For a natural integer $k\geqq 5$ and a prime $p\neq 2, 3$, dim $S_k(B(p))$ is given by the summation of the following quantities:*

$$t(\alpha_0, k)=(p+1)^2(p^2+1)(2k-2)(2k-3)(2k-4)/2^9 3^3 5,$$

$$t(\alpha_1, k)=(p+1)\left(1+\left(\frac{-1}{p}\right)\right)(-1)^k/2^6,$$

$$t(\alpha_2, k)+t(\alpha_3, k)=-(p+1)\left(1+\left(\frac{-3}{p}\right)\right)\Big/2\cdot 3^3\times[0, 1, -1; 3],$$

$$\sum_{i=4}^{6} t(\alpha_i, k)=[1, 0, 0, -1; 4]\begin{cases}1\cdots\text{if } p\equiv 1 \bmod 8,\\ 0\cdots\text{if } p\not\equiv 1 \bmod 8,\end{cases}$$

$$\sum_{i=7}^{12} t(\alpha_i, k)=\frac{4}{9}[1, 0, 0, -1, 0, 0; 6]\left(1+\left(\frac{-3}{p}\right)\right),$$

$$t(\alpha_{13}, k)+t(\alpha_{14}, k)=\frac{2}{3}[0, 1, -1; 3]\begin{cases}1\cdots\text{if } p\equiv 1 \bmod 12,\\ 0\cdots\text{otherwise},\end{cases}$$

$$\sum_{i=15}^{18} t(\alpha_i, k)=\frac{1}{5}[1, 0, 0, -1, 0; 5]\begin{cases}8\cdots\text{if } p\equiv 1 \bmod 5,\\ 1\cdots\text{if } p=5,\\ 0\cdots\text{otherwise},\end{cases}$$

$$\sum_{i=19}^{22} t(\alpha_i; k)=\frac{1}{6}\left(1+\left(\frac{-1}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right)$$
$$\times[1, 0, 0, -1, -1, -1, -1, 0, 0, 1, 1, 1; 12],$$

$$t(\beta_1, k)+t(\beta_2, k)=(p+1)\left(1+\left(\frac{-3}{p}\right)\right)[2k-3, -k+1, -k+2; 3]/2^2 3^3,$$

$$t(\beta_3, k)+t(\beta_4, k)=(p+1)\left(1+\left(\frac{-3}{p}\right)\right)\Big/2^2 3^2$$
$$\times[-1, -k+1, -k+2, k-1, k-2; 6],$$
$$t(\beta_5, k)+t(\beta_6, k)=(p+1)\left(1+\left(\frac{-1}{p}\right)\right)$$
$$\times[k-2, -k+1, -k+2, k-1; 4]/2^4 3,$$
$$t(\gamma_1, k)+t(\gamma_2, k)=5(p+1)\left(1+\left(\frac{-1}{p}\right)\right)(2k-3)/2^6 3,$$
$$t(\gamma_3, k)=(p+1)\left(1+\left(\frac{-3}{p}\right)\right)(2k-3)/3^3,$$
$$t(\delta_1, k)+t(\delta_2, k)=7(p+1)^2(-1)^k(2k-2)(2k-4)/2^8 3^2,$$
$$t(\hat{\beta}_1, k)+t(\hat{\beta}_2, k)=\left(1+\left(\frac{-3}{p}\right)\right)[0, 1, 1, 0, -1, -1; 6]/3,$$
$$t(\hat{\beta}_3, k)+t(\hat{\beta}_4, k)=-\left(1+\left(\frac{-3}{p}\right)\right)[2, -1, -1; 3]/3^2,$$
$$t(\hat{\beta}_5, k)+t(\hat{\beta}_6, k)=-\frac{4}{9}\left(1+\left(\frac{-1}{p}\right)\right)[1, -1, 0; 3],$$
$$t(\hat{\beta}_7, k)+t(\hat{\beta}_8, k)=-\left(1+\left(\frac{-1}{p}\right)\right)[1, -1, -1, 1; 4]/4,$$
$$t(\hat{\beta}_9, k)+t(\hat{\beta}_{10}, k)=-\left(1+\left(\frac{-1}{p}\right)\right)[1, -1, -1, 1; 4]/4,$$
$$t(\hat{\hat{\delta}}_1, k)+t(\hat{\hat{\delta}}_2, k)=(-1)^k/2,$$
$$t(\hat{\hat{\delta}}_3, k)+t(\hat{\hat{\delta}}_4, k)=\left(3-\left(\frac{-1}{p}\right)\right)\Big/2^3,$$
$$t(\hat{\delta}_1, k)+t(\hat{\delta}_2, k)=-(p+1)(-1)^k(2k-3)/2^2 3,$$
$$t(\varepsilon_1, k)=(p+1)/6,$$
$$t(\varepsilon_2, k)=0,$$
$$t(\varepsilon_3, k)=-(p+1)/2^2 3,$$
$$t(\varepsilon_4, k)=-(p+1)^2(2k-3)/2^4 3^2,$$
$$t(\hat{\gamma}_1, k)+t(\hat{\gamma}_2, k)=t(\hat{\gamma}_3, k)+t(\hat{\gamma}_4, k)=-\left(1+\left(\frac{-1}{p}\right)\right)\Big/2^2,$$
$$t(\hat{\gamma}_5, k)+t(\hat{\gamma}_6, k)+t(\hat{\gamma}_7, k)=-\frac{2}{3}\left(1+\left(\frac{-3}{p}\right)\right),$$

where $\left(\frac{*}{p}\right)$ *is the Legendre symbol and* $[t_0, \cdots, t_{r-1}; r]$ *means that we take the value* t_i *if* $k\equiv i \bmod r$.

Theorem 3.3. *For a natural integer* $k\geqq 5$ *and a prime number* $p\neq 2, 3$, $\dim S_k(U_2(p))=\dim S_k(U_0(p))$ *is given by the summation of the following quantities*:

$$t(\alpha_0, k)=(p+1)(p^2+1)(2k-2)(2k-3)(2k-4)/2^9 3^3 5,$$
$$t(\alpha_1, k)=(p+1)\left(1+\left(\frac{-1}{p}\right)\right)(-1)^k/2^7,$$
$$t(\alpha_2, k)+t(\alpha_3, k)=-(p+1)\left(1+\left(\frac{-3}{p}\right)\right)[0, 1, -1; 3]/2^2 3^3,$$
$$\sum_{i=4}^{6} t(\alpha_i, k)=\frac{1}{2}[1, 0, 0, -1; 4]\begin{cases}1\cdots\text{if } p\equiv 1 \bmod 8,\\ 0\cdots\text{otherwise},\end{cases}$$
$$\sum_{i=7}^{12} t(\alpha_i, k)=2\left(1+\left(\frac{-3}{p}\right)\right)[1, 0, 0, -1, 0, 0; 6]/3^2,$$
$$t(\alpha_{13}, k)+t(\alpha_{14}, k)=\frac{1}{3}[0, 1, -1; 3]\begin{cases}1\cdots\text{if } p\equiv 1 \bmod 12,\\ 0\cdots\text{otherwise},\end{cases}$$
$$\sum_{i=15}^{18} t(\alpha_i, k)=\frac{1}{5}[1, 0, 0, -1, 0; 5]\begin{cases}4\cdots\text{if } p\equiv 1 \bmod 5,\\ 1\cdots\text{if } p=5,\\ 0\cdots\text{otherwise},\end{cases}$$
$$\sum_{i=19}^{22} t(\alpha_i, k)=\left(2+\left(\frac{-1}{p}\right)+\left(\frac{-3}{p}\right)\right)$$
$$\times[1, 0, 0, -1, -1, -1, -1, 0, 0, 1, 1, 1; 12]/2^2 3,$$
$$t(\beta_1, k)+t(\beta_2, k)=\left(p+2+\left(\frac{-3}{p}\right)\right)[2k-3, -k+1, -k+2; 3]/2^3 3^3,$$
$$t(\beta_3, k)+t(\beta_4, k)=\left(p+2+\left(\frac{-3}{p}\right)\right)$$
$$\times[-1, -k+1, -k+2, 1, k-1, k-2; 6]/2^3 3^2,$$
$$t(\beta_5, k)+t(\beta_6, k)=\left(p+2+\left(\frac{-1}{p}\right)\right)[k-2, -k+1, -k+2, k-1; 4]/2^5 3,$$
$$t(\gamma_1, k)+t(\gamma_2, k)=5(p+1)\left(1+\left(\frac{-1}{p}\right)\right)(2k-3)/2^7 3,$$
$$t(\gamma_3, k)=(p+1)\left(1+\left(\frac{-3}{p}\right)\right)(2k-3)/2\cdot 3^3,$$
$$t(\delta_1, k)+t(\delta_2, k)=7(p+1)(-1)^k(2k-2)(2k-4)/2^8 3^2,$$
$$t(\hat{\beta}_1, k)+t(\hat{\beta}_2, k)=\left(3+\left(\frac{-3}{p}\right)\right)[0, 1, 1, 0, -1, -1; 6]/2^2 3,$$

$t(\hat{\beta}_3, k)+t(\hat{\beta}_4, k)=-\left(3+\left(\frac{-3}{p}\right)\right)[2, -1, -1; 3]/2^2 3^2,$

$t(\hat{\beta}_5, k)+t(\hat{\beta}_6, k)=\begin{cases} -4[1, -1, 0; 3]//3^2 \cdots \text{if } p\equiv 1 \bmod 3 \\ -[2, -1, -1; 3]/3^2 \cdots \text{if } p\equiv 2 \bmod 3, \end{cases}$

$t(\hat{\beta}_7, k)+t(\hat{\beta}_8, k)=-\left(3+\left(\frac{-1}{p}\right)\right)[1, -1, -1, 1; 4]/2^4,$

$t(\hat{\beta}_9, k)+t(\hat{\beta}_{10}, k)=-\left(3+\left(\frac{-1}{p}\right)\right)[1, -1, -1, 1, 4]/2^4,$

$t(\hat{\hat{\delta}}_1, k)+t(\hat{\hat{\delta}}_2, k)=(-1)^k/2^2,$

$t(\hat{\hat{\delta}}_3, k)+t(\hat{\hat{\delta}}_4, k)=\left(3-\left(\frac{-1}{p}\right)\right)\Big/2^4,$

$t(\hat{\delta}_1, k)+t(\hat{\delta}_2, k)=-(p+3)(-1)^k(2k-3)/2^4 3,$

$t(\varepsilon_1, k)=(p+1)/2^2 3,$

$t(\varepsilon_2, k)=0,$

$t(\varepsilon_3, k)=-(p+3)/2^4 3,$

$t(\varepsilon_4, k)=-(p+1)^2(2k-3)/2^5 3^2,$

$t(\hat{\gamma}_1, k)+t(\hat{\gamma}_2, k)=t(\hat{\gamma}_3, k)+t(\hat{\gamma}_4, k)=-\left(1+\left(\frac{-1}{p}\right)\right)\Big/2^3,$

$t(\hat{\gamma}_5, k)+t(\hat{\gamma}_6, k)+t(\hat{\gamma}_7, k)=-\left(1+\left(\frac{-3}{p}\right)\right)\Big/3,$

where the notations are same as in Theorem 3.2.

Numerical examples of $\dim S_k(B(p))$ *and* $\dim S_k(U_2(p))$ *for small* k *and* p.

In the following tables, we write $\dim S_k(B(p))$ in the second row, and $\dim S_k(U_2(p))$ in the third row.

(i) $p=5$

k	5	6	7	8	9	10	11	12	13	14	15	16
$B(p)$	2	15	10	43	27	90	64	166	116	267	203	412
$U_2(p)$	1	2	2	6	6	15	13	27	20	42	37	68

(ii) $p=7$

k	5	6	7	8	9	10	11	12	13	14	15	16
$B(p)$	11	45	43	125	123	277	263	505	471	825	791	1281
$U_2(p)$	2	5	7	15	17	34	37	63	61	100	104	160

(iii) $p=11$

k	5	6	7	8	9	10	11	12	13	14	15	16
$B(p)$	66	202	283	603	756	1340	1581	2501	2854	4190	4679	6503
$U_2(p)$	5	12	21	42	60	103	130	198	229	331	338	528

(iv) $p=13$

k	5	6	7	8	9	10	11	12	13	14	15	16
$B(p)$	141	387	578	1140	1507	2521	3120	4710	5557	7855	9094	12236
$U_2(p)$	12	27	45	80	113	180	232	337	403	556	662	875

Theorem 3.4. *For a natural integer* $k\geqq 5$ *and* $p\neq 2, 3$, *we have*

$$\dim S_k(B(p))-\dim S_k(U_0(p))-\dim S_k(U_1(p))-\dim S_k(U_2(p)) +\dim S_k(U_{02}(p))+2\dim S_k(Sp(2, \mathbf{Z}))=\sum_{i=1}^{12} T_i,$$

where T_i *is the contribution of semi-simple conjugacy classes whose principal polynomial is* $f_i(x)$ *or* $f_i(-x)$ *in* (3.1). *Non-elliptic conjugacy classes* (i.e. *of infinite order*) *has no contribution.* T_i ($i=1, \cdots, 21$) *are given explicitly as follows*:

$T_1=(p-1)(p^3-1)(2k-2)(2k-3)(2k-4)/2^9 3^3 5,$

$T_2=7(p-1)^2(-1)^k(k-1)(k-2)/2^7 3^2,$

$T_3=-(p-1)\left(1-\left(\frac{-1}{p}\right)\right)[k-2, -k+1, -k+2, k-1; 4]/2^5 3,$

$T_4=-(p-1)\left(1-\left(\frac{-3}{p}\right)\right)[2k-3, -k+1, -k+2; 3]/2^3 3^3,$

$T_5=-(p-1)\left(1-\left(\frac{-3}{p}\right)\right)[-1, -k+1, -k+2, 1, k-1, k-2; 6]/2^3 3^2,$

$T_6=(p-1)\left(1-\left(\frac{-1}{p}\right)\right)[-k+1, -k+2; 2]/2^6 3,$

$T_7=(p-1)\left(1-\left(\frac{-3}{p}\right)\right)[-2k+3, -2k+2, -2k+4; 3]/2^3 3^3,$

$T_8=\left(1-\left(\frac{-1}{p}\right)\right)\left(1-\left(\frac{-3}{p}\right)\right)$
$\times[1, 0, 0, -1, -1, -1, -1, 0, 0, 1, 1; 12]/2^2 3,$

$$T_9=\left(1-\left(\frac{-3}{p}\right)\right)^2[1, 0, 0, -1, 0, 0; 6]/3^2,$$

$$T_{10}=\frac{1}{5}[1, 0, 0, -1, 0; 5]\begin{cases}1\cdots\text{if } p=5\\ 2\cdots\text{if } p\equiv 2, 3 \bmod 5,\\ 4\cdots\text{if } p\equiv 4 \bmod 5,\\ 0\cdots\text{if } p\equiv 1 \bmod 5,\end{cases}$$

$$T_{11}=\frac{1}{2}[1, 0, 0, -1; 4]\begin{cases}1\cdots\text{if } p\equiv 7 \bmod 8,\\ 0\cdots\text{otherwise,}\end{cases}$$

$$T_{12}=\frac{1}{6}[1, 0, 0, -1, 2, -2; 6]\begin{cases}1\cdots\text{if } p\equiv 11 \bmod 12,\\ 0\cdots\text{otherwise,}\end{cases}$$

where the notations are same as in Theorem 3.2.

Proof. One can get this Theorem 3.4, by straightforward calculation, using Theorems 3.2, 3.3 in this paper, Theorems 6.2, 7.1 in [16], and Theorem 4 in [24]. q.e.d.

Next, we treat $B'(p)$. In this case, every element of G' is semi-simple, and if it is of finite order, then its principal polynomial is one of $f_i(x)$ or $f_i(-x)$ in (3.1). For any open compact subgroup U of G'_A, we denote by $H_i(U)$ the contribution to dim $M_0(U)$ of elements of G' whose principal polynomial is $f_i(x)$ or $f_i(-x)$. For $g \in Sp(2)$, it is well known that tr $\rho_\nu(g)$ depends only on the principal polynomial of g and that tr $\rho_\nu(g)=$ tr $\rho_\nu(-g)$. We fix an element $g_i \in Sp(2)$ whose principal polynomial is $f_i(x)$. Now, we state our results.

Theorem 3.5. *For any U as above and any integer $\nu\geqq 0$, we have*

$$\dim M_\nu(U)=\sum_{i=1}^{12} H_i(U) \operatorname{tr} \rho_\nu(g_i).$$

For any prime $p\neq 2$, 3, and $U=U_1'(p)$, $U_0'(p)$, or $B'(p)$, $H_i(U)$ is given as follows:

$$H_1(B'(p))=(p^4-1)/2^6 3^2 5,$$
$$H_1(U_1'(p))=(p-1)(p^2+1)/2^6 3^2 5,$$
$$H_1(U_0'(p))=(p^2-1)/2^6 3^2 5,$$
$$H_2(B'(p))=H_2(U_0'(p))=0,$$
$$H_2(U_1'(p))=7(p-1)^2/2^6 3^2,$$
$$H_3(B'(p))=H_3(U_0'(p))=0,$$

$$H_3(U_1'(p))=(p-1)\left(1-\left(\frac{-1}{p}\right)\right)\Big/2^6 3^2,$$
$$H_4(B'(p))=H_4(U_0'(p))=0,$$
$$H_4(U_1'(p))=(p-1)\left(1-\left(\frac{-3}{p}\right)\right)\Big/2^3 3^2,$$
$$H_5(B'(p))=H_5(U_0'(p))=0,$$
$$H_5(U_1'(p))=(p-1)\left(1-\left(\frac{-3}{p}\right)\right)\Big/2^3 3^2,$$
$$H_6(B'(p))=(p+1)\left(1-\left(\frac{-1}{p}\right)\right)\Big/2^5+5(p-1)\left(1+\left(\frac{-1}{p}\right)\right)\Big/2^5 3,$$
$$H_6(U_1'(p))=5(p-1)/2^5 3+\left(1-\left(\frac{-1}{p}\right)\right)\Big/2^5,$$
$$H_6(U_0'(p))=(p+1)\left(1-\left(\frac{-1}{p}\right)\right)\Big/2^6+5(p-1)\left(1+\left(\frac{-1}{p}\right)\right)\Big/2^6 3,$$
$$H_7(B'(p))=(p+1)\left(1-\left(\frac{-3}{p}\right)\right)\Big/2^2 3^2+(p-1)\left(1+\left(\frac{-3}{p}\right)\right)\Big/2\cdot 3^2,$$
$$H_7(U_1'(p))=(p-1)/2\cdot 3^2+\left(1-\left(\frac{-3}{p}\right)\right)\Big/2^2 3^2,$$
$$H_7(U_0'(p))=(p+1)\left(1-\left(\frac{-3}{p}\right)\right)\Big/2^3 3^2+(p-1)\left(1+\left(\frac{-3}{p}\right)\right)\Big/2^2 3^2,$$
$$H_8(B'(p))=H_8(U_0'(p))=0,$$
$$H_8(U_1'(p))=\left(1-\left(\frac{-1}{p}\right)\right)\left(1-\left(\frac{-3}{p}\right)\right)\Big/2^2 3,$$
$$H_9(B'(p))=H_9(U_0'(p))=0,$$
$$H_9(U_1'(p))=\left(1-\left(\frac{-3}{p}\right)\right)^2\Big/3^2,$$

$$H_{10}(B'(p))=\begin{cases}1/5\cdots\text{if } p=5,\\ 0\ \cdots\text{otherwise,}\end{cases}$$

$$H_{10}(U_1'(p))=\begin{cases}1/5\cdots\text{if } p=5,\\ 4/5\cdots\text{if } p\equiv 4 \bmod 5,\\ 0\ \cdots\text{otherwise,}\end{cases}$$

$$H_{10}(U_0'(p))=\begin{cases}1/5\cdots\text{if } p=5,\\ 2/5\cdots\text{if } p\equiv 2,3 \bmod 5,\\ 0\ \cdots\text{otherwise,}\end{cases}$$

$$H_{11}(B'(p))=\left(1-\left(\frac{2}{p}\right)\right)\Big/2^2,$$

$$H_{11}(U_1'(p))=\begin{cases}0 & \cdots \text{if } p\equiv 1 \bmod 8,\\ 1/4 & \cdots \text{if } p\equiv 3,5 \bmod 8,\\ 1/2 & \cdots \text{if } p\equiv 7 \bmod 8,\end{cases}$$

$$H_{11}(U_0'(p))=\left(1-\left(\frac{2}{p}\right)\right)\Big/2^3,$$

$$H_{12}(B'(p))=\begin{cases}1/3 & \cdots \text{if } p\equiv 5 \bmod 12,\\ 0 & \cdots \text{otherwise},\end{cases}$$

$$H_{12}(U_1'(p))=\begin{cases}1/6 & \cdots \text{if } p\equiv 5 \bmod 6,\\ 0 & \cdots \text{otherwise},\end{cases}$$

$$H_{12}(U_0'(p))=\begin{cases}1/6 & \cdots \text{if } p\equiv 5 \bmod 12,\\ 0 & \cdots \text{otherwise}.\end{cases}$$

Remark. The above results for $U_1'(p)$ and $U_0'(p)$ have been already given in [19], including the case where $p=2, 3$. We reproduced them here for the convenience of the readers. The Weyl character formula gives explicit values of $\mathrm{tr}\,\rho_\nu(g_i)$, which has been calculated in [19] (I) Theorem 3 (p. 596).

Theorem 3.6. *For any integer $\nu\geq 0$, put $k=\nu+3$. For any prime $p\neq 2, 3$, and for the above k, define T_i $(i=1, \cdots, 12)$ as in Theorem 3.4 (, although k might be 3 or 4). Then, we get*

$$(H_i(B'(p))-H_i(U_1'(p))-H_i(U_0'(p)))\ \mathrm{tr}\,\rho_\nu(g_i)=T_i$$

for all $i=1, \cdots, 12$.

Proof. This is obtained by straightforward calculation. q.e.d.

We see very easily that $\mathfrak{M}_\nu(U_1'(p))\cap\mathfrak{M}_\nu(U_0'(p))=0$, unless $\nu=0$, and $\dim(\mathfrak{M}_0(U_1'(p))\cap\mathfrak{M}_0(U_0'(p)))=1$, if $\nu=0$, so the dimensions of new forms belonging to $B'(p)$ is given by:

$$\dim\mathfrak{M}_\nu^0(B'(p))=\dim\mathfrak{M}_\nu(B'(p))-\dim\mathfrak{M}_\nu(U_1'(p))-\dim\mathfrak{M}_\nu(U_0'(p))+\delta,$$

where $\delta=0$, if $\nu\neq 0$, and $\delta=1$, if $\nu=0$.

Numerical examples of dimensions of $\mathfrak{M}_\nu(B'(p))$, $\mathfrak{M}_\nu(U_i(p))$ $(i=0, 1)$, and new forms $\mathfrak{M}_\nu^0(B'(p))$.

(i) $p=5$

ν	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$B'(p)$	2	1	5	8	15	22	34	47	67	87	115	146	184	225
$U_1'(p)$	2	0	3	0	6	0	14	3	23	6	33	10	53	21
$U_0'(p)$	1	0	1	1	2	2	3	3	5	5	7	8	10	11
new forms	0	1	1	7	7	20	17	41	39	76	75	128	121	193

(ii) $p=7$

ν	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$B'(p)$	2	6	14	28	50	80	122	176	244	328	430	550	692	856
$U_0'(p)$	2	0	5	0	16	3	29	8	55	21	85	37	133	67
$U_0'(p)$	1	1	1	2	3	4	5	6	8	10	13	15	18	22
new forms	0	5	8	26	31	73	88	162	181	297	332	498	541	767

(iii) $p=11$

ν	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$B'(p)$	7	27	74	156	285	467	718	1044	1457	1965	2582	3314	4175	5171
$U_1'(p)$	5	1	16	3	45	16	99	48	186	106	296	182	474	318
$U_0'(p)$	1	1	2	3	5	6	9	12	16	20	26	32	40	48
new forms	2	25	56	150	235	445	610	984	1255	1839	2260	3100	3661	4805

(iv) $p=13$

ν	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$B'(p)$	13	53	144	304	555	911	1400	2036	2841	3833	5036	6464	8143	10087
$U_1'(p)$	4	0	23	7	70	32	154	88	288	184	483	333	750	546
$U_0'(p)$	2	2	3	5	8	10	14	18	24	30	39	47	58	70
new forms	8	51	118	292	477	869	1232	1930	2529	3619	4514	6084	7335	9471

§ 4. Arithmetic general formulae for dimensions*)

4-0. This section is mostly an exposition of [15], [16], and [19]. Our purpose is to describe the general "arithmetic" formulae for the dimensions of our space $S_k(\Gamma)$, $\mathfrak{M}_\nu(U_A)$ or automorphic forms for arithmetic subgroups of $Sp(n, \boldsymbol{R})$, and $Sp(n)$. Here n is an arbitrary positive integer. These formulae, Theorem A in Section 4–2 and Theorem B in Section 4–4, enable us to compute explicitly the dimensions of $S_k(\Gamma)$, $\mathfrak{M}_\nu(U_A)$ for the special groups considered in Sections 1, 2, and 3, as we shall carry out in Sections 5, 6, which lead us to our main results in this paper.

In the split case (i.e. for $Sp(n, \boldsymbol{R})$), our formula is based on Selberg's trace formula; and the derivation of our formula from Selberg's formula consists of two main parts i.e.,

(i) evaluation of certain integrals (analytic part), and

(ii) classification of conjugacy classes in Γ and their centralizers (arithmetic part)

(ii) (bis) when the conjugacy classes in question are semi-simple, we need only G_A-conjugacy classes instead of Γ-conjugacy classes, and certain G-Maß (see Theorem (4.31)).

On the other hand, in the compact case (i.e. for $Sp(n)$), our formula can be obtained in quite elementary way as a special case of the trace formula for the Brandt matrices $B_\rho(n)$ (c.f. [15]), which generalizes the method of Eichler [9, 10] and Shimizu [43]. Here the analytic part (i) is quite simple; it is nothing but the character computation of the finite dimensional representation ρ, which is now a classical result of H. Weyl [50]. Therefore, the essential part of the derivation of our formula consists of only (ii) (bis), although explicit computations are not so easy.

Moreover, as we shall see, the arithmetic part (ii) (bis) can be handled *in a unified manner in both $Sp(n, \boldsymbol{R})$ and $Sp(n)$ cases.* So we first describe this part in the following paragraph, where a certain arithmetic invariant $H(g, U_A)$ will be defined for a semisimple conjugacy class of G_Q^1, a $\boldsymbol{Q}$-form of *either $Sp(n, \boldsymbol{R})$ or $Sp(n)$*, and a closed formula for it will be given. It would be convenient, however, to describe here the motivation to introducing such a invariant by sketching the special meaning of it in the compact case.

In the compact case, our space $\mathfrak{M}_\nu(U_A)$ of automorphic forms is isomorphic to $\bigoplus_{i=1}^{H} V^{\Gamma_i}$ (c.f. § 1), so we have

* In this section, G will denote *either one* of the groups G or G' of Section 1, Section 2, unless otherwise stated.

$$\begin{aligned}\dim \mathfrak{M}_\nu(U_A) &= \sum_{i=1}^{H} \dim V^{\Gamma_i} \\ (4.1)\qquad &= \sum_{i=1}^{H} \frac{1}{\#\Gamma_i} \sum_{g \in \Gamma_i} \mathrm{tr}\,(\rho_\nu(g)) \\ &= \sum_f \mathrm{tr}\,(\rho_\nu(g)) \sum_{i=1}^{H} \frac{\#[\Gamma_i \cap [f]]}{\#\Gamma_i}\end{aligned}$$

where the first sum is extended over the set of principal polynomials $f=f(x)$ of torsion elements of G_Q^1 (or $G_R^1 = Sp(n)$), and $[f]$ denotes the set of elements g which belong to $f(x)$. Note that, $\mathrm{tr}\,(\rho_\nu(g))$ depends only on f to which g belongs, and that the inner sum does not involve ρ_ν. Thus the computation of $\dim \mathfrak{M}_\nu(U_A)$ reduces to that of:

$$(4.2)\qquad H(f, U_A) := \sum_{i=1}^{H} \frac{\#\Gamma_i \cap [f]}{\#\Gamma_i}.$$

In general, the set $[f]$ in G_Q consists of *infinitely* many G_Q-conjugacy classes, while obviously only a finite number of them make nontrivial contributions to $\dim \mathfrak{M}_\nu(U_A)$. This leads to the following

Definition 4.3. A conjugacy class $\{g\}_{G_Q}$ in G_Q^1 is called "*locally integral*" (with respect to U_A) if $\Gamma_i \cap \{g\}_{G_Q} \neq \phi$ for some i $(1 \leq i \leq H)$. For each G_Q-conjugacy class $\{g\}_{G_Q}$, we define an invariant similar as (4.2):

$$(4.4)\qquad H(g, U_A) := \sum_{i=1}^{H} \frac{\#[\Gamma_i \cap \{g\}_{G_Q}]}{\#\Gamma_i}.$$

Clearly, $\{g\}_{G_Q}$ is locally integral if and only if $H(g, U_A) \neq 0$. Note also that this implies g is of finite order,

4-1. *A formula for $H(g, U_A)$.* Let D be a quaternion algebra over $\boldsymbol{Q}$ (definite or indefinite), and let the group G_Q be defined by

$$\begin{aligned} (4.5)\qquad G_Q &:= \text{the group of similitudes of the hermitian space } (D^n, F), \\ & \qquad F(x, y) = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n, \\ &= \{g \in GL_n(D);\ g\,{}^t\bar{g} = n(g)\cdot 1_n,\ n(g) \in \boldsymbol{Q}_+^\times\},\end{aligned}$$

where for $g=(g_{ij})$, we write ${}^t\bar{g}=(\bar{g}_{ji})$, $a \mapsto \bar{a}$ being the canonical involution of D. We may regard it as $\boldsymbol{Q}$-rational points of an algebraic group G defined over $\boldsymbol{Q}$. G is reductive. We denote its semi-simple part by $G^1 := \{g \in G;\ n(g)=1\}$. In G_A, we consider an open subgroup U_A which, as we assume throughout this paper, is of the form

$$\begin{aligned}(4.6)\qquad U_A &= \underline{R}_A^\times \cap G_A \\ &= G_{\boldsymbol{R}} \times \prod_p U_p \quad (U_p = \underline{R}_p^\times \cap G_p)\end{aligned}$$

for some Z-order $\underline{R}$ of the Q-algebra $M_n(D)$. Then G_A is decomposed into a disjoint union of finite number of $U_A - G_Q$ double cosets:

$$G_A = \coprod_{i=1}^{H} U_A g_i G_Q. \tag{4.7}$$

By an "*arithmetic subgroup* Γ" of G_Q, or G_Q^1, we mean a system of subgroups $(\Gamma_i)_{i=1}^H$, where

$$\Gamma_i = G_Q \cap g_i^{-1} U_A g_i \quad (= G_Q^1 \cap g_i^{-1} U_A g_i). \tag{4.8}$$

It is this system of groups $(\Gamma_i)_{i=1}^H$ with respect to which our spaces of automorphic forms are defined. If D is definite, then Γ_i are all finite groups, since they are contained in the discrete subgroup G_Q^1 and the compact group $G_A^1 \cap g_i^{-1} U_A g_i$. On the other hand, if D is indefinite, we have a natural isomorphism

$$\begin{aligned} &G_Q^1 \xrightarrow{\sim} Sp(n, R), \qquad g = (g_{ij}) \longmapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \\ &g_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}, \quad A = (a_{ij}), \quad B = (b_{ij}) \quad \text{etc.} \end{aligned} \tag{4.9}$$

where we identify $D_R = D \otimes_Q R$ and $M_2(R)$; and Γ_i's are arithmetic discrete subgroups of $Sp(n, R)$, which act on the Siegel upper half plane $\mathfrak{H}_n$ properly discontinuously in the usual manner. In this case, if $\underline{R}$ is sufficiently large, we have $H=1$ by the strong approximation theorem (c.f. Kneser [32]), which is the case for all arithmetic subgroups of $GSp(2, Q) = G_Q$ treated in this paper. However, to treat uniformly two cases (D=definite, or indefinite), we do not assume that $H=1$.

We take and fix, once for all, an open subgroup U_A and a semi-simple conjugacy class $\{g\}_{G_Q}$ contained in G_Q^1. Put

$$\begin{aligned} Z(g) &:= \text{commutor algebra of } g \text{ in } M_n(D) \\ &= \{z \in M_n(D);\ zg = gz\}, \\ Z_G(g) &= Z(g)^\times \cap G_Q \quad (= \text{the centralizer of } g \text{ in } G_Q). \end{aligned}$$

Then $Z(g)$ is a semi-simple algebra over Q, and $Z_G(g)$ is an algebraic group, reductive, over Q. In the set $\{\Lambda\}$ of Z-orders of $Z(g)$, we define two equivalence relations

$$\begin{aligned} &\Lambda_1 \sim \Lambda_2 \Longleftrightarrow \Lambda_2 = a\Lambda_1 a^{-1} \qquad \text{for some } a \in Z_G(g) \\ &\Lambda_1 \approx \Lambda_2 \Longleftrightarrow \Lambda_2 = a\Lambda_1 a^{-1} := \bigcap_p (a_p \Lambda_{1p} a_p^{-1} \cap Z(g)) \\ &\qquad\qquad\qquad\qquad \text{for some } a = (a_p) \in Z_G(g)_A \end{aligned} \tag{4.10}$$

An equivalence class in the second relation is usually called a *G-genus*, which we denote by $L_G(\Lambda)$ if it contains Λ; it consists of finitely many classes with respect to the first equivalence relation. We have a disjoint decomposition of $\Gamma_i \cap \{g\}_{G_Q}$ for each i:

$$\Gamma_i \cap \{g\}_{G_Q} = \coprod_{\Lambda/\sim} C(g, \Lambda, i) \cap \Gamma_i \tag{4.11}$$

where we put

$$\begin{aligned} C(g, \Lambda, i) &:= \{x^{-1}gx;\ x \in G_Q,\ Z(g) \cap x\underline{R}_i x^{-1} \sim \Lambda\}, \\ \underline{R}_i = g_i^{-1}\underline{R} g_i &:= \bigcap_p (g_{ip}^{-1}\underline{R}_p g_{ip} \cap M_n(D)) \end{aligned} \tag{4.12}$$

and the union is extended over the (actually a finite) set of Z-orders Λ of $Z(g)$, modulo the equivalence $\sim$. Note that the set $C(g, \Lambda, i)$ is stable under the conjugation by Γ_i, and it consists of a finite number of Γ_i-conjugacy classes. Now we define our arithmetic invariants

$$\begin{aligned} H(g, \Lambda, U_A) &:= \sum_{i=1}^{H} \sharp[C(g, \Lambda, i) \cap \Gamma_i / \widetilde{\Gamma_i}], \\ H(g, U_A) &:= \sum_{\Lambda/\sim} \mathrm{vol}\,(\Lambda^\times \cap G_Q^1 \backslash G_R^1 \cap Z_G(g)_R) \cdot H(g, \Lambda, U_A). \end{aligned} \tag{4.13}$$

Remark 4.14. Note that these are invariants of the G_Q-conjugacy class $\{g\}_{G_Q}$. Since $H(g, \Lambda, U_A) \neq 0$ for only a finite number of classes of Λ, the last sum is actually a finite sum. Here the volume of the quotient $\Lambda^\times \cap G_Q^1 \backslash G_R^1 \cap Z_G(g)_R$ is measured by a suitably normalized (fixed) Haar measure of $G_R^1 \cap Z_G(g)_R$. In the case $G_R^1 = Sp(n)$ (i.e. D=definite), we may take the measure so normalized that $\mathrm{vol}\,(G_R^1 \cap Z_G(g)_R) = 1$. Then we see that our invariant $H(g, U_A)$ coincides with the one given by (4.4), since we have the following

Lemma 4.15. (D=*definite or indefinite*)
If $a \in C(g, \Lambda, i) \cap \Gamma_i$, *then we have*

$$\begin{aligned} C(a; \Gamma_i) &:= \text{centralizer of } a \text{ in } \Gamma_i \\ &= \Lambda^\times \cap G_Q^1 \quad (= \text{independent of } a, i\ !). \end{aligned}$$

For the proof, see [15], Lemma 4. Thus we see that our invariant $H(g, U_A)$ is the weighted average of the number of elements in Γ_i which are conjugate in G_Q to g; and $H(g, \Lambda, U_A)$ is a refinement of it. We want to (indeed we should) give them some expressions which do not involve H, the class number of U_A. For this purpose, we put

$$\begin{aligned} M(g, \Gamma_i, \Lambda) &= \{x \in G_Q;\ x^{-1}gx \in \Gamma_i,\ Z(g) \cap x\underline{R}_i x^{-1} \sim \Lambda\}, \\ M_A(g, U_A, \Lambda) &= \{x \in G_A;\ x^{-1}gx \in U_A,\ Z(g) \cap x\underline{R} x^{-1} \sim \Lambda\}, \\ M_p(g, U_p, \Lambda) &= \{x \in G_p;\ x^{-1}gx U_p,\ Z(g)_p \cap x\underline{R}_p x^{-1} \sim \Lambda_p\}. \end{aligned} \tag{4.16}$$

Then we obviously have the following

Lemma 4.17. *The map $x^{-1}gx \mapsto x$ induces the following bijection for each i ($1 \le i \le H$):*

$$C(g, \Lambda, i) \cap \Gamma_i / \widetilde{\Gamma_i} \xrightarrow{\sim} Z_G(g) \backslash M(g, \Gamma_i, \Lambda)/\Gamma_i$$

(c.f. [15], Lemma 3).

The next lemma plays a key role in our problem, since it enables us to get rid of H from $H(g, U_A)$:

Lemma 4.18. *For each double coset $G_Q g_i^{-1} U_A$ in (4.7), we have a bijection induced from the map $a\, g_i^{-1} u \mapsto a$ ($a \in G_Q$, $u \in U_A$):*

$$Z_G(g) \backslash M_A(g, U_A, \Lambda) \cap G_Q g_i^{-1} U_A / U_A \xrightarrow{\sim} Z_G(g) \backslash M(g, \Gamma_i, \Lambda)/\Gamma_i$$

(loc. cit., Lemma 5).

Corollary 4.19. *We have*

$$H(g, \Lambda, U_A) = \sum_{i=1}^{H} \#[Z_G(g) \backslash M(g, \Gamma_i, \Lambda)/\Gamma_i] = \#[Z_G(g) \backslash M_A(g, U_A, \Lambda)/U_A].$$

To proceed further, we note that $M_A(g, U_A, \Lambda)$ is not stable under the action of $Z_G(g)_A$ from the left, and therefore put

$$M_A^*(g, U_A, \Lambda) := \bigcup_{\Lambda' \in L_G(\Lambda)} M_A(g, U_A, \Lambda'). \tag{4.20}$$

Now consider the natural projection:

$$\phi\colon Z_G(g) \backslash M_A(g, U_A, \Lambda)/U_A \longrightarrow Z_G(g)_A \backslash M_A^*(g, U_A, \Lambda)/U_A \cong \prod_p [Z_G(g)_p \backslash M_p(g, U_p, \Lambda_p)/U_p].$$

Lemma 4.21. *The map ϕ above is $h_0(\Lambda; G)$-to-one, where $h_0(\Lambda; G)$ is the two-sided G-class number of Λ defined by the following formula*

$$\begin{aligned} h_0(\Lambda; G) &:= \#[Z_G(g) \backslash Z_G(g) \cdot I(\Lambda)/(\Lambda_A^\times \cap G_A)] \\ I(\Lambda) &= \{z \in Z_G(g)_A;\ z\Lambda z^{-1} = \Lambda\}. \end{aligned} \tag{4.22}$$

(loc. cit. (18)).

Definition. Let $Z_G(g)_A$ be decomposed into a disjoint union

$$Z_G(g)_A = \coprod_{j=1}^{h} Z_G(g) y_j (\Lambda_A^\times \cap G_A), \tag{4.23}$$

and put

$$\Lambda_j := y_j \Lambda y_j^{-1} = \bigcap_p (y_{jp} \Lambda_p y_{jp}^{-1} \cap Z(g)).$$

(i) The number $h = h(\Lambda; G)$ of cosets in (4.23) is called the G-class number of Λ, or $L_G(\Lambda)$ (note that it depends only on the G-genus $L_G(\Lambda)$).

(ii) The invariant of $L_G(\Lambda)$ defined by

$$M_G(\Lambda) := \sum_{j=1}^{h} \mathrm{vol}\,(\Lambda_j^\times \cap G_Q \backslash G_R^1 \cap Z_G(g)_R) \tag{4.24}$$

is called the "G-Maß (or G-measure)" of Λ, or $L_G(\Lambda)$.

Note that these invariants do not depend on the choice of (y_j) in the decomposition (4.23). It is not difficult to prove the following

Lemma 4.25. *We have*

$$h(\Lambda\colon G) = \sum_{\Lambda^{(k)} \in L_G(\Lambda)/\sim} h_0(\Lambda^{(k)}; G)$$

$$M_G(\Lambda) = \sum_{\Lambda^{(k)} \in L_G(\Lambda)/\sim} h_0(\Lambda^{(k)}; G)\ \mathrm{vol}\,(\Lambda^{(k)\times} \cap G_Q \backslash G_R^1 \cap Z_G(g)_R)$$

(loc. cit., Lemma 7).

Combining these results, we finally get the following expression of our invariant $H(g, U_A)$

Theorem 4.26. *We have*

$$H(g, U_A) = \sum_{L_G(\Lambda)} M_G(\Lambda) \prod c_p(g, U_p, \Lambda_p), \tag{4.26}$$

where

$$c_p(g, U_p, \Lambda_p) = \#[Z_G(g)_p \backslash M_p(g, U_p, \Lambda_p)/U_p].$$

Proof. By (4.19), (4.21), we have

$$\begin{aligned} H(g, U_A) &= \sum_{\Lambda/\sim} h_0(\Lambda; G)\ \mathrm{vol}\,(\Lambda^\times \cap G_Q \backslash G_R^1 \cap Z_G(g)_R) \\ &\quad \times \#[Z_G(g)_A \backslash M_A^*(g, U_A, \Lambda)/U_A] \\ &= \sum_{L_G(\Lambda)} \sum_{\Lambda' \in L_G(\Lambda)/\sim} h_0(\Lambda'; G)\ \mathrm{vol}\,(\Lambda'^\times \cap G_Q \backslash G_R^1 \cap Z_G(g)_R) \\ &\quad \times \prod_p \#[Z_G(g)_p \backslash M_p(g, U_p, \Lambda_p)/U_p]. \end{aligned}$$

Here we used the fact that $M_A^*(g, U_A, \Lambda)$ depends only on the G-genus $L_G(\Lambda)$. The assertion now follows from Lemma 4.25. q.e.d.

We note that the sum in (4.26), which is seemingly extended over all G-genera of Z-orders in $Z(g)$, is actually a finite sum (c.f. Remark (4.14)). Moreover, the products are always finite; thus we have

(i) If Λ is fixed, $c_p(g, U_p, \Lambda_p)=0$, or 1 for all but finitely many p.

(ii) For a given p, $c_p(g, U_p, \Lambda_p)\neq 0$ only for finitely many classes $\Lambda_p/\sim$; moreover such a $\Lambda_p/\sim$ is unique for all but finitely many p.

Remark 4.27. The G-Maß $M_G(\Lambda)$ can be evaluated in a well-known manner by using theory of Tamagawa numbers (c.f. Weil [49], see also [19], (I), § 3).

It would be worth noting that, in the same way as Theorem 4.26, we can combine (4.11), (4.13), (4.19) and (4.21) to obtain a closed formula for the sum of the number of Γ_i-conjugacy classes in $\Gamma_i\cap\{g\}_{G_Q}$ $(1\leq i\leq H)$:

Theorem 4.28. *Notations being as above, we have, for a semi-simple element $g\in G_Q^1$:*

$$\sum_{i=1}^{H}\#[\Gamma_i\cap\{g\}_{G_Q}/\widetilde{\Gamma_i}]=\sum_{L_G(\Lambda)}h(\Lambda; G)\prod_p c_p(g, U_p, \Lambda_p).$$

If, in particular, G_R^1 is not compact, then we have $H=1$ and this gives a formula for the number of semi-simple conjugacy calsses in the arithmetic subgroup Γ of $Sp(n, \boldsymbol{R})$. In general, it is known that the set of semi-simple conjugacy classes in the classical groups *over fields* are parametrized by the isomorphism classes of various kinds of hermitian forms. Moreover, the centralizers of them are the unitary groups of the corresponding hermitian forms. Thus, the above theorem may be viewed as an *integral* version of this fact, since the essential part $h(\Lambda; G)$ in the formula is nothing but the class number of the unitary group $Z_G(g)$, with respect to the genus $L_G(\Lambda)$ (c.f. [19], § 2, and see also Remark 5.45).

4-2. *General Dimension Formula* (Compact Case). Assume that D is definite. Then our space $\mathfrak{M}_\rho(U_A)$ of automorphic forms of weight ρ for an open subgroup U_A of G_A is defined as in Section 1. Combining the results in the preceeding paragraph and (4.1), we immediately have the following general formula for dim $\mathfrak{M}_\rho(U_A)$ (a special case $\sigma=1$, $e=1$ of [15]):

Theorem A. *For a finite dimensional representation ρ of $Sp(n)$, we have*

$$\dim \mathfrak{M}_\rho(U_A)=\sum_f\sum_{g\in[f]/\widetilde{G_Q}}\mathrm{tr}\,(\rho(g))\sum_{L_G(\Lambda)}M_G(\Lambda)\prod_p c_p(g, U_p, \Lambda_p), \tag{4.30}$$

where the first sum is extended over the set of polynomials $f(x)$ of degree $2n$ which are products of some cyclotomic polynomials, and the second is over the set of locally integral G_Q-conjugacy classes belonging to $f(x)$, and the third is over the G-genera of Z-orders $L_G(\Lambda)$ of $Z(g)$ for each representative g of $[f]/\widetilde{G_Q}$.

4-3. *Parametrization of semi-simple Conjugacy Classes.* In the actual calculation of dimensions using (4.30), or (4.40) in the next paragraph, a fundamental rôle is played by the following

Theorem 4.31. (Hasse Principle for conjugacy classes in G_Q, G_Q^1). *Two elements g_1, g_2 of G_Q (resp. G_Q^1) are G_Q- (resp. G_Q^1-) conjugate if and only if they are conjugate in G_p (resp. G_p^1) for all p.*

(c.f. Asai [2], and [19], § 2).

For each monic polynomial $f(x)\in Q[x]$ of degree $2n$ such that $x^{2n}f(x^{-1})=f(x)$, we denote by $G[f]$ the set of semi-simple elements of G^1 whose principal polynomial is $f(x)$. Then the above theorem means that the following natural map induced by the inclusion map is injective:

$$G[f]/\widetilde{G_Q}\hookrightarrow G_A[f]/\widetilde{G_A}. \tag{4.32}$$

This reduces our problem to classify the G_Q-conjugacy classes to those for G_p-conjugacy classes, if we can determine the image of this map. The latters are much easier than the former, because there are only finitely many $(\leq 4$, if $n=2)$ G_p-conjugacy classes in each $G_p[f]$, and we can choose a representative of classes in $G_p[f]$ to have a very simple form which enable us to compute $c_p(g, U_p, \Lambda_p)$.

If the map (4.32) is surjective (hence bijective), we need nothing more than just putting local data together. However, this is not always the case; so we shall describe here the image of this map, under the following conditions: $n=2$, $f(x)=f_i(x)$ $(1\leq i\leq 12)$ are as in Section 3 (for details as well as the general case, see [19], Section 2).

Proposition 4.33.

(i) *If $f(x)$ is either one of $f_1(\pm x)$, $f_2(\pm x)$, $f_3(\pm x)$, $f_4(\pm x)$, $f_5(\pm x)$, $f_8(\pm x)$, or $f_{10}(\pm x)$, then (4.32) is surjective.*

(ii) *If $f(x)=f_6(x)$ or $f_7(\pm x)$, then the centralizer of each element $g\in G_Q[f]$ is expressed as*

$$Z_G(g)=Q(g)^\times\cdot Z_0(g)^\times, \tag{4.34}$$

where $Z_0(g)$ is a quaternion algebra over Q such that

$$Z_0(g)\otimes_Q F\cong D\otimes_Q F\quad (F=Q[x]/\sqrt{f(x)}\cong Q(g)),$$

and the product formula $\prod_p \mathrm{inv}_p(Z_0(g))=1$ *for the invariants of* $(Z_0(g)_p)$ *determines the image of* (4.32) *which has index* 2 *in* $G_A[f]/_{\widetilde{G_A}}$.

(iii) *If* $f(x)=f_9(x)$, $f_{11}(x)$ *or* $f_{12}(x)$, *then for each element* $g \in G[f]$, g^2 *belongs to either* $f_6(x)$ *or* $f_7(\pm x)$, *and the image of* (4.32) *is determined by* $Z_0(g^2)$ *as in* (ii) *above, which has also index* 2 *in* $G_A[f]/_{\widetilde{G_A}}$.

4-4. *General Dimension Formula* (Split Case). Let Γ be an arithmetic subgroup of $G_Q^1=Sp(n, Q)$, or a Q-form of $Sp(n, R)$. The dimension of $S_k(\Gamma)$ is first expressed by Godement [13] as an integral of an infinite series

$$\dim S_k(\Gamma)=\frac{a_n(k)}{\sharp Z(\Gamma)}\int_{\Gamma\backslash\mathfrak{H}_n}\sum_{\gamma\in\Gamma}H_\gamma(Z)\,dZ, \tag{4.35}$$

where $k>2n$, and

$$a_n(k)=\frac{1}{2^n(2\pi)^{n(n+1)/2}}\prod_{j=0}^{n-1}\frac{\Gamma(k-(n-1)/2+j/2)}{\Gamma(k-n+j/2)},$$

$$H_\gamma(Z)=\det\left(\frac{Z-\gamma\bar{Z}}{2i}\right)^{-k}\det(C\bar{Z}+D)^{-k}\det(Y)^k \quad \left(\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\right),$$

and $dZ=(\det Y)^{-n-1}dx\,dy$ $(Z=X+iY)$ is an invariant measure on $\mathfrak{H}_n$; $Z(\Gamma):=$center of Γ.

Our purpose here is to sketch briefly how one reforms it to a more manageable formula, suitable for an explicit computation. This has been done in the case $n=2$ by Christian [6], Morita [39], and Arakawa [1] (Q-rank one case), for the special case of principal congruence subgroups $\Gamma=\Gamma(N)$, $N\geq 3$, and by the first named author for arbitrary Γ ([16], (I)).

The main idea of the reformulation is well-known and a routine; we should exchange the integral and infinite sum in (4.35) and then combine the integrals in each conjugacy classes of Γ, to get a closed expression as a sum extended over the set of conjugacy classes of Γ. But this is allowed only if $\Gamma\backslash\mathfrak{H}_n$ is compact, which never occurs in our case with $n\geq 2$, since the Q-rank of G_Q^1 is n or $[n/2]$, according as $D=M_2(Q)$ or not, while $\Gamma\backslash\mathfrak{H}_n$ is compact if and only if Q-rank of G_Q is 0. However, we can overcome this difficulty by introducing certain dumping factors and replacing $H_\gamma(Z)$ by $H_\gamma(Z; s)=H_\gamma(Z)\times$(a dumping factor in s). In order to justify this argument we have to choose dumping factors and make estimations of sums of $H_\gamma(Z: s)$ to apply Lebesgue's theorem for various subsets of Γ. Substantial part of these estimations has been established by Christian [5]. We omit the details and refer to [16], Section 2, where the case $n=2$ was discussed using results of [5]. The second difficulty is the fact that $C(\gamma; \Gamma)$, the centralizer of γ in Γ, is *not* always a lattice of $C(\gamma; G_R^1)$ (see Example 4.37). This means that $\mathrm{vol}\,(C(\gamma; \Gamma)\backslash C(\gamma; G_R^1))$ is not always finite. To overcome this point, we first observe:

Proposition 4.36. *For any* $\gamma\in\Gamma$, *there exists a connected closed subgroup* $C_0(\gamma; G_R^1)$ *of* $C(\gamma; G_R^1)$ *which is characterized, modulo a compact semi-direct factor, by the following properties*

(i) $C_0(\gamma; \Gamma)=: C_0(\gamma; G_R^1)\cap\Gamma$ *is a lattice of* $C_0(\gamma; G_R^1)$

(ii) $[C(\gamma; \Gamma): C_0(\gamma; \Gamma)]<\infty$.

Example 4.37. Let $\Gamma=Sp(2, Z)$ and $\gamma=\begin{pmatrix}1 & S\\ 0 & 1\end{pmatrix}\in\Gamma$ with $S={}^tS\in M_2(Z)$. Then $C(\gamma; G_R^1)\cong O(S)\times R^3$, and $C(\gamma; \Gamma)$ is a lattice of $C(\gamma: G_R^1)$ if and only if $O_Z(S)=\{A\in GL_2(Z); AS\,{}^tA=S\}$ is a lattice of $O(S)$; it is easy to see that this is equivalent to that either S is definite, or $-\det(S)\notin(Q^\times)^2$. Thus we have, removing the compact factor $O(S)$ if S is definite.

$$C_0(\gamma; G_R^1)=\begin{cases}C(\gamma; G_R^1) & \text{if } S \text{ is indefinite, } -\det(S)\notin(Q^\times)^2,\\ \left\{\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}; X={}^tX\right\} & \text{if } S \text{ is definite, or } -\det(S)\in(Q^\times)^2.\end{cases}$$

Definition 4.38. Two elements γ_1, γ_2 of Γ are said to belong to the same "family", if (i) $C_0(\gamma_1; G_R^1)=C_0(\gamma_2; G_R^1)$ and (ii) $\gamma_{1s}=\gamma_{2s}$, where $\gamma_i=\gamma_{is}\gamma_{iu}$ $(i=1, 2)$ is the Jordan decomposition.

Now we divide the set Γ into disjoint union of three subsets $\Gamma^{(e)}$, $\Gamma^{(h)}$, and $\Gamma^{(p)}$:

(i) $\Gamma^{(e)}$ consists of elliptic elements and ± 1. (An element $\gamma\neq\pm 1$ of G_R^1 is called elliptic, if it has a fixed point in $\mathfrak{H}_n$; or equivalently (under the condition $\gamma\in\Gamma$), it is of finite order.)

(ii) $\Gamma^{(h)}$ consists of those elements $\gamma\in\Gamma$ which are of "hyperbolic" type i.e., γ has a real eigenvalue $\neq\pm 1$.

(iii) $\Gamma^{(p)}$ consists of "p-unipotent (or parabolic)" elements of Γ i.e., those elements $\gamma\in\Gamma-\Gamma^{(e)}$ whose semi-simple factors γ_s belong to $\Gamma^{(e)}$; equivalently, γ is p-unipotent if and only if some power of γ is unipotent, from which the name comes.

We denote the contributions to (4.35) of each of these subsets by $T_k(\Gamma^{(e)})$, $T_k(\Gamma^{(h)})$, and $T_k(\Gamma^{(p)})$ respectively.

Proposition 4.39. *For any semi-simple element* γ *of* Γ, $C(\gamma; \Gamma)$ *is always a lattice of* $C(\gamma; G_R^1)$. *Moreover, for the subset* $\Gamma^{(e)}$, *the termwise integrability is valid without dumping factors.*

Note that, in the case $\gamma\in\Gamma^{(e)}$, the integral $I(\gamma)=I(\gamma, s)|_{s=0}$ defined by (4.40) depends only on the conjugacy class $\{g\}_{G_R}$; and this has been evaluated by Langlands [33] in a more general context (see § 4–5 and

Remark (4.55)). After these remarks, we can immediately apply results of Section 4–1 to obtain the following

Theorem $\mathbf{B}^{(e)}$ (Elliptic Contributions). *With the notations of Section 4–1, we have for* $k>2n$

$$T_k(\Gamma^{(e)})=\sum_{\Gamma}\sum_{g\in[\Gamma]/\widetilde{G_Q}} t_k(g)\sum_{L_G(\Lambda)} M_G(\Lambda)\prod_{p} c_p(g, U_p, \Lambda_p), \tag{4.40}$$

where we put

$$\begin{aligned} t_k(g)&=a_k(k)\cdot I(g)\\ &=a_n(k)\int_{C(g;G^1_R)\backslash\mathfrak{H}_n} H_g(Z)\,dZ. \end{aligned}$$

Note that above formula for $T_k(\Gamma^{(e)})$ is completely analogous to the dimension formula (4.30) in Theorem A. In both cases, the factors $\mathrm{tr}(\rho(g))$ and $t_k(g)$, being invariants of G_R-conjugacy classes $\{g\}_{G_R}$, may be regarded as an "archimedean local factor $c_\infty(g, U_\infty, \Lambda_\infty)$", with $U_\infty= G_R$, $\Lambda_\infty=C(g, G_R)=Z_G(g)_R$. As for the explicit formulae for them, see Section 4–5.

Let us next consider $\Gamma^{(h)}$ and $\Gamma^{(p)}$:

Theorem $\mathbf{B}^{(h)}$. *For* $\Gamma^{(h)}$, *we have* $T_k(\Gamma^{(h)})=0$, *since for any* $\gamma\in\Gamma^{(h)}$,

$$I_0(\gamma; s) :=\int_{C_0(\gamma;G^1_R)\backslash\mathfrak{H}_n} H_\gamma(Z; s)dZ=0. \tag{4.41}$$

This is known in general as the "Selberg's Principle" (c.f. Warner [48]).

Theorem $\mathbf{B}^{(p)}$ (Parabolic Contributions). *For* $\Gamma^{(p)}$, *we have*

$$T_k(\Gamma^{(p)})=\frac{1}{\sharp Z(\Gamma)}\sum_{F} v(F)\lim_{s\downarrow 0}\zeta(s; F), \tag{4.42}$$

where the sum is extended over a complete set of Γ*-conjugacy classes of families* $F\subseteq\Gamma^{(p)}$, *and* $v(F)=\mathrm{vol}\,(C_0(\gamma; \Gamma)\backslash C_0(\gamma; G^1_R))$ *for* $\gamma\in F$. *The zeta-function* $\zeta(s; F)$ *attached to the family* F *is given by*

$$\zeta(s; F)=\sum_{\gamma\in F/\widetilde{\Gamma}}\frac{a_n(k)\cdot I_0(\gamma, s)}{[C(\gamma; \Gamma)\colon \pm C_0(\gamma; \Gamma)]}, \tag{4.43}$$

with $I_0(\gamma, s)$ *as in* (4.41).

Remark 4.44. From the finiteness of the number of cusps of Γ, it follows that the set of non-conjugate families in $\Gamma^{(p)}$ is finite, so that the

sum in (4.42) is a finite sum. Roughly speaking, $\zeta(s; F)$ is a zetafunction corresponding to F which is (a part of) a lattice, not necessarily homogeneous, in a vector space contained in the unipotent radical of a parabolic subgroup. The typical cases (i.e., purely unipotent elements) have been treated by Shintani [45]. In general, however, it is not easy to evaluate $\lim_{s\downarrow 0}\zeta(s; F)$.

4-5. *Formulae for* $\mathrm{tr}\,\rho(g)$, $t_k(g)$ *and their Relations.* Here we shall describe the explicit formulae for "∞-factors" $\mathrm{tr}\,\rho_k(g)$ and $t_k(g)$ of our dimension formulae (4.30), (4.40), for a semi-simple (elliptic) element g. In our group $G^1_R=Sp(n)$ or $Sp(n, R)$, we take the standard compact Cartan subgroup

$$H=\left\{g(\theta)=\begin{bmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n}\end{bmatrix}\in Sp(n);\ \theta_1, \cdots, \theta_n\in R\right\}$$

(4.45)

$$H=\left\{g(\theta)=\left[\begin{array}{ccc|ccc} \cos\theta_1 & & & \sin\theta_1 & & \\ & \ddots & & & \ddots & \\ & & \cos\theta_n & & & \sin\theta_n \\ \hline -\sin\theta_1 & & & \cos\theta_1 & & \\ & \ddots & & & \ddots & \\ & & -\sin\theta_n & & & \cos\theta_n \end{array}\right]\in Sp(n, R);\ \theta_j\in R\right\}.$$

Here in $Sp(n)$, we identify C with the subalgebra of $H=R+Ri+Rj+Rij$ by $\sqrt{-1}\mapsto i$. Note that any (resp. elliptic) element of $Sp(n)$ (resp. $Sp(n, R)$) is conjugate to an element of H. We first assume that $g=g(\theta)$ is regular i.e., $C(g; G^1_R)=H$; equivalently, $\theta_i+\theta_j\notin 2\pi Z$ for any i, j. Then we have

Theorem (Wely [50]). *The irreducible character of* $Sp(n)$ *which corresponds to the Young diagram*

$$\left.\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & k\\ \hline & & \cdots & \\ \hline 1 & 2 & \cdots & k\\ \hline \end{array}\right\} n$$

takes the following value at the regular element $g=g(\theta)$:

$$\mathrm{tr}\,\rho_k(g(\theta))=\frac{\det\,[\sin\,(k+n+1-j)\theta_i]}{\det\,[\sin\,(n+1-j)\theta_i]} \tag{4.46}$$

Its degree is given by

$$d_n(k)=\prod_{i\leq j}\frac{(2k+2n+2-i-j)}{(2n+2-i-j)}. \tag{4.47}$$

We note the relation:

$$\begin{aligned} &d_n(k)=c_n\cdot a_n(k+n+1),\\ &c_n=\frac{1}{2^{n(n+2)}\pi^{n(n+1)/2}}\prod_{i\leq j}(2n+2-i-j). \end{aligned} \tag{4.48}$$

Theorem (Langlands [33], see also Harish-Chandra [14]). *Assume that $k>2n$, and $g=g(\theta)\in Sp(n, \boldsymbol{R})$ has an isolated fixed point on H_n, which is the case for a regular element. Then the integral $t_k(g)=a_n(k)\cdot I(g)$ in* (4.40) *is given by*

$$t_k(g(\theta))=\frac{\prod_{j=1}^{n}e^{-ik\theta_j}}{\prod_{j\leq l}(1-e^{-i(\theta_j+\theta_l)})}\qquad (i=\sqrt{-1}). \tag{4.49}$$

Here, in the integral (4.40), we are taking the measure of $C(g; G_{\boldsymbol{R}}^1)$ such that its volume is equal to 1. (Note that the condition on the isolated fixed point implies that $C(g; G_{\boldsymbol{R}}^1)$ is compact.)

Assuming that g is regular, we note that there are 2^n conjugacy classes in $Sp(n, \boldsymbol{R})$, each represented by $g(\pm\theta_1, \cdots, \pm\theta_n)$, which are conjugate to g in $Sp(n, \boldsymbol{C})$, the complexification of $Sp(n, \boldsymbol{R})$, while in $Sp(n)$, all $g(\pm\theta_1, \cdots, \pm\theta_n)$ are conjugate to g. By comparing the above two formulae, it is easy to observe the following

Theorem (Character Relation; regular case)*).

$$\operatorname{tr}\rho_k(g(\theta_1, \cdots, \theta_n))=(-1)^{n(n+1)/2}\sum_{\varepsilon_i=\pm1}t_{k+n+1}(g(\varepsilon_1\theta_1, \cdots, \varepsilon_n\theta_n)). \tag{4.50}$$

This kind of character relations seem to be more or less well-known to the experts in more general context, as long as regular elements are concerned. It seems less known, however, that a similar relation remains to hold also for singular elliptic elements, under a suitable formulation, e.g., normalization of Haar measures. In fact, the relation (4.48) may be viewed as giving such a relation in the extremely singular case $g=\pm1$.

Here we note that the relation (4.50), which has been noticed (as well as (4.49)) in the case $n=2$ by Y. Ihara around 1962, was one of the motivations to his conjectural question in [28], which is our main problem in this paper.

*) If g is regular, $t_k(g)$ is in fact a character of a representation belonging to a discrete series (cf. [14]).

For singular elements g, we need much more involved notations to state the formulae for $\operatorname{tr}\rho_k(g)$ and $t_k(g)$; therefore we shall only give them in the case $n=2$ below. As for the character relation, we content ourselves with the following description.

Theorem (Character Relation; general case). *Under a suitable normalization of the Haar measures, the following relation holds for arbitrary elliptic element $g(\theta)$:*

$$\operatorname{tr}\rho_k(g(\theta))=(-1)^{n(n+1)/2}\sum_{\varepsilon}(-1)^{b(\varepsilon\theta)}t_{k+n+1}(g(\varepsilon\theta)), \tag{4.51}$$

where $\varepsilon=(\varepsilon_i)$ runs over all possible values in $(\pm1)^n$ so that

$$g(\varepsilon\theta)=g(\varepsilon_1\theta_1, \cdots, \varepsilon_n\theta_n)$$

are all non-conjugate, and $b(\varepsilon\theta)$ denotes the complex dimension of the fixed points set of $g(\varepsilon\theta)$ in $\mathfrak{H}_n$.

Remark 4.52. It is easy to see that $b(\theta)$ is given by

$$b(\theta)=\sharp\{(i,j);\ 1\leq i\leq j\leq n,\ \theta_i+\theta_j\in 2\pi\boldsymbol{Z}\}.$$

Moreover, from Langlands' formula for $t_k(g)$ in [33], it is observed that $t_k(g(\theta))$ is a polynomial of k of degree $b(\theta)$, modulo some factors $e^{2\pi ik/m}$ $(m\in\boldsymbol{Z})$. This observation is used to get asymptotic formulae for $\dim S_k(\Gamma)$ as a function of k (c.f. [17]).

Now assume $n=2$. The Weyl's formula for $\operatorname{tr}\rho_k(g(\theta))$ for a singular element is derived from the formula (4.46) by taking limits; we have

$$\begin{aligned} &\operatorname{tr}\rho_k(g(0, \theta))=\frac{(k+2)\sin(k+1)\theta-(k+1)\sin(k+2)\theta}{2\sin\theta(1-\cos\theta)},\\ &\operatorname{tr}\rho_k(g(\theta, \theta))=\frac{[(k+1)\cos(k+1)\theta\sin(k+2)\theta-(k+2)\cos(k+2)\theta\sin(k+1)\theta]}{2\sin^3\theta},\\ &\operatorname{tr}\rho_k(g(0, \pi))=\frac{(-1)^k}{2}(k+1)(k+2). \end{aligned} \tag{4.53}$$

The basic idea is similar also in the split case; here, however, the limits should be taken as distributions (i.e., the limit formula of Harish-Chandra, see [33], [14]). The results are as follows:

$$t_k(g(0,\theta))=\frac{i[(k-1)e^{-i(k-2)\theta}-(k-2)e^{-i(k-1)\theta}]}{2^5\pi^2\sin\theta(1-\cos\theta)}.$$

(4.54) $$t_k(g(\theta,-\theta))=\frac{(2k-3)}{2^5\pi^2\sin^2\theta},$$

$$t_k(g(0,\pi))=\frac{(-1)^k(2k-2)(2k-4)}{2^9\pi^4}.$$

Remark 4.55. In [16], the first author has computed the integral $t_k(g)$ by a completely elementary method and obtained the above results. The constants in the denominators are due to the usual normalization of the Haar measure of $C(g;G_R^1)$, which will be cancelled by multiplying vol $(C(g;\Gamma)\backslash C(g;G_R^1))$ $(g\in\Gamma)$. Also, we note that the above result for $t_k(g(0,\pi))$ does not agree with Langlands' formula ([33], (2), p. 101); this is because the factor $e^{\rho_\Gamma(H)}$ is missing in the denominator of (2), [33].

4-6. *P-unipotent (parabolic) contributions* ($n=2$). We assume $n=2$, and describe briefly the zetafunction $\zeta(s;F)$ attached to each family F of p-unipotent elements of Γ. There are seven cases to be distinguished according as the types of their zetafunction.

(i) *elliptic/parabolic.* After normalizing by G_R-conjugation simultaneously, we may assume that

(4.56) $$\gamma=\hat\beta(\theta,t)=\begin{pmatrix}\cos\theta&0&\sin\theta&0\\0&1&0&t\\-\sin\theta&0&\cos\theta&0\\0&0&0&1\end{pmatrix}\quad(\sin\theta\neq0,\ t\neq0),$$

and the family $F=F(\gamma)$ is given by

$$F(\gamma)=\{\hat\beta(\theta,a+n);\ n\in Z,\ a+n\neq0\}\quad(0\leq a<1).$$

We have $C_0(\gamma;G_R^1)=\{\hat\beta(0,u);\ u\in R\}$.

Theorem P-1 ([16], Theorem I–5). *Under these notations, we have*

(4.57) $$\zeta(s;F)=\frac{-e^{-i(k-3/2)\theta}}{2^3\pi\sin\theta\sin\theta/2}\times[e^{-i\pi(s+1)/2}\zeta(s+1,a)+e^{i\pi(s+1)/2}\zeta(s+1,1-a)],$$
$$\lim_{s\downarrow0}\zeta(s;F)=\frac{1}{2^2\sin\theta\sin\theta/2}\left[\cos\left(k-\frac{3}{2}\right)\theta+\cot^*(\pi a)\sin\left(k-\frac{3}{2}\right)\theta\right].$$

Here, $\zeta(s,a)=\sum_{n=0}^{\infty}(n+a)^{-s}$ is the Hurwitz zetafunction, and

$$\cot^*(x)=\begin{cases}\cot(x)&\text{if } x\notin Z\pi,\\0&\text{if } x\in Z\pi.\end{cases}$$

(ii) *paraelliptic.* The normalized form of an element of this type is

(4.58) $$\gamma=\hat\gamma(\theta,t)=\begin{pmatrix}\cos\theta&\sin\theta&t\cos\theta&t\sin\theta\\-\sin\theta&\cos\theta&-t\sin\theta&t\cos\theta\\0&0&\cos\theta&\sin\theta\\0&0&-\sin\theta&\cos\theta\end{pmatrix}\quad(t,\ \sin\theta\neq0),$$

and the family is given by

$$F(\gamma)=\{\hat\gamma(\theta,a+n);\ n\in Z,\ a+n\neq0\}\quad(0\leq a<1).$$

we have $C_0(\gamma;G_R^1)=\{\hat\gamma(0,u);\ u\in R\}$.

Theorem P-2 (loc. cit. Theorem I–6). *Under these notations, we have*

(4.59) $$\zeta(s;F)=\frac{1}{2^3\pi\sin^2\theta}[e^{-i\pi(s+1/2)}\zeta(2s+1,a)+e^{i\pi(s+1/2)}\zeta(2s+1,1-a)]$$
$$\lim_{s\downarrow0}\zeta(s;F)=-\frac{1}{2^3\sin^2\theta}(1+i\cot^*(\pi a)).$$

(iii) *δ-parabolic (nondegenerate case)*

(4.60) $$\gamma=\hat\delta(s_1,s_2)=\begin{pmatrix}1&0&s_1&0\\0&-1&0&s_2\\0&0&1&0\\0&0&0&-1\end{pmatrix}\quad(s_1,s_2\neq0),$$
$$F(\gamma)=\{\hat\delta\ (m+c,\ am+bn+ac);\ m,n\in Z,\ m+c,\ am+bn+ac\neq0\}$$
$$(a,b\in Z,\ (a,b)=1,\ b>0,\ 0\leq c<1).$$

We have

$$C_0(\gamma;G_R^1)=\left\{\begin{pmatrix}1&0&t_1&0\\0&1&0&t_2\\0&0&1&0\\0&0&0&1\end{pmatrix};\ t_1,t_2\in R\right\}.$$

Theroem P-3 (loc. cit. Theorem I–7). *Under these notations, we have*

$$\zeta(s;F)=\frac{(-1)^k}{2^3\pi^2b^2}\sum_{j=0}^{b-1}\left[e^{-i\pi(s+1)/2}\zeta\left(s+1,\frac{j+c}{b}\right)+e^{i\pi(s+1)/2}\zeta\left(s+1,\frac{b-j-c}{b}\right)\right]$$

$$(4.61)\qquad \times\left[e^{i\pi(s+1)/2}\zeta\left(s+1,\frac{a(j+c)}{b}\right)+e^{-i\pi(s+1)/2}\zeta\left(s+1,\frac{b-a(j+c)}{b}\right)\right]$$

$$\lim_{s\downarrow 0}\zeta(s;F)=\frac{(-1)^k}{2^3b^2}\sum_{j=0}^{b-1}\left[1+i\cot^*\left(\frac{(j+c)\pi}{b}\right)\right]\left[1-i\cot^*\left(\frac{a(j+c)\pi}{b}\right)\right].$$

(iv) *δ-parabolic (degenerate case)*

(4.62) $\gamma=\hat{\delta}(t,0)$, $(t\neq 0, \hat{\delta}$; as in (4.60)), $F(\gamma)=\{\hat{\delta}(n,0); n\in Z\}$.

We have

$$C_0(\gamma;G_R^1)=\left\{\begin{pmatrix}1&0&u&0\\0&a&0&b\\0&0&1&0\\0&c&0&d\end{pmatrix}; u,a,b,c,d\in R,\ ad-bc=1\right\}.$$

Theorem P-4 (loc. cit. Theorem I–8). *Under the above notation, we have*

$$(4.63)\qquad \begin{aligned}&\zeta(s;F)=\frac{(-1)^k(2k-3)}{2^5\pi^3}\zeta(s+1)\cos\left(\frac{s+1}{2}\right)\pi,\\ &\lim_{s\downarrow 0}\zeta(s;F)=-\frac{(-1)^k(2k-3)}{2^6\pi^2}.\end{aligned}$$

(v) To describe the purely unipotent contributions, we need some preparations. We note first that, if $I_0(\gamma,s)\neq 0$ for a unioptent element of Γ, then γ is conjugate in G_Q to an element of the following form

$$(4.64)\qquad \gamma=\gamma(S)=\begin{pmatrix}1&S\\0&1\end{pmatrix}\quad S={}^tS,$$

with either (i) $\det S=0$, (ii) $-\det S\in(Q^\times)^2$, or $S\gtrless 0$ i.e., $S=$definite. This, in particular, means that such γ belongs to the unipotent radical P_U of a parabolic Q-subgroup P of G_Q which corresponds to a point cusp that γ fixes. If $\det S\neq 0$, we can associate in this way a lattice $L=P_U\cap\Gamma$, which we also regard as a lattice of $SM_2(R)$, the 2×2 symmetric real matrices via a fixed isomorphism $P_U(R)\cong SM_2(R)$. We have an action of a Levi subgroup P_M of P on P_U, which may be assumed as

$$T\longmapsto AT{}^tA\quad (T\in SM_2(R),\ A\in GL_2(R))$$

under an isomorphism $P_M(R)\cong GL_2(R)$. Moreover, for simplicity, we assume that

$$(4.65)\qquad P\cap\Gamma=(P_M\cap\Gamma)\cdot(P_U\cap\Gamma).$$

We denote by $(P_M\cap\Gamma)_0$ the image of $P_M\cap\Gamma$ in $GL_2(R)$, and put

$$(P_M\cap\Gamma)_0^+:=(P_M\cap\Gamma)\cap SL_2(R).$$

Also put, for $\gamma=g\gamma(S)g^{-1}$ as above,

$$(4.66)\qquad O_\Gamma(S)=\{A\in(P_M\cap\Gamma)_0;\ AS{}^tA=S\}.$$

If S is as in (ii), (4.64), the family F represented by γ is given by

$$F(\gamma)=g\,\{\gamma(S');\ S'\in L,\ -\det(S')\in(Q^\times)^2 \text{ or } S\gtrless 0\}\,g^{-1}.$$

We divide F into two parts $F^\pm$ and F^s according as S' satisfies $S'\gtrless 0$, or $-\det(S')\in(Q^\times)^2$.

Theorem P-5 (loc. cit. Theorem I–9). *Notations being as above, we have*

$$(4.67)\qquad \begin{aligned}&\zeta(s;F^\pm)=\frac{s}{2\pi}\sum_{S\in L^+ \bmod (P_M\cap\Gamma)_0}\frac{1}{\sharp O_\Gamma(S)(\det S)^{s+3/2}}\\ &\qquad\qquad (L^+=\{S\in L;\ S>0\})\\ &\lim_{s\downarrow 0}\zeta(s;F^\pm)=\frac{1}{2^2\pi}\,\frac{\mathrm{vol}\,((P_M\cap\Gamma)_0^+\backslash\mathfrak{H}_1)}{[(P_M\cap\Gamma)_0:(P_M\cap\Gamma)_0^+]\,\mathrm{vol}\,(L\backslash SM_2(R))}.\end{aligned}$$

Theorem P-6 (loc. cit. Theorem I–10).

$$(4.68)\qquad \begin{aligned}&\zeta(s;F^s)=-\frac{1}{2^4\pi^2}\sum_{j=1}^{t}\sum_{S\in L_j^s \bmod B_j}\frac{1}{|\det S|^{s+3/2}},\\ &\lim_{s\downarrow 0}\zeta(s;F^s)=-\frac{1}{2^5 3}\sum_{j=1}^{t}\frac{c_j}{d_j^3},\end{aligned}$$

Here notations are as follows: let $\beta_1,\cdots,\beta_t$ *be the set of nonequivalent cusp of* $(P_M\cap\Gamma)_0^+$ *in* $\mathfrak{H}_1$. *Take* $V\in SL_2(Q)$ *such that* $V\langle\beta_j\rangle=\infty$, *and put* $L_j^s:=V^{-1}\begin{pmatrix}*&*\\ *&0\end{pmatrix}{}^tV^{-1}\cap L$. B_j *is the parabolic subgroup of* $(P_M\cap\Gamma)_0^+$ *which stabilizes* β_j. *The module* $VL_j^s\,{}^tV$ *has a unique basis of the form*

$$\begin{pmatrix}t_1^{(j)}&0\\0&0\end{pmatrix},\ \begin{pmatrix}t_2^{(j)}&d_j\\d_j&0\end{pmatrix};\quad d_j>0,\ t_1^{(j)}>|t_2^{(j)}|\geq 0,$$

and c_j *is defined by*

$$B_j=V^{-1}\left\{\pm\begin{pmatrix}1&(t_1^{(j)}c_j/2d_j)Z\\0&1\end{pmatrix}\right\}V.$$

Finally in the case (i) $\det S=0$, the family F represented by $\gamma=$

$g\gamma(S)g^{-1}$ may be assumed to be given by

$$F(\gamma)=g\left\{\begin{pmatrix}1&0&dn&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix};\ n\in \boldsymbol{Z}-\{0\}\right\}g^{-1}\subseteq g\begin{pmatrix}1&L\\0&1\end{pmatrix}g^{-1}.\quad (d\in \boldsymbol{Q}_+^\times)$$

and we have $C_0(\gamma;G_{\boldsymbol{R}}^1)=C(\gamma;G_{\boldsymbol{R}}^1)$.

Theorem P-7 (loc. cit. Theorem I-11). *Notations being as above, we have*

$$\zeta(s;F)=-\frac{(2k-3)}{2^5\pi^4}\sum_{n=-\infty}^{\infty}\frac{1}{|dn|^{s+2}},$$

$$\lim_{s\downarrow 0}\zeta(s;F)=-\frac{2k-3}{2^3 3\pi^2 d^2}.$$

§ 5. Conjugacy classes of $U_2(p)$ $(=\Gamma_0'(p))$ and $B(p)$ (Proof of Theorem 3-2, 3-3).

5-1. In this section, we shall use the usual notation:

$$U_1(p)=\Gamma_0(p),\quad U_2(p)=\Gamma_0'(p),\quad \text{and}\quad U_{12}(p)=Sp(2,\boldsymbol{Z}).$$

We shall describe the conjugacy classes in $\Gamma_0'(p)$ and $B(p)$ of those elements (or families) which make nontrivial contributions to $\dim S_k(\Gamma_0'(p))$, $\dim S_k(B(p))$, in such a form that is sufficient to work out the explicit formulae for them, as presented in Section 3, if we put all data given here to our general formulae (4.30), (4.40), and (4.42). Since $\Gamma_0'(p)$ (resp. $B(p)$) is a subgroup of $Sp(2,\boldsymbol{Z})$ (resp. $\Gamma_0(p)$), and the list of conjugacy classes of the latter group has been given in [16], Sections 6, 7, we need not begin at the beginning. So, we mainly apply the global method (i.e., argument on Γ-conjugacy classes) also for semi-simple elements. Of course, in that case we can replace it by the local method described in Theorem $B^{(e)}$, as executed in [24] for $U_{02}(p)$ $(=K(p))$, and in [16], (II) for other arithmetic subgroups in $\boldsymbol{Q}$-rank one case.

In general, for two lattices Γ_1,Γ_2 of $G_{\boldsymbol{R}}^1$ such that $\Gamma_1\supseteq\Gamma_2$, $[\Gamma_1:\Gamma_2]<\infty$, we have a bijection in the same way as (4.17)

$$\{\gamma\}_{\Gamma_1}\cap\Gamma_2/\underset{\Gamma_2}{\sim}\xrightarrow{\ \sim\ }C(\gamma;\Gamma_1)\backslash M(\gamma,\Gamma_1,\Gamma_2)/\Gamma_2 \tag{5.1}$$

for any $\gamma\in\Gamma_1$, where we put $M(\gamma,\Gamma_1,\Gamma_2)=\{x\in\Gamma_1;\ x^{-1}\gamma x\in\Gamma_2\}$. Let $\gamma_1,\cdots,\gamma_d$ $(d=d(\gamma)=\#[C(\gamma;\Gamma_1)\backslash M(\gamma,\Gamma_1,\Gamma_2)/\Gamma_2])$ be a complete set of representatives of Γ_2-conjugacy classes in $\{\gamma\}_{\Gamma_1}\cap\Gamma_2$. We define "*relative Maß*" of γ with respect to Γ_1/Γ_2, by

$$m(\gamma;\Gamma_1/\Gamma_2)=\sum_{i=1}^{d}[C(\gamma_i;\Gamma_1):C(\gamma_i;\Gamma_2)]. \tag{5.2}$$

Then the elliptic contributions to $\dim S_k(\Gamma_i)$ $(i=1,2)$ are related as follows: namely for $\gamma\in\Gamma_1^{(e)}$

$$T_k(\{\gamma\}_{\Gamma_1}\cap\Gamma_2^{(e)})=m(\gamma;\Gamma_1/\Gamma_2)T_k(\{\gamma\}_{\Gamma_1}) \tag{5.3}^{*)}$$

Thus to compute the elliptic contributions for $\Gamma_2=\Gamma_0'(p)$, $B(p)$, it suffices to calculate the relative Maß's for each conjugacy class $\{\gamma\}_{\Gamma_1}$ of $\Gamma_1=Sp(2,\boldsymbol{Z})$, $\Gamma_0(p)$. On the other hand, the p-unipotent (parabolic) contributions require more careful treatment.

Lemma 5.4. *As a complete set of representatives of the coset space $Sp(2,\boldsymbol{Z})/\Gamma_0'(p)$ (resp. $\Gamma_0(p)/B(p)$), we can take the following*

$$[Sp(2,\boldsymbol{Z}):\Gamma_0'(p)]=(p+1)(p^2+1)\quad (\textit{resp.}\ [\Gamma_0(p):B(p)]=p+1)$$

elements:

(i) $Sp(2,\boldsymbol{Z})/\Gamma_0'(p)$:

$$X_1(a,b,c):=\begin{pmatrix}1&0&0&0\\a&1&0&0\\b&c&1&-a\\c&0&0&1\end{pmatrix},\qquad X_2(a,b):=\begin{pmatrix}0&1&0&0\\1&0&0&0\\a&0&0&1\\b&a&1&0\end{pmatrix},$$

$$X_3(a):=\begin{pmatrix}0&-a&-1&0\\0&1&0&0\\1&0&0&0\\a&0&0&1\end{pmatrix},\qquad X_4:=\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

(ii) $\Gamma_0(p)/B(p)$:

$$Z_1(t):=\begin{pmatrix}1&0&0&0\\t&1&0&0\\0&0&1&-t\\0&0&0&1\end{pmatrix},\qquad Z_2:=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

Here a, b, c, and t run over the integers modulo p.

By using this lemma and the list of conjugacy classes of $Sp(2,\boldsymbol{Z})$, $\Gamma_0(p)$ given in [16], we can find a complete set of representatives $x_1,\cdots,x_d$ of the double cosets of (5.1), where x_i are taken from the above set of representatives. This can be done in a completely elementary way, and we omit the details of the calculations. In the following, we describe only

*) If Γ_1, Γ_2 are defined by U_{1A}, U_{2A} respectively as in (4.8), we have the following relation:

$$H(\gamma;U_{2A})=m(\gamma;\Gamma_1/\Gamma_2)\cdot H(\gamma;U_{1A}).$$

the list of these x_i's with the invariants attached to each conjugacy classes such as $m(\gamma; \Gamma_1/\Gamma_2)$, which are necessary to obtain explicit formulae for $\dim S_k(\Gamma_0'(p))$, and $\dim S_k(B(p))$.

5-2. *Conjugacy classes of* $\Gamma_0'(p)$. (p=prime, $\neq 2, 3$). We use the notations of [16], Theorem 6–1. However, for the convenience of readers, we reproduce here the matrix representatives of each conjugacy classes of $Sp(2, \boldsymbol{Z})$, and those of $Sp(2, \boldsymbol{R})$ taken in the standard Cartan subgroup H as in (4.45) for elliptic elements. The symbol $\pm\gamma$ means that $-\gamma$ should be added, though we write $+\gamma$ alone.

(5.5) $\gamma=\pm\alpha_0,\quad \alpha_0=1_4\sim g(0, 0),\quad d(\gamma)=1,\quad x=X_1(0, 0, 0),$

$m(\gamma; Sp(2, Z)/\Gamma_0'(p))=[Sp(2, Z): \Gamma_0'(p)]=(p+1)(p^2+1).$

(5.6) $$\gamma=\pm\alpha_1=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\sim g(\pi/2, \pi/2),$$

$$d(\gamma)=(p+1)\left(1+\left(\frac{-1}{p}\right)\right),$$

$x=X_1(a, b, ab),\quad X_2(0, b, 0)\quad$ with $b^2+1\equiv 0 \pmod p$,

$$m(\gamma; Sp(2, Z)/\Gamma_0'(p))=(p+1)\left(1+\left(\frac{-1}{p}\right)\right).$$

(5.7) $$\gamma=\pm\alpha_2=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}\sim g(2\pi/3, 2\pi/3);\ \text{and}\ \alpha_3=\alpha_2^{-1}.$$

$$d(\gamma)=(p+1)\left(1+\left(\frac{-3}{p}\right)\right),$$

$x=X_1(a, b, ab),\quad X_2(0, b, 0)\quad$ with $b^2+b+1\equiv 0 \pmod p$,

$$m(\gamma; Sp(2, Z)/\Gamma_0'(p))=(p+1)\left(1+\left(\frac{-3}{p}\right)\right).$$

(5.8) $$\gamma=\alpha_4=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\sim g(-\pi/4, 3\pi/4);\ \text{and}\ \alpha_5=\alpha_4^3.$$

$$d(\gamma)=\begin{cases} 4 & \text{if } p\equiv 1 \pmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

$x=X_1(a, a^2, -a^{-1})\quad$ with $a^4+1\equiv 0 \pmod p$,

$m(\gamma; Sp(2, Z)/\Gamma_0'(p))=d(\gamma).$

Here, and throughout the following, we *are* confusing the integers mod p with elements of the finite field $\boldsymbol{F}_p$, writing a^{-1} the integer $x \pmod p$ such that $ax\equiv 1 \pmod p$.

(5.9) $$\gamma=\pm\alpha_6=\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\sim g(\pi/4, 3\pi/4),$$

$$d(\gamma)=\begin{cases} 4 & \text{if } p\equiv 1 \pmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

$$x=X_1\left(\frac{1-b}{b^2-3b+3}, b, \frac{-(1-b)^2}{b^2-3b+3}\right)\ \text{with}\ (b-1)^4+1\equiv 0 \pmod p,$$

$m(\gamma: Sp(2, Z)/\Gamma_0'(p))=d(\gamma).$

(5.10) $$\gamma=\pm\alpha_7=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\sim g(-\pi/3, -2\pi/3),$$

$$d(\gamma)=2\left(1+\left(\frac{-3}{p}\right)\right),$$

$x=X_1(a, a, -1),\quad X_1(a, -a, 1)\quad$ with $a^2-a+1\equiv 0 \pmod p$,

$$m(\gamma; Sp(2, Z)/\Gamma_0'(p))=2\left(1+\left(\frac{-3}{p}\right)\right).$$

(5.11) $$\gamma=\pm\alpha_8=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\sim g(-2\pi/3, -\pi/3),$$

$$d(\gamma)=2\left(1+\left(\frac{-3}{p}\right)\right),$$

$x=X_1(0, b, 0),\quad X_2(0, b)\quad$ with $b^2+b+1\equiv 0 \pmod p$,

$$m(\gamma; Sp(2, Z)/\Gamma_0'(p))=2\left(1+\left(\frac{-3}{p}\right)\right).$$

(5.12) $$\gamma=\alpha_9=\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\sim g(2\pi/3, -\pi/3);\ \text{and}\ \alpha_{10}=\alpha_9^{-1}.$$

$$d(\gamma)=2\left(1+\left(\frac{-3}{p}\right)\right),$$

$$x=\begin{cases}X_1(a,a^2,1) & \cdots\text{with } a^2+a+1\equiv 0\\ X_1(a,a^2,-1) & \cdots\text{with } a^2-a+1\equiv 0\end{cases}\pmod p,$$

$$m(\gamma;\, Sp(2,\boldsymbol{Z})/\Gamma_0'(p))=2\left(1+\left(\frac{-3}{p}\right)\right).$$

(5.13) $$\gamma=\alpha_{11}=\begin{pmatrix}-1&0&-1&0\\0&1&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\sim g(-2\pi/3,\,\pi/3);\ \text{and}\ \alpha_{12}=\alpha_{11}^{-1},$$

$$d(\gamma)=2\left(1+\left(\frac{-3}{p}\right)\right),$$

$x=X_1(0,b,0)$, $X_2(0,b)$ with $b^2+b+1\equiv 0 \pmod p$,

$$m(\gamma;\, Sp(2,\boldsymbol{Z})/\Gamma_0'(p))=2\left(1+\left(\frac{-3}{p}\right)\right).$$

(5.14) $$\gamma=\alpha_{13}=\begin{pmatrix}0&0&0&-1\\1&0&0&0\\0&1&0&1\\0&0&1&0\end{pmatrix}\sim g(-\pi/6,\,5\pi/6);\ \text{and}\ \alpha_{14}=\alpha_{13}^{-1},$$

$$d(\gamma)=\begin{cases}4 & \text{if } p\equiv 1 \pmod{12},\\ 0 & \text{otherwise},\end{cases}$$

$x=X_1(a,-a^{-2},-a^{-1})$ with $a^4-a^2+1\equiv 0 \pmod p$,

$m(\gamma;\, \mathrm{Sp}(2,\boldsymbol{Z})/\Gamma_0'(p))=d(\gamma)$.

(5.15) $$\gamma=\pm\alpha_{15}=\begin{pmatrix}0&1&1&1\\0&0&1&0\\0&0&0&1\\-1&0&0&-1\end{pmatrix}\sim g(2\pi/5,\,-4\pi/5);$$

and $\alpha_{16}=\alpha_{15}^2$, $\alpha_{17}=\alpha_{15}^3$, $\alpha_{18}=\alpha_{15}^4$,

$$d(\gamma)=\begin{cases}4 & \text{if } p\equiv 1 \pmod 5,\\ 1 & p=5\\ 0 & \text{otherwise},\end{cases}$$

If $p=5$, $x\equiv X_1(2,2,2)$,

if $p\equiv 1 \pmod 5$, $x=X_1(a,b,c)$ with $a^2+a-1\equiv 0$,

$$b\equiv 1+(1+a)c,\quad c^2+c+\frac{1}{a+2}\equiv 0 \pmod p,$$

$m(\gamma;\, Sp(2,\boldsymbol{Z})/\Gamma_0'(p))=d(\gamma)$.

(5.16) $$\gamma=\pm\alpha_{19}=\begin{pmatrix}-1&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}\sim g(-2\pi/3,\,-\pi/2);$$

and $\alpha_{20}=\alpha_{19}^{-1}$, $\alpha_{21}=\alpha_{19}^7$, $\alpha_{22}=\alpha_{19}^5$,

$$d(\gamma)=2+\left(\frac{-1}{p}\right)+\left(\frac{-3}{p}\right),$$

$$x=\begin{cases}X_1(0,b,0) & \text{with } b^2+b+1\equiv 0, \text{ and}\\ X_2(0,b) & \text{with } b^2+1\equiv 0 \pmod p,\end{cases}$$

$$m(\gamma;\, Sp(2,\boldsymbol{Z})/\Gamma_0'(p))=2+\left(\frac{-1}{p}\right)+\left(\frac{-3}{p}\right).$$

(5.17) $$\gamma=\pm\beta_1=\begin{pmatrix}0&0&1&0\\0&1&0&0\\-1&0&-1&0\\0&0&0&1\end{pmatrix}\sim g(-2\pi/3,\,0);\ \text{and}\ \beta_2=\beta_1^{-1},$$

$$\gamma=\pm\beta_3=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&1&0\\0&0&0&1\end{pmatrix}\sim g(-\pi/3,\,0);\ \text{and}\ \beta_4=\beta_3^{-1},$$

$$d(\gamma)=2+\left(\frac{-3}{p}\right)=m(\gamma;\, Sp(2,\boldsymbol{Z})/\Gamma_0'(p)),$$

$x=X_2(0,0)$ and $X_1(0,b,0)$ with $b^2+b+1\equiv 0 \pmod p$,

(5.18) $$\gamma=\pm\beta_5=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\sim g(-\pi/2,\,0);\ \text{and}\ \beta_6=\beta_5^{-1},$$

$$d(\gamma)=2+\left(\frac{-1}{p}\right)=m(\gamma;\, Sp(2,\boldsymbol{Z})/\Gamma_0'(p)),$$

$x=X_2(0,0)$ and $X_1(0,b,0)$ with $b^2+1\equiv 0 \pmod p$

(5.19) $$\gamma=\gamma_1=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\sim g(\pi/2,\,-\pi/2),$$

$$\gamma=\gamma_2=\begin{pmatrix}0&-1&1&0\\1&0&0&-1\\0&0&0&-1\\0&0&1&0\end{pmatrix}\sim g(\pi/2,\,-\pi/2),$$

$$d(\gamma)=1+\left(\frac{-1}{p}\right),$$

$x=X_1(a,0,0)$ with $a^2+1\equiv 0 \pmod p$,

$m(\gamma; Sp(2,Z)/\Gamma_0'(p))=(p+1)\left(1+\left(\frac{-1}{p}\right)\right)$,

(5.20) $$\gamma=\pm\gamma_3=\begin{pmatrix}0&-1&0&0\\1&-1&0&0\\0&0&-1&-1\\0&0&1&0\end{pmatrix}\sim g(2\pi/3,-2\pi/3),$$

$d(\gamma)=1+\left(\frac{-3}{p}\right)$,

$x=X_1(a,0,0)$ with $a^2-a+1\equiv 0 \pmod p$

$m(\gamma; Sp(2,Z)/\Gamma_0'(p))=(p+1)\left(1+\left(\frac{-3}{p}\right)\right)$.

(5.21) $$\gamma=\delta_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} \text{ and } \delta_2=\begin{pmatrix}1&0&0&-1\\0&-1&1&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\sim g(0,\pi),$$

$d(\gamma)=2$,

$x=X_1(0,0,0)$, $X_2(0,0)$,

$m(\gamma; Sp(2,Z)/\Gamma_0'(p))=2(p+1)$.

P-unipotent classes. We first note that $\Gamma_0'(p)$ has two point cusps and three one dimensional cusps, corresponding to the following parabolic subgroups:

Point cusps:

$$P_0^{(j)}=x_j^{-1}\begin{pmatrix}*&*\\0&*\end{pmatrix}x_j,\quad x_1=1_4,\quad x_2=X_1(0,1,0).$$

One dimensional cusps:

$$P_1^{(j)}=x_j^{-1}\begin{pmatrix}*&0&*&*\\ *&*&*&*\\ *&0&*&*\\0&0&0&*\end{pmatrix}x_j,$$

$$x_1=1_4,\quad x_2=X_1(0,0,-1),\quad x_3=X_2(0,0).$$

Each family of p-unipotent elements belongs to (at least) one of these parabolic subgroups, up to $\Gamma_0'(p)$-conjugation. In the following, we give a typical element of each family of $Sp(2,Z)$ listed in [16], Theorem 6–1, and describe the decomposition of it into $\Gamma_0'(p)$-conjugacy classes. We put, for each class $x^{-1}\gamma x$ of $\Gamma_0'(p)$,

$$i_0(x):=[C_0(x^{-1}\gamma x; Sp(2,Z)): C_0(x^{-1}\gamma x; \Gamma_0'(p))].$$

(5.22) $$\gamma=\pm\hat\beta_1(n)=\begin{pmatrix}0&0&1&0\\0&1&0&n\\-1&0&1&0\\0&0&0&1\end{pmatrix}\sim\hat\beta(\pi/3,n);$$

and $\hat\beta_2(n)=\hat\beta_1^{-1}(-n)$, $n\in Z-\{0\}$,

$d(\gamma)=3+\left(\frac{-3}{p}\right)$,

$$x=\begin{cases}X_2(0,0),\ X_1(0,b,0) \text{ with } b^2+b+1\equiv 0 \pmod p, & \cdots\ n\text{: arbitrary},\\ X_4 & \cdots\ n\equiv 0 \pmod p,\end{cases}$$

$i_0(x)=1$ for each x.

(5.23) $$\gamma=\pm\hat\beta_3(n)=\begin{pmatrix}0&0&-1&0\\0&1&0&n\\1&0&-1&0\\0&0&0&1\end{pmatrix}\sim\hat\beta(4\pi/3,n);$$

and $\hat\beta_4(n)=\hat\beta_3^{-1}(-n)$, $n\in Z-\{0\}$,

$d(\gamma)=3+\left(\frac{-3}{p}\right)$,

$$x=\begin{cases}X_2(0,0),\ X_1(0,b,0) \text{ with } b^2+b+1\equiv 0 \pmod p, & \cdots\ n\text{: arbitrary},\\ X_4 & \cdots\ n\equiv 0 \pmod p,\end{cases}$$

$i_0(x)=1$ for each x.

(5.24) $$\gamma=\pm\hat\beta_5(n)=\begin{pmatrix}-1&0&1&1\\-1&1&0&n\\-1&0&0&0\\0&0&0&1\end{pmatrix}\sim\hat\beta(2\pi/3,n-1/3);$$

and $\hat\beta_6(n)=\hat\beta_5^{-1}(-n)$, $n\in Z$.

$d(\gamma)=3+\left(\frac{-3}{p}\right)$,

$$x=\begin{cases}X_2(0,0),\ X_1(a,b,0) \text{ with } b^2-b+1=0,\ a=(2-b)^{-1}, & \cdots\ n\text{: arbitrary},\\ X_1(0,-1,3) & \cdots\ 3n-1\equiv 0 \pmod p,\end{cases}$$

$i_0(x)=1$ for each x.

(5.25) $$\gamma=\pm\hat\beta_7(n)=\begin{pmatrix}0&0&-1&0\\0&1&0&n\\1&0&0&0\\0&0&0&1\end{pmatrix}\sim\hat\beta(-\pi/2,n);$$

and $\hat\beta_8(n)=\hat\beta_7^{-1}(-n)$, $n\in Z-\{0\}$,

$$d(\gamma)=3+\left(\frac{-1}{p}\right),$$

$$x=\begin{cases} X_2(0,0),\ X_1(0,b,0) & \text{with } b^2+1\equiv 0 \pmod p),\\ & \cdots\ n\text{: arbitrary},\\ X_4 & \cdots\ n\equiv 0 \pmod p), \end{cases}$$

$i_0(x)=1$ for each x.

(5.26) $$\gamma=\pm\hat{\beta}_9(n)=\begin{pmatrix} 0 & 0 & 1 & -1\\ 1 & 1 & 0 & n\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\sim\hat{\beta}(\pi/2, n-1/2);$$

and $\hat{\beta}_{10}(n)=\hat{\beta}_9^{-1}(-n)$, $n\in \boldsymbol{Z}$,

$$d(\gamma)=3+\left(\frac{-1}{p}\right),$$

$$x=\begin{cases} X_2(0,0),\ X_1\left(\dfrac{-b}{b+1}, b, 0\right) & \text{with } b^2+1\equiv 0 \pmod p),\\ & \cdots\ n\text{: arbitrary},\\ X_1(0,-1,-2) & \cdots\ 2n-1\equiv 0 \pmod p), \end{cases}$$

$i_0(x)=1$ for each x.

(5.27) $$\gamma=\hat{\gamma}_1(n)=\begin{pmatrix} 0 & -1 & 0 & -n\\ 1 & 0 & n & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\sim\hat{\gamma}(-\pi/2, n),\quad n\in \boldsymbol{Z}-\{0\},$$

$$d(\gamma)=2\left(1+\left(\frac{-1}{p}\right)\right),$$

$$x=\begin{cases} X_1(a,0,0) & \text{with } a^2+1\equiv 0,\ n\text{: arbitrary},\ i_0(x)=1,\\ X_1(a,-a,1) & \text{with } a^2+1\equiv 0,\ n\equiv 0 \pmod p),\ i_0(x)=p, \end{cases}$$

(5.28) $$\gamma=\hat{\gamma}_2(n)=\begin{pmatrix} 0 & -1 & 0 & -n\\ 1 & 0 & n+1 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\sim\hat{\gamma}(-\pi/2, n+1/2),\quad n\in \boldsymbol{Z},$$

$$d(\gamma)=2\left(1+\left(\frac{-1}{p}\right)\right),$$

$$x=\begin{cases} X_1(a,0,0) & \text{with } a^2+1\equiv 0,\ n\text{: arbitrary},\ i_0(x)=1,\\ X_1(0,-2,c) & \text{with } c^2+4\equiv 0,\ 2n+1\equiv 0 \pmod p),\ i_0(x)=p, \end{cases}$$

(5.29) $$\gamma=\hat{\gamma}_3(n)=\begin{pmatrix} 0 & -1 & 1 & -n\\ 1 & 0 & n & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\sim\hat{\gamma}(-\pi/2, n),\quad n\in \boldsymbol{Z}-\{0\},$$

$$d(\gamma)=2\left(1+\left(\frac{-1}{p}\right)\right),$$

$$x=\begin{cases} X_1(a,0,0) & \text{with } a^2+1\equiv 0,\ n\text{: arbitrary},\ i_0(x)=1,\\ X_1(0,b,1) & \text{with } b^2+1\equiv 0,\ n\equiv 0 \pmod p),\ i_0(x)=p. \end{cases}$$

(5.30) $$\gamma=\hat{\gamma}_4(n)=\begin{pmatrix} 0 & -1 & 1 & -n\\ 1 & 0 & n+1 & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\sim\hat{\gamma}(-\pi/2, n+1/2),\quad n\in \boldsymbol{Z},$$

$$d(\gamma)=2\left(1+\left(\frac{-1}{p}\right)\right),$$

$$x=\begin{cases} X_1(a,0,0) & \text{with } a^2+1\equiv 0,\ n\text{: arbitrary},\ i_0(x)=1,\\ X_1(1/2,b,1) & \text{with } b^2+1\equiv 0,\ 2n+1\equiv 0 \pmod p),\ i_0(x)=p. \end{cases}$$

(5.31) $$\gamma=\pm\hat{\gamma}_5(n)=\begin{pmatrix} 0 & -1 & -n & -2n\\ 1 & -1 & n & -n\\ 0 & 0 & -1 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\sim\hat{\gamma}(2\pi/3, n),\quad n\in \boldsymbol{Z}-\{0\},$$

$$d(\gamma)=2\left(1+\left(\frac{-3}{p}\right)\right),$$

$$x=\begin{cases} X_1(a,0,0) & \text{with } a^2-a+1\equiv 0,\ n\text{: arbitrary},\ i_0(x)=1,\\ X_1(a,-a,1) & \text{with } a^2-a+1\equiv 0,\ n\equiv 0 \pmod p),\ i_0(x)=p, \end{cases}$$

(5.32) $$\gamma=\pm\hat{\gamma}_6(n)=\begin{pmatrix} 0 & -1 & -n & -2n\\ 1 & -1 & n+1 & -n\\ 0 & 0 & -1 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\sim\hat{\gamma}(2\pi/3, n+1/3),\quad n\in \boldsymbol{Z},$$

$$d(\gamma)=2\left(1+\left(\frac{-3}{p}\right)\right),$$

$$x=\begin{cases} X_1(a,0,0) & \text{with } a^2-a+1\equiv 0,\ n\text{: arbitrary},\ i_0(x)=1,\\ X_1(2,3b,3) & \text{with } b^2+b+1\equiv 0,\ 3n+1\equiv 0 \pmod p),\\ & \qquad i_0(x)=1, \end{cases}$$

(5.33) $$\gamma=\pm\hat{\gamma}_7(n)=\begin{pmatrix} 0 & -1 & -n & -2n\\ 1 & -1 & n+2 & -n\\ 0 & 0 & -1 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\sim\hat{\gamma}(2\pi/3, n+2/3),\quad n\in \boldsymbol{Z},$$

$$d(\gamma)=2\left(1+\left(\frac{-3}{p}\right)\right),$$

$$x=\begin{cases} X_1(a,0,0) & \text{with } a^2-a+1\equiv 0,\ n\text{: arbitrary},\ i_0(x)=1,\\ X_1(2,3b/2,3/2) & \text{with } b^2+b+1\equiv 0,\ 3n+2\equiv 0 \pmod p),\\ & \qquad i_0(x)=p, \end{cases}$$

(5.34) $\gamma=\hat{\hat{\delta}}_1(m, n)=\begin{pmatrix}1 & 0 & m & 0\\ 0 & -1 & 0 & n\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$ $m, n \in Z-\{0\}$,

$d(\gamma)=4$,

$$x=\begin{cases} \left.\begin{array}{l} X_1(0, 0, 0)\\ X_2(0, 0)\end{array}\right\} m, n\text{: arbitrary}, & i_0(x)=1,\\ X_1(0, 1, 0)\cdots m\equiv 0 \pmod p, & i_0(x)=p,\\ X_2(0, 1) \quad \cdots n\equiv 0 \pmod p, & i_0(x)=p.\end{cases}$$

(5.35) $\gamma=\hat{\hat{\delta}}_2(m, n)=\begin{pmatrix}1 & 0 & m & -1\\ 0 & -1 & 1 & n\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$ $m, n \in Z-\{0\}$,

$d(\gamma)=4$,

$$x=\begin{cases} \left.\begin{array}{l} X_1(0, 0, 0)\\ X_2(0, 0)\end{array}\right\} m, n\text{: arbitrary}, & i_0(x)=1,\\ X_2(2, 0) \quad \cdots m\equiv 0 \pmod p, & i_0(x)=p,\\ X_1(0, 0, 2)\cdots n\equiv 0 \pmod p, & i_0(x)=p.\end{cases}$$

(5.36) $\gamma=\hat{\hat{\delta}}_3(m, n)=\begin{pmatrix}1 & 0 & 2m & m+2\\ 1 & -1 & m-2 & n\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & -1\end{pmatrix}$ $m, n \in Z$, $m, 2n-m\neq 0$,

$d(\gamma)=4$,

$$x=\begin{cases} \left.\begin{array}{l} X_1(0, 0, 0)\\ X_2(0, 0)\end{array}\right\} m, n\text{: arbitrary}, & i_0(x)=1,\\ X_2(2, 0) \quad \cdots m\equiv 0 \pmod p, & i_0(x)=p,\\ X_1(0, 0, 2)\cdots n\equiv 0 \pmod p, & i_0(x)=p,\end{cases}$$

(5.37) $\gamma=\hat{\hat{\delta}}_4(m, n)=\begin{pmatrix}1 & 0 & 2m-1 & m\\ 1 & -1 & m-1 & n\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & -1\end{pmatrix}$ $m, n \in Z$,

$d(\gamma)=4$,

$$x=\begin{cases} \left.\begin{array}{l} X_1(0, 0, 0)\\ X_2(0, 0)\end{array}\right\} m, n\text{: arbitrary}, & i_0(x)=1,\\ X_1(0, 2, 0) \quad \cdots 2m-1\equiv 0 \pmod p, & i_0(x)=p,\\ X_1(0, 2, -4)\cdots 4n-2m+1\equiv 0 \pmod p, & i_0(x)=p.\end{cases}$$

(5.38) $\gamma=\pm\hat{\delta}_1(m)=\begin{pmatrix}1 & 0 & m & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$ $m \in Z-\{0\}$,

$d(\gamma)=3$,

$$x=\begin{cases} X_1(0, 0, 0)\cdots m\text{: arbitrary}, & i_0(x)=1,\\ X_2(0, 0) \quad \cdots m\text{: arbitrary}, & i_0(x)=p+1,\\ X_1(0, 1, 0)\cdots m\equiv 0 \pmod p, & i_0(x)=p.\end{cases}$$

(5.39) $\gamma=\pm\hat{\delta}_2(m)=\begin{pmatrix}1 & 0 & m & -1\\ 0 & -1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$ $m \in Z-\{0\}$,

$d(\gamma)=3$,

$$x=\begin{cases} X_1(0, 0, 0)\cdots m\text{: arbitrary}, & i_0(x)=1,\\ X_2(0, 0) \quad \cdots m\text{: arbitrary}, & i_0(x)=p+1,\\ X_2(0, 2) \quad \cdots m\equiv 0 \pmod p, & i_0(x)=p.\end{cases}$$

(5.40) $\gamma=\pm\varepsilon_1(S), \varepsilon_3(S)=\begin{pmatrix}1 & 0 & s_1 & s_{12}\\ 0 & 1 & s_{12} & s_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$; $S=\begin{pmatrix}s_1 & s_{12}\\ s_{12} & s_2\end{pmatrix}$, $\det S\neq 0$,

$d(\gamma)=2$,

$$x=\begin{cases} X_1(0, 0, 0)\cdots S\text{: arbitrary};\ L=SM_2(Z), \\ \qquad\qquad (P_M\cap\Gamma)_0\cong G\Gamma_0(p),\\ X_1(0, 1, 0)\cdots s_1, s_{12}\equiv 0 \pmod p;\\ \qquad L=\begin{pmatrix}pZ & pZ\\ pZ & Z\end{pmatrix}\cap SM_2(Z),\quad (P_M\cap\Gamma)_0\cong G\Gamma_0^*(p),\end{cases}$$

where

$$G\Gamma_0(p)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2(Z);\ c\equiv 0 \pmod p\right\},$$

$$G\Gamma_0^*(p)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2(Z);\ b\equiv 0 \pmod p\right\}.$$

Note that $G\Gamma_0(p)$, $G\Gamma_0^*(p)$ both have two cusps 0, $i\infty$. The invariants described in (4.68) are given as follows:

(i) For $x=X_1(0, 0, 0)=1_4$,

$\beta=i\infty$: $L_\beta=\begin{pmatrix}pZ & pZ\\ pZ & 0\end{pmatrix}\cap SM_2(Z)$, $B_\beta=\pm\begin{pmatrix}1 & Z\\ 0 & 1\end{pmatrix}$; $(c, d)=(2, 1)$,

$$\beta=0:\ L_\beta=\begin{pmatrix}0 & pZ\\ pZ & Z\end{pmatrix}\cap SM_2(Z),\quad B_\beta=\pm\begin{pmatrix}1 & 0\\ pZ & 1\end{pmatrix};\ (c,d)=(2p,1),$$

(ii) For $x=X_1(0,1,0)$,

$$\beta=i\infty:\ L_\beta=\begin{pmatrix}pZ & pZ\\ pZ & 0\end{pmatrix}\cap SM_2(Z),\quad B_\beta=\pm\begin{pmatrix}1 & pZ\\ 0 & 1\end{pmatrix};\ (c,d)=(2p,p),$$

$$\beta=0:\ L_\beta=\begin{pmatrix}0 & pZ\\ pZ & Z\end{pmatrix}\cap SM_2(Z),\quad B_\beta=\pm\begin{pmatrix}1 & 0\\ Z & 1\end{pmatrix};\ (c,d)=(2p,p).$$

Remark. In the case (ii) above, the Levi-component P_M should be chosen carefully, so that (4.65) holds: namely

$$P_M=x^{-1}\left\{\begin{pmatrix}A & 0\\ 0 & {}^tA^{-1}\end{pmatrix};\ A\in GL_2(Q)\right\}x,\quad x=\begin{pmatrix}1_2 & 1_2\\ 0 & 1_2\end{pmatrix}\cdot X_1(0,1,0).$$

(5.41) $$\gamma=\pm\varepsilon_4(n)=\begin{pmatrix}1&0&0&0\\0&1&0&n\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad n\in Z-\{0\},$$

$d(\gamma)=3$,

$$x=\begin{cases}X_1(0,0,0) & \cdots\ n\text{: arbitrary}, & i_0(x)=p(p+1),\\ X_2(0,0) & \cdots\ n\text{: arbitrary}, & i_0(x)=1,\\ X_4 & \cdots\ n\equiv 0 \pmod p, & i_0(x)=p^3.\end{cases}$$

5-3. *Conjugacy classes of $B(p)$ (p=prime$\neq 2,3$).* We first recall, for the convenience of readers, that the coset space $Sp(2,Z)/\Gamma_0(p)$ has the following complete set of representatives:

$$Y_1(a,b)=\begin{pmatrix}a&-b&-1&0\\0&1&0&0\\1&0&0&0\\b&0&0&1\end{pmatrix}\quad\left(\text{or } Y_1'(a,b)=\begin{pmatrix}-b&a&0&-1\\1&0&0&0\\0&1&0&0\\0&b&1&0\end{pmatrix}\right),$$

$$Y_2(a,b,c)=\begin{pmatrix}a&b&-1&0\\b&c&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad Y_3(a)=\begin{pmatrix}1&0&0&0\\0&a&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix},$$

$$Y_4=1_4,\quad ([Sp(2,Z):\Gamma_0(p)]=(p+1)(p^2+1)),$$

where a, b, c run over the set of integers modulo p. We shall make full use of the results of [16], Section 7 where the decomposition of $Sp(2,Z)$-conjugacy classes into $\Gamma_0(p)$-conjugacy classes is described. In the following list, $d_y(\gamma)$ is the number of $B(p)$-conjugacy classes contained in the set $\{y^{-1}\gamma y\}_{\Gamma_0(p)}\cap B(p)$, where y is as above; and $d(\gamma)$ denotes the sum of $d_y(\gamma)$.

(5.42) $\gamma=\pm\alpha_0$: $d(\gamma)=1$, $x=Z_1(0)$ ($y=Y_4$),
$m(\gamma; Sp(2,Z)/B(p))=(p+1)^2(p^2+1)$,

In the next two cases, θ denotes the integer mod p, which generates $F_p^\times$.

(5.43) $\gamma=\pm\alpha_1$

$$A=\left\{\theta^j;\ 1\leq j\leq\frac{p-1}{4},\ j\neq\frac{p-1}{8}\right\}\Big/\sim,\quad u\sim v\Leftrightarrow(uv)^4=1$$

y	p (mod 8)	$d_y(\alpha_1)\cdots x$	order of centralizer in $B(p)$
$Y_2(i,0,i)$ and $Y_2(-i,0,-i)$, $i=\theta^{(p-1)/4}$	1	$(p-9)/8\cdots Z_1(t),\ t\in A$ $1\cdots Z_1(\theta^{(p-1)/8})$ $1\cdots Z_1(1)$ $1\cdots Z_1(0)$	4 8 8 16
	5	$(p-5)/8\cdots Z_1(t),\ t\in A$ $1\cdots Z_1(1)$ $1\cdots Z_1(0)$	4 8 16
	3, 7	0	
$Y_2(i,0,-i)$	1, 5 3, 7	$2\cdots Z_1(0),\ Z_2$ 0	16
$Y_2(a,b,-a)$ with $b\not\equiv 0$, $a^2+b^2+1\equiv 0$	1	2 ($a=0$) ($Z_1(t)$: 2 ($a=1$) $(bt+a)^2+1\equiv 0$) 2 (otherwise; there are $(p-9)/8$ such pairs (a,b))	8 8 4
	5	2 ($a=0$) $Z_1(t)$: $(bt+a)^2+1\equiv 0$ 2 (otherwise; there are $(p-5)/8$ such pairs (a,b))	8 4
	3, 7	0	

From this table, we get (see also Remark 5.45):

$$m(\gamma; Sp(2,Z)/B(p))=2(p+1)\left(1+\left(\frac{-1}{p}\right)\right).$$

(5.44) $\gamma=\pm\alpha_2, \alpha_3$

$$B=\left\{\theta^j;\ 1\leq j\leq\frac{p-1}{6},\ j\neq\frac{p-1}{12}\right\}\Big/\sim,\quad u\sim v\Leftrightarrow(uv)^6=1$$

y	p (mod 12)	$d_y(\alpha_2)\cdots x$	order of centralizer in $B(p)$
$Y_2(\omega, 0, \omega)$ and $Y_2(\bar\omega, 0, \bar\omega)$, $\omega=\theta^{(p-1)/6}$	1	$(p-13)/12 \cdots Z_1(t),\ t\in B$	6
		$1 \cdots Z_1(\theta^{(p-1)/12})$	12
		$1 \cdots Z_1(1)$	12
		$1 \cdots Z_1(0)$	36
	7	$(p-7)/12 \cdots Z_1(t),\ t\in B$	6
		$1 \cdots Z_1(1)$	12
		$1 \cdots Z_1(0)$	36
	5, 11	0	
$Y_2(\omega, 0, \bar\omega)$	1, 7	$2 \cdots Z_1(0),\ Z_2$	36
	5, 11	0	
$Y_2(a, b, -1-a)$ with $b\not\equiv0$, $a^2+a+1+b^2\equiv0$	1	2 ($a=0$) $\left(Z_1(t)\colon bt^2+(2a+1)t-b\equiv0\right)$	12
		2 ($a=-2$)	12
		2 (otherwise; there are $(p-13)/12$ such pairs (a, b))	6
	7	2 ($a=-2$)	12
		2 (otherwise; there are $(p-7)/12$ such pairs (a, b))	6
	5, 11	0	

From this table, we get (see also Remark 5.45):

$$m(\gamma;\ Sp(2, Z)/B(p))=2(p+1)\left(1+\left(\frac{-3}{p}\right)\right).$$

Remark 5.45. The above lists for $B(p)$-conjugacy classes belonging to α_1, α_2, and α_3 are obtained after somewhat complicated calculations. We gave these lists in order to make our description consistent. Indeed, it is worth noting that the relative Maß of α_1 (resp. α_2, α_3) coincides with that of γ_1, γ_2 (resp. γ_3), for which the calculation is quite easy. This fact can be proved without computing the former, by the method described in Section 4–1 (see footnote to (5.3)). Of course a similar observation can be made for the group $\Gamma_0(p)$. The complicated situation for $\gamma=\alpha_1, \alpha_2$, and α_3 comes from the fact that the quaternion algebras $Z_0(\gamma)$ attached to their centralizers are *definite*, so that the class numbers $h(\Lambda; G)$ of their Z-orders are big (c.f. Theorem 4.28), while for $\gamma=\gamma_1, \gamma_2, \gamma_3$, $Z_0(\gamma)$ are indefinite, and we have $h(\Lambda; G)=1$ by the strong approximation theorem [32].

(5.46) $\gamma=\alpha_4, \alpha_5$

$$d_y(\gamma)=\begin{cases}2 & \text{if } p\equiv1 \pmod 8,\\ 0 & \text{otherwise,}\end{cases}$$

$$x=\begin{cases}Z_1(t)\cdots t^2+a\equiv0 & \text{for } y=Y_2(c, a, a)\colon a^2+1\equiv0,\\ Z_1(t)\cdots t^2+bt-1\equiv0 & \text{for } y=Y_2(-1, b, 1)\colon b^2+2\equiv0,\end{cases}$$

$$m(\gamma;\ Sp(2, Z)/B(p))=\begin{cases}8 & \text{if } p\equiv1 \pmod 8,\\ 1 & \text{otherwise.}\end{cases}$$

(5.47) $\gamma=\pm\alpha_6$

$$d_y(\gamma)=\begin{cases}2 & \text{if } p\equiv1 \pmod 8,\\ 0 & \text{otherwise,}\end{cases}$$

$$x=\begin{cases}Z_1(t)\cdots t^2+t+\dfrac{a}{a+1}\equiv0 & \text{for } y=Y_2(2a+1, a, 2a+1)\colon\ 3a^2+2a+1\equiv0,\\ Z_1(t)\cdots t^2+(b+1)t+\dfrac{b-1}{2}\equiv0 & \text{for } y=Y_2(b, b, 1)\colon\ b^2+1\equiv0,\end{cases}$$

$$m(\gamma;\ Sp(2, Z)/B(p))=\begin{cases}8 & \text{if } p\equiv1 \pmod 8,\\ 0 & \text{otherwise.}\end{cases}$$

(5.48) $\gamma=\pm\alpha_7$

$$d_y(\gamma)=1+\left(\frac{-3}{p}\right),$$

$$x=\begin{cases}Z_1(1),\ Z_1(-1)\cdots & \text{for } y=Y_1(0, b)\colon b^2+b+1\equiv0,\\ Z_1(t)\cdots t^2+t+1\equiv0 & \text{for } y=Y_2(2a, a, 2a)\colon 3a^2+1\equiv0,\end{cases}$$

$$m(\gamma;\ Sp(2, Z)/B(p))=\left(1+\left(\frac{-3}{p}\right)\right)^2.$$

(5.49) $\gamma=\pm\alpha_8$

$$d_y(\gamma)=1+\left(\frac{-3}{p}\right),$$

$x=Z_1(0),\ Z_2$ for $y=Y_2(a,0,c)$: $a^2+a+1\equiv c^2+c+1\equiv 0$,

$m(\gamma;Sp(2,Z)/B(p))=\left(1+\left(\frac{-3}{p}\right)\right)^3.$

(5.50) $\gamma=\alpha_9,\ \alpha_{10}$

$d_y(\gamma)=1+\left(\frac{-3}{p}\right),$

$$x=\begin{cases} Z_1(t)\cdots at^2-1\equiv 0 & \text{for } y=Y_2(a,0,0):\ a^2+a+1\equiv 0,\\ Z_1(t)\cdots t\equiv(-b+1)/2 & \text{for } y=Y_2(-2,b,1):\ b^2+3\equiv 0,\end{cases}$$

$m(\gamma;Sp(2,Z)/B(p))=\left(1+\left(\frac{-3}{p}\right)\right)^3.$

(5.51) $\gamma=\alpha_{11},\ \alpha_{12}$

$d_y(\gamma)=1+\left(\frac{-3}{p}\right),$

$x=Z_1(0),\ Z_2$ for $y=Y_2(a,0,c)$: $a^2+a+1\equiv c^2+c+1\equiv 0$,

$m(\gamma;Sp(2,Z)/B(p))=\left(1+\left(\frac{-3}{p}\right)\right)^3.$

(5.52) $\gamma=\alpha_{13},\ \alpha_{14}$

$$d_y(\gamma)=\begin{cases}2 & \text{if } p\equiv 1 \pmod{12},\\ 0 & \text{otherwise,}\end{cases}$$

$$x=\begin{cases} Z_1(t)\cdots t^2\equiv\dfrac{1}{a+1} & \text{for } y=Y_2(a,0,a):\ a^2+a+1\equiv 0,\\ Z_1(t)\cdots t^2+bt-1\equiv 0 & \text{for } y=Y_2(-1,b,0):\ b^2+1\equiv 0,\end{cases}$$

$m(\gamma;Sp(2,Z)/B(p))=2\left(1+\left(\frac{-1}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right).$

(5.53) $\gamma=\pm\alpha_{15},\cdots,\alpha_{18}$

$$d_y(\gamma)=\begin{cases}2 & \text{if } p\equiv 1 \pmod 5, \quad \text{or } p=5,\\ 0 & \text{otherwise,}\end{cases}$$

$x=Z_1(t)\cdots t^2+\dfrac{a}{a+1}t+a\equiv 0$ for $y=Y_2(a,b,c)$:

$$a^4+a^3+a^2+a+1\equiv 0,\quad b\equiv\frac{-1}{a+1},\quad c\equiv\frac{-1}{a},$$

$$m(\gamma;Sp(2,Z)/B(p))=\begin{cases}8 & \text{if } p\equiv 1 \pmod 5,\\ 1 & \text{if } p=5,\\ 0 & \text{otherwise.}\end{cases}$$

(5.54) $\gamma=\pm\alpha_{19},\cdots,\alpha_{22}$

$$d_y(\gamma)=\begin{cases}2 & \text{if } p\equiv 1 \pmod{12},\\ 0 & \text{otherwise,}\end{cases}$$

$x=Z_1(0),\ Z_2$ for $y=Y_2(a,0,c)$: $a^2+a+1\equiv c^2+1\equiv 0$,

$m(\gamma;Sp(2,Z)/B(p))=2\left(1+\left(\frac{-1}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right).$

(5.55) $\gamma=\pm\beta_1,\cdots,\beta_4$

$d_y(\gamma)=1+\left(\frac{-3}{p}\right),$

$x=Z_1(0),\ Z_2$ for $y=Y_1(a,0)$: $a^2+a+1\equiv 0$,

$m(\gamma;Sp(2,Z)/B(p))=2(p+1)\left(1+\left(\frac{-3}{p}\right)\right).$

(5.56) $\gamma=\pm\beta_5,\ \beta_6$

$d_y(\gamma)=1+\left(\frac{-1}{p}\right),$

$x=Z_1(0),\ Z_2$ for $y=Y_1(a,0)$: $a^2+1\equiv 0$,

$m(\gamma;Sp(2,Z)/B(p))=2(p+1)\left(1+\left(\frac{-1}{p}\right)\right).$

(5.57) $\gamma=\gamma_1$ (resp. γ_2)

$$d_y(\gamma)=\begin{cases}1+\left(\frac{-1}{p}\right) & \text{for } y=Y_4,\\ 1 & \text{for } y=Y_1'(0,b)\ (\text{resp. } Y_1'(b,b)):\ b^2+1\equiv 0,\end{cases}$$

$$x=\begin{cases} Z_1(t)\cdots t^2+1\equiv 0 & \text{for } y=Y_4,\\ Z_1(0)\cdots & \text{for } y=Y_1'(0,b)\ (\text{resp, } Y_1'(b,b)),\end{cases}$$

$m(\gamma;Sp(2,Z)/B(p))=2(p+1)\left(1+\left(\frac{-1}{p}\right)\right),$

(5.58) $\gamma=\pm\gamma_3$

$$d_y(\gamma)=\begin{cases}1+\left(\frac{-3}{p}\right) & \text{for } y=Y_4,\\ 1 & \text{for } y=Y_1'(0,b):\ b^2+b+1\equiv 0,\end{cases}$$

$$x=\begin{cases} Z_1(t)\cdots t^2-t+1\equiv 0 & \text{for } y=Y_4,\\ Z_1(0)\cdots & \text{for } y=Y_1'(0,b),\end{cases}$$

$m(\gamma;Sp(2,Z)/B(p))=2(p+1)\left(1+\left(\frac{-3}{p}\right)\right).$

(5.59) $\gamma=\delta_1, \delta_2$

$d_y(\gamma)=2$ for $y=Y_4$,

$x=Z_1(0), Z_2$,

$m(\gamma: Sp(2, Z)/B(p))=2(p+1)^2$.

P-unipotent classes. The group $B(p)$ has four point cusps and four one-dimensional cusps; the parabolic subgroups corresponding to these cusps are given as follows:

Point cusps:

$$P_0^j=x_j^{-1}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}x_j,$$

$x_1=Y_4$, $x_2=Y_2(0, 0, 0)$, $x_3=Y_1(0, 0)$, and $x_4=Y_3(0)\cdot Z_2$.

One dimensional cusps:

$$P_1^j=x_j^{-1}\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}x_j,$$

$x_1=Y_4$, $x_2=Y_2(0, 0, 0)$, $x_3=Z_2$, and $x_4=Y_2(0, 0, 0)\cdot Z_2$.

We put, for each element γ' of $B(p)$,

$$i_0=i_0(\gamma')=[C_0(\gamma'; Sp(2, Z)): C_0(\gamma'; B(p))].$$

(5.60) $\gamma=\pm\hat{\beta}_1(n), \hat{\beta}_2(n)$

$d_y(\gamma)=1+\left(\frac{-3}{p}\right)$; $i_0=1$,

$x=Z_1(0), Z_2$ for $y=Y_1(c, 0)$, $Y_2(a, 0, 0)$:

$a^2-a+1\equiv c^2-c+1\equiv 0$,

$m(\gamma; Sp(2, Z)/B(p))=2\left(1+\left(\frac{-3}{p}\right)\right)^2$.

(5.61) $\gamma=\pm\hat{\beta}_3(n), \hat{\beta}_4(n)$

$d_y(\gamma)=1+\left(\frac{-3}{p}\right)$; $i_0=1$,

$x=Z_1(0), Z_2$ for $y=Y_1(c, 0)$, $Y_2(a, 0, 0)$:

$a^2-a+1\equiv c^2-c+1\equiv 0$,

$m(\gamma; Sp(2, Z)/B(p))=2\left(1+\left(\frac{-3}{p}\right)\right)^2$.

(5.62) $\gamma=\pm\hat{\beta}_5(n), \hat{\beta}_6(n)$

$d_y(\gamma)=1+\left(\frac{-3}{p}\right)$; $i_0=1$,

$$x=\begin{cases} Z_1\left(\frac{c+1}{3}\right),\ Z_2 & \text{for } y=Y_1(c, 0):\ c^2-c+1\equiv 0, \\ Z_1(0),\ Z_1(-3) & \text{for } y=Y_2(3a-1, a, 0):\ 3a^2-3a+1\equiv 0. \end{cases}$$

$m(\gamma; Sp(2, Z)/B(p))=2\left(1+\left(\frac{-3}{p}\right)\right)^2$.

(5.63) $\gamma=\pm\hat{\beta}_7(n), \hat{\beta}_8(n)$

$d_y(\gamma)=1+\left(\frac{-1}{p}\right)$; $i_0=1$,

$x=Z_1(0), Z_2$ for $y=Y_1(c, 0)$, $Y_2(a, 0, 0)$: $c^2+1\equiv a^2+1\equiv 0$,

$m(\gamma; Sp(2, Z)/B(p))=2\left(1+\left(\frac{-1}{p}\right)\right)^2$.

(5.64) $\gamma=\pm\hat{\beta}_9(n), \hat{\beta}_{10}(n)$

$d_y(\gamma)=1+\left(\frac{-1}{p}\right)$; $i_0=1$,

$$x=\begin{cases} Z_1\left(\frac{c+1}{2}\right),\ Z_2 & \text{for } y=Y_1(c, 0):\ c^2+1\equiv 0, \\ Z_1(0),\ Z_1(2) & \text{for } y=Y_2(a, 0, 0):\ a^2+1\equiv 0, \end{cases}$$

$m(\gamma; Sp(2, Z)/B(p))=2\left(1+\left(\frac{-1}{p}\right)\right)^2$.

(5.65) $\gamma=\hat{\gamma}_1(n)$

$d_y(\gamma)=1+\left(\frac{-1}{p}\right)$,

$$x=\begin{cases} Z_1(t) & \text{with } t^2+1\equiv 0 \text{ for } y=Y_4,\ Y_2(0, 0, 0):\ i_0=1,\ n\text{: arbitrary}, \\ Z_1(0) & \text{for } y=Y_1'(0, b):\ b^2+1\equiv 0,\ i_0=1,\ n\text{: arbitrary}, \\ Z_1(1) & \text{for } y=Y_1'(0, b):\ i_0=p,\ n\equiv 0 \pmod p), \end{cases}$$

(5.66) $\gamma=\hat{\gamma}_2(n)$

$d_y(\gamma)=1+\left(\frac{-1}{p}\right)$,

$$x=\begin{cases} Z_1(t) \text{ with } t^2+1\equiv 0 \text{ for } y=Y_4,\ Y_2(0,0,1/2);\ i_0=1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n\text{: arbitrary}, \\ Z_1(0) \text{ for } y=Y_1'(-1/2,b)\text{: } b^2+1\equiv 0,\ i_0=1,\quad n\text{: arbitrary}, \\ Z_1(1) \text{ for } y=Y_1'(-1/2,b)\text{: } i_0=p,\ 2n+1\equiv 0 \pmod p. \end{cases}$$

(5.67) $\gamma=\hat{\gamma}_3(n)$

$$d_y(\gamma)=1+\left(\frac{-1}{p}\right),$$

$$x=\begin{cases} Z_1(t) \text{ with } t^2+1\equiv 0 \text{ for } y=Y_4,\ Y_2(0,1/2,0),\ i_0=1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n\text{: arbitrary}, \\ Z_1(0) \text{ for } y=Y_1'(b,b)\text{: } b^2+1\equiv 0,\ i_0=1,\quad n\text{: arbitrary}, \\ Z_1(1) \text{ for } y=Y_1'(b,b)\text{: } i_0=p,\ n\equiv 0 \pmod p. \end{cases}$$

(5.68) $\gamma=\hat{\gamma}_4(n)$

$$d_y(\gamma)=1+\left(\frac{-1}{p}\right),$$

$$x=\begin{cases} Z_1(t) \text{ with } t^2+1\equiv 0 \text{ for } y=Y_4,\ Y_2(0,1/2,1/2),\ i_0=1, \\ Z_1(0) \text{ for } y=Y_1'(b-1/2,b)\text{: } b^2+1\equiv 0,\ i_0=1,\ n\text{: arbitrary}, \\ Z_1(1) \text{ for } y=Y_1'(b-1/2,b)\text{: } i_0=p,\ 2n+1\equiv 0 \pmod p), \end{cases}$$

(5.69) $\gamma=\pm\hat{\gamma}_5(n)$

$$d_y(\gamma)=1+\left(\frac{-3}{p}\right),$$

$$x=\begin{cases} Z_1(t) \text{ with } t^2-t+1\equiv 0 \text{ for } y=Y_4,\ i_0=1,\quad n\text{: arbitrary}, \\ Z_1(t) \text{ with } t^2+t+1\equiv 0 \text{ for } y=Y_2(0,0,0),\ i_0=1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n\text{: arbitrary}, \\ Z_1(0) \text{ for } y=Y_1'(0,b);\ b^2+b+1\equiv 0,\ i_0=1,\quad n\text{: arbitrary}, \\ Z_1(1) \text{ for } y=Y_1'(0,b),\ i_0=p,\ n\equiv 0 \pmod p). \end{cases}$$

(5.70) $\gamma=\pm\hat{\gamma}_6(n)$

$$d_y(\gamma)=1+\left(\frac{-3}{p}\right),$$

(5.71)

$$x=\begin{cases} Z_1(t) \text{ with } t^2-t+1\equiv 0 \text{ for } y=Y_4,\ i_0=1,\quad n\text{: arbitrary}, \\ Z_1(t) \text{ with } t^2+t+1\equiv 0 \text{ for } y=Y_2(-1/3,0,1/3),\ i_0=1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n\text{: arbitrary}. \\ Z_1(0) \text{ for } y=Y_1'\left(\frac{-b}{2b+1},b\right)\text{: } b^2+b+1\equiv 0,\ i_0=1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n\text{: arbitrary}, \\ Z_1(1) \text{ for } y=Y_1'\left(\frac{-b}{2b+1},b\right),\ i_0=p,\ 3n\equiv\frac{2b}{2b+1} \pmod p. \end{cases}$$

(5.71) $\gamma=\pm\hat{\gamma}_7(n)$

$$d_y(\gamma)=1+\left(\frac{-3}{p}\right),$$

$$x=\begin{cases} Z_1(t) \text{ with } t^2-t+1\equiv 0 \text{ for } y=Y_4,\ i_0=1,\quad n\text{: arbitrary}, \\ Z_1(t) \text{ with } t^2+t+1\equiv 0 \text{ for } y=Y_2(-2/3,0,2/3),\ i_0=1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n\text{: arbitrary}, \\ Z_1(0) \text{ for } y=Y_1'\left(\frac{-2b}{2b+1},b\right)\text{: } b^2+b+1\equiv 0,\ i_0=1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n\text{: arbitrary}, \\ Z_1(1) \text{ for } y=Y_1'\left(\frac{-2b}{2b+1},b\right),\ i_0=1,\ 3n\equiv\frac{4b}{2b+1} \pmod p. \end{cases}$$

(5.72) $\gamma=\hat{\hat{\delta}}_1(m,n)$ (resp. $\hat{\hat{\delta}}_2(m,n)$)

$d_y(\gamma)=2$ for $y=Y_4,\ Y_2(0,0,0),\ Y_3(0),\ Y_1'(0,0),$
(resp. $Y_4,\ Y_2(0,1/2,0),\ Y_3(0),\ Y_1'(0,0)$),

$x=Z_1(0),\ Z_2$ (resp. $Z_1(0),\ Z_1(2)$), $i_0=1,$

condition on (m,n):

Y_4	$Y_2(0,0,0),\ Y_2(0,\frac{1}{2},0)$	$Y_3(0)$	$Y_1'(0,0)$
arbitrary	$m\equiv n\equiv 0 \pmod p$	$n\equiv 0$	$m\equiv 0$

(5.73) $\gamma=\hat{\hat{\delta}}_3(m,n)$ (resp. $\hat{\hat{\delta}}_4(m,n)$)

$d_y(\gamma)=2$ for $y=Y_4,\ Y_2(2,0,0),\ Y_1'(0,0),\ Y_1'(0,-2),$
(resp. $Y_4,\ Y_2(1/2,0,0),\ Y_1'(0,0),\ Y_1'(0,-2)$),

$x=Z_1(0),\ Z_1(-1)$ (resp. $Z_1(0),\ Z_1(-4)$), $i_0=1,$

condition on (m,n):

Y_4	$Y_2(2,0,0,),\ Y_2(\frac{1}{2},0,0)$	$Y_1'(0,0)$	$Y_1'(0,-2)$
arbitrary	$m\equiv n\equiv 0 \pmod p$	$m\equiv 0$ resp. $2m\equiv 1$	$m\equiv 2n$ resp. $2m-4n\equiv 1$

(5.74) $\gamma=\pm\hat{\delta}_1(n)$ (resp. $\hat{\delta}_2(n)$)

$d_y(\gamma)=2$ for $y=Y_4,\ Y_2(0,0,0)$ (resp. $Y_4,\ Y_1'(0,0)$),

$$x=\begin{cases} Z_1(0),\ Z_2 & \text{for } Y_4,\ Y_2(0,0,0),\ i_0=1, \\ Z_1(0),\ Z_1(2) & \text{for } Y_1'(0,0),\ i_0=1, \end{cases}$$

Y_4	$Y_2(0,0,0)$, $Y_1'(0,0)$
n: arbitrary	$n\equiv 0 \pmod p$

(5.75) $\gamma=\pm\varepsilon_1(S),\ \varepsilon_3(S)$

$$d_y(\gamma)=\begin{cases} 1 & \text{for } y=Y_4,\ Y_2(0,0), \\ 2 & \text{for } y=Y_3(0), \end{cases}$$

$$x=\begin{cases} Z_1(0) & \text{for } y=Y_4,\ Y_2(0,0) \\ Z_1(0),\ Z_2 & \text{for } Y_3(0) \end{cases}$$

y / x	Y_4 / $Z_1(0)$	$Y_2(0,0)$ / $Z_1(0)$	$Y_3(0)$ / $Z_1(0)$	$Y_3(0)$ / $Z_1(0)$
L	$\begin{pmatrix} Z & Z \\ & Z \end{pmatrix}$	$\begin{pmatrix} pZ & pZ \\ & pZ \end{pmatrix}$	$\begin{pmatrix} Z & Z \\ & pZ \end{pmatrix}$	$\begin{pmatrix} Z & pZ \\ & pZ \end{pmatrix}$
i_0	1	p^3	p	p^2

For each family, the corresponding Levi-component is isomorphic to

$$P_M\cap B(p)\cong G\Gamma_0(p)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(Z);\ c\equiv 0 \pmod p\right\}.$$

Therefore we have the following invariants (c, d) for each cusp of $G\Gamma_0(p)$ (c.f. Theorem P-6, (4.68)).

L	$\begin{pmatrix} Z & Z \\ & Z \end{pmatrix}$	$\begin{pmatrix} pZ & pZ \\ & pZ \end{pmatrix}$	$\begin{pmatrix} Z & Z \\ & pZ \end{pmatrix}$	$\begin{pmatrix} Z & pZ \\ & pZ \end{pmatrix}$
$\beta=i\infty$, $B_\beta=\begin{pmatrix} 1 & Z \\ 0 & 1\end{pmatrix}$	$t_1=1$ $t_2=0$ $d=1$ $c=2$	p 0 p 2	1 0 1 2	1 0 p $2p$
$\beta=0$, $B_\beta=\begin{pmatrix} 1 & 0 \\ pZ & 1\end{pmatrix}$	$t_1=1$ $t_2=0$ $d=1$ $c=2p$	p 0 p $2p$	p 0 1 2	p 0 p $2p$

(5.76) $\gamma=\pm\varepsilon_4(m)$

$d_y(\gamma)=2$ for $y=Y_4,\ Y_2(0,0,0)$,

$x=Z_1(0),\ Z_2$,

	Y_4	$Y_2(0,0,0)$
$Z_1(0)$	$i_0=p$ m: arbitrary	$i_0=1$ $m\equiv 0 \pmod p$
Z_2	$i_0=1$ m: arbitrary	$i_0=p$ $m\equiv 0 \pmod p$

§ 6. Local data for $B'(p)$

In this section, we shall give local data which are necessary to calculate dim $\mathfrak{M}_\nu(B'(p))$. The local data for $q\neq p$ have been given in [19], so we shall calculate $c_p(g, R_p, \Lambda_p)$ and masses, where g is a torsion element of G_p^* and

$$R_p=\begin{pmatrix} O_p & O_p \\ \pi O_p & O_p\end{pmatrix}.$$

Throughout this section, we assume that $p\neq 2, 3$.

Proposition 6.1. *Put* $g=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$, or $\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}$. *Then,*

$$c_p(g, R_p, \Lambda_p)=\begin{cases} 1\cdots \textit{if } \Lambda_p\sim R_p, \\ 0\cdots \textit{otherwise}. \end{cases}$$

Let Λ be the order of $M_2(O)$ such that $\Lambda_p=R_p$ and $\Lambda_q=M_2(O_q)$ $(q\neq p)$. Then,

$$M_G(\Lambda)=(p^4-1)/2^7 3^2 5.$$

Proof. This is obvious, because $[P_1': B']=p+1$ (cf. [19] (I) Proposition 9).

Proposition 6.2. *If the principal polynomial of $g\in G_p^*$ is $f_i(x)$ or $f_i(-x)$ for some $i=2, 3, 4, 5, 8,$ or 9, then $c_p(g, R_p, \Lambda_p)=0$ for all orders of $Z(g)_p$.*

Proof. We assume $p\neq 2, 3$, so it is known that

$$c_p\left(g, \begin{pmatrix} O_p & \pi^{-1}O_p \\ \pi O_p & O_p \end{pmatrix}, \Lambda_p\right)=0,$$

for any above g and any Λ_p ([19], (III)). Thus, our Proposition is obvious. q.e.d.

Next, we treat elements $g \in G_p^*$ such that $f_6(g)=0$ or $f_7(\pm g)=0$. Put $Z_0(g)_p=Z_0(g)\otimes_Q Q_p$, where $Z_0(g)$ is the quaternion algebra over Q defined as in [19], I, (12) (p. 562). For $\Lambda_p \subset Z(g)_p$, we define $d_p(\Lambda_p)$ and $e_p(\Lambda_p)$ as in [19], (I), Proposition 12 (p. 572). Put $F=Q[g]$ and $o=Z[g]$. As we have assumed $p\neq 2, 3$, $\left(\frac{F}{p}\right)\neq 0$, where $\left(\frac{F}{p}\right)$ is the Legendre symbol. By definition, we have $Z_0(g)_p\otimes_Q F \cong D_p\otimes_Q F$, so $\left(\frac{F}{p}\right)=-1$, if $Z_0(g)_p$ is not division.

Proposition 6.3. *If $Z_0(g)_p$ is split and $\left(\frac{F}{p}\right)=-1$, we get*

$$c_p(g, R_p, \Lambda_p)=\begin{cases} 2\cdots \textit{if } \Lambda_p \sim \begin{pmatrix} o_p & o_p \\ po_p & o_p \end{pmatrix}=\Lambda, \\ 0\cdots \textit{otherwise}. \end{cases}$$

where $o_p=o\otimes_Z Z_p$. We have $e_p(\Lambda)=1$ and $d_p(\Lambda)=p+1$.

Proof. If $Z_0(g)_p$ is split and $\left(\frac{F}{p}\right)=-1$, then g is G_p^*-conjugate to $\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$, where $\omega \in O_p$ is of order 3, 4, or 6. So, we put $g=\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$. By virtue of [19], (III), Proposition 2.5 (i), if $x^{-1}gx \in B' \subset P_0'$, then $x \in Z_{G_p^*}(g)\cdot P_0'$. We can put $O_p=Z_p+Z_p\varepsilon+Z_p\pi+Z_p\pi\varepsilon$, where $F_p=F\otimes_Q Q_p=Q_p[\varepsilon]$, $\varepsilon^2 \in Q_p^\times$, $\pi^2 \in Q_p^\times$, and $\varepsilon\pi=-\pi\varepsilon$. Then,

$$P_0'=\coprod_a \begin{pmatrix} 1 & \pi^{-1}a \\ 0 & 1 \end{pmatrix} B' \coprod \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix} B',$$

where $a \in Z_p[\varepsilon]$ runs through a set of complete representatives of $Z_p[\varepsilon]/pZ_p[\varepsilon]$. If $x \in Z_{G_p^*}(g)\begin{pmatrix} 1 & \pi^{-1}a \\ 0 & 1 \end{pmatrix}B'$, then

$$\begin{pmatrix} 1 & -\pi^{-1}a \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1 & \pi^{-1}a \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} \omega, & (\omega-\bar\omega)\pi^{-1}a \\ 0 & \omega \end{pmatrix} \in B',$$

so $a \in po_p$. Thus, we get $x \in Z_{G_p^*}(g)B'$ in this case. For $x=\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}$, we get $x^{-1}gx={}^tg \in B'$. Now, assume that $\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}=hk$, where $h \in Z_{G_p^*}(g)$ and $k \in B'$. Then, $h \in Z_{G_p^*}(g)\cap P_0' \subset \begin{pmatrix} o_p & o_p \\ po_p & o_p \end{pmatrix}$, which is a contradiction, because we must have $h \in B'$. Thus, we get $c_p=2$. We have $\Lambda_0=\begin{pmatrix} Z_p & \varepsilon Z_p \\ p\varepsilon Z_p & Z_p \end{pmatrix}$ and we get $d_p(\Lambda)=p+1$. q.e.d.

Proposition 6.4. *If $Z_0(g)$ is division, we get*

(i) *if $\left(\frac{F}{p}\right)=1$, then*

$$c_p(g, R_p, \Lambda_p)=\begin{cases} 2\cdots \textit{if } \Lambda_p \sim Z(g)_p\cap R_p=\Lambda, \\ 0\cdots \textit{otherwise}, \end{cases}$$

and $d_p(\Lambda)=e_p(\Lambda)=1$,

(ii) *if $\left(\frac{F}{p}\right)=-1$, then*

$$c_p(g, R_p, \Lambda_p)=0 \quad \textit{for any } \Lambda_p.$$

Proof. By virtue of Proposition 2.6, (ii) in [19], (III) (p. 398), the above (ii) is obvious. So, assume that $\left(\frac{F}{p}\right)=1$. We can assume that $g=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G_p^*$, where a, $b \in Q_p$ are different roots of $f_6(x)=0$, $f_7(x)=0$, or $f_7(-x)=0$. If $x^{-1}gx \in B'$, then $x \in Z_{G_p^*}(g)P'$, by virtue of [19], (III), Proposition 2.6, (i). In the similar way as in the proof of Proposition 6.3, we can show that $x \in Z_{G_p^*}(g)B'$ or $Z_{G_p^*}(g)\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}B'$, and these two double cosets are disjoint. q.e.d.

For $g \in G_p^*$ such that $f_i(\pm g)=0$ ($i=10$, 11, or 12), it is obvious that $c_p(g, R_p, \Lambda_p)=0$, unless $\Lambda_p=Z_p[g]$. From now on, we put $c_p(g)=c_p(g, R_p, Z_p[g])$. For a fixed $i=10$, 11, or 12, denote by t the number of G_p^*-conjugacy classes in $\{g \in G_p^*; f_i(g)=0\}$.

Proposition 6.5. *Let $g \in G_p^*$ be of order 5 or 10.*

(i) *If $p=5$, then $c_p(g)=1$, and $t=1$ for $i=10$.*

(ii) *If $p\neq 5$, then $c_p(g)=0$.*

Proof. By virtue of [19], (I), Proposition 19, (ii) and (III), Proposition 2.8, (ii), the above (ii) is obvious. Assume $p=5$. Then, we can put $g=\begin{pmatrix} 0 & -\varepsilon \\ \varepsilon^{-1} & \omega \end{pmatrix}$, where ω is an element of O_p such that $\omega^2-\omega-1=0$, and $\varepsilon \in O_p$, $\varepsilon^2=-3$, $\varepsilon\omega=\bar\omega\varepsilon$. If $x^{-1}gx \in B'$, then $x \in Q_p(g)P_1'$, by virtue of [19], Proposition 19, (iv). It is easy to see $x \in Q_p(g)\begin{pmatrix} 1 & 0 \\ -2\varepsilon & 1 \end{pmatrix}B'$. q.e.d.

Proposition 6.6. *Let g be of order* 8. *Then,*
(i) *if* $p\equiv\pm1 \bmod 8$, *then* $c_p(g)=0$,
(ii) *if* $p\equiv3$, *or* $5 \bmod 8$, *then* $c_p(g)=4$ *and* $t=1$.

Proposition 6.7. *Let g be an element of* G_p^* *such that* $f_{12}(g)=0$. *Then,*
(i) *if* $p\equiv\pm1 \bmod 12$, *then* $c_p(g)=0$,
(ii) *if* $p\equiv5 \bmod 12$, *then* $c_p(g)=4$, $t=1$, *and* $Z_0(g^2)_p=$*split*,
(iii) *if* $p\equiv7 \bmod 12$, *then* $c_p(g)=4$, $t=1$, *and* $Z_0(g^2)_p=$*division*.

Proof of Propositions 6.6 *and* 6.7. By virtue of [19], (I) Propositions 20, 21, and (III), Propositions 2.9, 2.10, the above (i) of Proposition 6.6 and 6.7 are obvious. If $p\equiv3$, $5 \bmod 8$ (resp. $p\equiv5$, $7 \bmod 12$), we can write $f_{11}(x)$ (resp. $f_{12}(x)$) as a product of quadratic polynomials in $\boldsymbol{Q}_p[x]$:

$$f_i(x)=(x^2+ax+b)(x^2+ab^{-1}x+b^{-1}) \quad (i=11 \text{ or } 12),$$

where $b\neq1$. We can take $\omega\in O_p$ such that $\omega^2+a\omega+b=0$. Put $\omega_1=\omega$ and $\omega_2=b^{-1}\omega$. Then, g is G_p^*-conjugate to $\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix}$. If $x^{-1}gx\in B'$ for some $x\in G_p^*$, then $x\in Z_{G_p^*}(g)P_1'$ or $Z_{G_p^*}(g)\begin{pmatrix}\pi & 0\\ 0 & \pi\end{pmatrix}P_1'$, by virtue of [19], (I), Proposition 21. We have

$$P_1'=\coprod_a \begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}B' \amalg \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}B',$$

where a runs through a set of representatives of

$$\{x\in O_p;\ \mathrm{tr}(x)=0\}/\{x\in\pi O_p;\ \mathrm{tr}(x)=0\}.$$

In the similar way as in the proof of Proposition 6.4, (ii), we have

$$x\in Z_{G_p^*}(g)y_iB' \quad (i=1, 2, 3, \text{ or } 4),$$

where

$$y_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad y_2=\begin{pmatrix}\pi & 0\\ 0 & \pi\end{pmatrix},\quad y_3=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \text{and}\quad y_4=\begin{pmatrix}0 & \pi\\ \pi & 0\end{pmatrix}.$$

These four double cosets are disjoint. q.e.d.

§ 7. Related topics

Here, we would like to take this opportunity to write briefly on some related topics.

(1) *Ihara lifting* For $n=2$, Ihara [28] has shown that there exists a kind of lifting of automorphic forms of $S_{2\nu+4}(\Gamma_0(p))$ to $\mathfrak{M}_\nu(U_1'(p))$, where

$$\Gamma_0(p)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\boldsymbol{Z});\ c\equiv0 \bmod p\right\}.$$

(Actually, he did not assume that the discriminant of D is a prime. As for this, see his paper.)

More exactly, we can take the representation space V_ν of ρ_ν (for $n=2$) as follows: We identify $\boldsymbol{H}$ (the Hamilton quaternions) with $\boldsymbol{R}^4$. V_ν is the set of real valued homogeneous polynomials $f(x, y)$ on $\boldsymbol{H}^2\cong\boldsymbol{R}^8$ such that

1) $f(ax, ay)=N(a)^\nu f(x, y)$ for all $a\in\boldsymbol{H}^\times$, and
2) $\Delta f=0$,

where $N(a)$ is the reduced norm of a and Δ is the usual Laplacian. $Sp(2)$ acts on V_ν by

$$f(x, y)\longmapsto f((x, y)g) \quad \text{for all } g\in Sp(2).$$

For the sake of simplicity, we assume here that the class number of $U_1(p)$ is one, i.e. $\dim\mathfrak{M}_\nu(U_1(p))=1$ for $\nu=0$, although, as Ihara has kindly shown us, his theory works completely in the same way without any such restriction. Put $\Gamma=U_1(p)\cap G'$. Then, under the above assumption, we get

$$\mathfrak{M}_\nu(U_1'(p))=\{f\in V_\nu;\ f((x, y)\gamma)=f(x, y) \text{ for all } \gamma\in\Gamma\}.$$

Let $f\in\mathfrak{M}_\nu(U_1'(p))$ be a common eigen form of all the Hecke operators $T(m)$. For such f, put

$$\vartheta_f(\tau)=\sum_{(x,y)\in O^2} f(x, y)e^{2\pi i(N(x)+N(y))\tau},\quad \tau\in\mathfrak{H}_1.$$

Then, $\vartheta_f(\tau)\in S_{2\nu+4}(\Gamma_0(p))$.

Theorem 7.1 (Ihara [28]). *Assume that* $f(1, 0)\neq0$. *Then,* ϑ_f *is also a common eigen form* (*of the Hecke operators of* $\Gamma_0(p)$), *and we get*

$$L(s, f)=\zeta(s-\nu-1)\zeta(s-\nu-2)L(s, \vartheta_f)$$

up to the Euler p-factors.

This Ihara's result was the first one among results on lifting obtained later by many mathematicians. For example, the Saito-Kurokawa lifting may be regarded as a similar version of Ihara lifting for the split group $Sp(2, \boldsymbol{R})$. The second author has extended Theorem 7.1 to general n: under a similar condition on f as $f(1, 0)\neq 0$ for $n=2$, he expressed the eigen values of f by some group theoretical numbers and coefficients of some one dimensional automorphic forms, and at least for $n=3$, gave $L(s, f)$ explicitly.

Some examples of Theorem 7.1 have been given by Ihara (loc. cit.). We give here another example. We assume $n=2$. Put $p=2$ and $\nu=2$. Then, $\dim \mathfrak{M}_\nu(U_1'(2))=1$ and this space is spanned by:

$$f(x, y)=N(x)^2-3N(x)N(y)+N(y)^2.$$

Then, by Theorem 7.1, we have

$$L(s, f)=\zeta(s-5)\zeta(s-6)L(s, h),$$

up to Euler 2 factors, where h is the unique normalized cusp form of $S_8(\Gamma_0(2))$. On the other hand, Maaß [38] has shown that

$$L(s, F)=\zeta(s-5)\zeta(s-6)L(s, h),$$

for some $F \in S_5(\Gamma_e(1))$ (unique up to constant), where $\Gamma_e(1)$ is the unique index two subgroup of $Sp(2, \boldsymbol{Z})$ which contains the level two principal congruence subgroup. So, (f, F) gives an example for the Langlands philosophy. But, this example is less essential than the examples in [21], because this is a relation through one dimensional forms and does not satisfy the Ramanujan Conjecture. In our set-up in Conjecture 1.11, there is no relation, at least apparently, between old forms and those forms obtained from lifting. As for another aspect between lifting and old forms, see [24].

(2) *Construction of automorphic forms.* For $n=1$, it is well-known that we can construct the forms in $S_k(\Gamma_0(p))$ from $\mathfrak{M}_{k-2}(O_A)$ through the Weil representation: We can embed $Sp(1)\cong SU(2)$ to $SO(4)$, and roughly speaking, we can get forms in $S_k(\Gamma_0(p))$ through theta functions

$$\vartheta(f)=\sum_{n\in Z^4} P(n)e^{2\pi i Q(n)\tau}, \quad \tau\in\mathfrak{H}_1,$$

where Q are quadratic forms of four variables and P are spherical functions (i.e. automorphic forms of $Sp(1)$). In our case of $n=2$, the situation is fairly different. We can embed $Sp(2)$ to $SO(8)$ for example, and get a Siegel modular form in a similar way, but the weight of this form cannot be $\nu+3$. On the other hand, we have $Sp(2)/\pm 1=SO(5)$. By using this isomorphism, it has been shown in [25], that we can construct automorphic forms on $\widetilde{Sp(2, \boldsymbol{R})}$ (the non-trivial double cover of $Sp(2, \boldsymbol{R})$)

from forms belonging to $Sp(2)$, and that this construction preserves L-functions. Let $\rho(f_1, f_2)$ be the representation of $Sp(2)$ whose Young diagram is

1	·	·	·	f_1
1	·	·	f_2	

.

Then, $\rho(f_1, f_2)$ factors through $SO(5)$ if and only if $f_1+f_2=$even. We assume this. Then, from any form $\varphi\in\mathfrak{M}_{\rho(f_1,f_2)}(U_1'(p))$ $(n=2)$, we can construct a vector valued Siegel modular form $\sigma(\varphi)$ of weight $\det^{(f_1-f_2+5)}\otimes \mathrm{Sym}(f_2)$, where $\mathrm{Sym}(f_2)$ is the symmetric tensor representation of $GL(2)$ of degree f_2. We can develop the Hecke theory on $\widetilde{Sp(2, \boldsymbol{R})}$ and define L-series. By some local theory similar to Yoshida [51], we can show that $L(s, \varphi)=L(s, \sigma(\varphi))$ up to finitely many Euler factors. It is very plausible that there exists a similar mapping from forms of $\widetilde{Sp(2, \boldsymbol{R})}$ to those of $Sp(2, \boldsymbol{R})$. So, the above results might be regarded as the first half of an explicit mapping of forms of $Sp(2)$ to those of $Sp(2, \boldsymbol{R})$.

(3) *A relation to supersingular abelian varieties.* We have some geometrical interpretation of $\dim \mathfrak{M}_0(U_i'(p))$ $(i=0, 1)$ and the Hecke operators. Let H_n be the class number of the principal genus of the definite quaternion hermitian space D^n with metric $N(x_1)+\cdots+N(x_n)$ for $(x_1, \cdots, x_n)\in D^n$. Put

$$U=G_\infty' \prod_q (GL_n(O_q)\cap G_q').$$

Then, $\dim \mathfrak{M}_0(U)=H_n$, so $\dim \mathfrak{M}_0(U_1'(p))=H_2$ (cf. Shimura [44]). For $n=1$, it is known by Deuring [8] that H_1 is equal to the number of isomorphism classes of super singular elliptic curves E over fields of characteristic p. It is clear that the Brandt matrices defined by Eichler [9] coincide with matrices which consists of numbers of isogenies between supersingular elliptic curves. Now, we assume $n\geqq 2$. We have a similar (but slightly different) relation also for these cases: H_n is equal to the number of principal polalizations of E^n up to Aut (E^n) (cf. Ibukiyama-Katsura-Oort [26], J-P. Serre [42]). Combining this fact for $n=2$ with some geometrical consideration, the number of supersingular curves of genus two with prescribed automorphism groups have been counted (Ibukiyama-Katsura-Oort. loc. cit.). This gives an example of explicit descriptions of Γ_i in (1.3) up to isomorphisms. Next, let $C_i(i=1,\cdots, H_n)$ be the complete set of representatives of the principal polarizations of E^n up to Aut (E^n). For natural integers m, put

$$s_{ij}(m)=\#\{\varphi\in \mathrm{End}(E_n);\ \varphi^*(C_j)\equiv mC_i\}/\mathrm{Aut}(E_n, C_i),$$

where $\equiv$ denotes the algebraic equivalence. Put $S(m)=(s_{ij}(m))$. On the other hand, denote by $H(m)=(h_{ij}(m))$ the Brandt matrix, i.e., the matrix induced from the Hecke operator $T(m)$ on the right hand side of (1.4) (c.f. [15], § 1). Then, changing the numbering, if necessary, we get $H(m)=S(m)$. The class number H_2' of the non-principal genus in D^2 is equal to dim $\mathfrak{M}_0(U_0'(p))$ (cf. Shimura [44]). It is known by Katsura-Oort [31] that H_2' is equal to the number of irreducible components of the set of principally polarized supersingular abelian surfaces in the coarse moduli scheme $A_{2,1}$.

References

[1] T. Arakawa, The dimension of the space of cusp forms on the Siegel upper half plane of degree two related to a quaternion unitary groups, J. Math. Soc. Japan, **33** (1981), 125–145.

[2] T. Asai, The conjugacy classes in the unitary, symplectic, and orthogonal groups over an algebraic number field, J. Math. Kyoto Univ., **16** (1976), 325–350.

[3] A. Borel, Automorphic L-functions, Proc. Symp. Pure Math., **XXXIII**, part, 2 (1979), 27–62.

[4] W. Casselman, Introduction to the theory of admissible representations of p-adic reductive groups, preprint.

[5] U. Christian, Untersuchung einer Poincaréschen Reihe I, J. reine angew. Math., **233** (1968), 37–88; II, J. reine angew. Math., **237** (1969), 12–25.

[6] ——, Berechnung des Ranges der Schar der Spitzenformen zur Modulgruppe zweiten Grades und Stufe $q>2$, J. reine angew. Math. **277** (1975), 130–154; Zur Berechnung des Ranges der Schar der Spitzenformen zur Modulgruppe zweiten Grades und Stufe $q>2$, J. reine angew. Math., **296** (1977), 108–118.

[7] P. Deligne and D. Kazhdan, On representations of local division algebras, preprint (1982).

[8] M. Deuring, Die Typen der Multiplikatorenringe elliptischer Funktionenkörper, Abh. Math. Sem. Univ. Hamburg, **14** (1941), 197–272.

[9] M. Eichler, Zur Zahlentheorie der Quaternion-Algebren, Math. Z., **43** (1938), 127–151.

[10] ——, Über die Darstellbarkeit von Modulformen durch Thetareihen, J. reine angew. Math., **195** (1956), 159–171.

[11] ——, Quadratische Formen und Modulformen. Acta arith., **4** (1958), 217–239.

[12] D. Flath, A comparison of the automorphic representations of $GL(3)$ and its twisted forms, Pacific J. Math., **97** (1982).

[13] R. Godement, Généralités sur les formes modulaires I, II Exposé 7, 8, Séminaire Cartan, 1957/1958.

[14] Harish-Chandra, Discrete series for semi-simple Lie groups II, Acta Math., **116** (1966), 1–111.

[15] K. Hashimoto, On Brandt matrices associated with the positive definite quaternion hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **27** (1980), 227–245.

[16] ——, The dimension of the space of cusp forms on Siegel upper half plane of degree two (I), J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30** (1983), 403–488. (II), Math. Ann., **266** (1984), 539–559.

[17] ——, On the first, second, and third asymptotic formulas for the dimension of the spaces of Siegel modular forms of degree n, Séminaire Théorie des nombres de Paris, 1982–83, 103–113, Birkhäuser Boston, Inc.

[18] ——, Class numbers of positive definite ternary quaternion hermitian forms, Proc. Japan Acad., 59 Ser. A. no. 10 (1983), 490–493.

[19] K. Hashimoto and T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms (I), J. Fac. Sci. Univ. Tokyo Sect. IA Math., **27** (1980), 549–601; (II); J. Fac. Sci. Univ. Tokyo Sect. IA Math., **28** (1982), 695–699; (III), J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30** (1983), 393–401.

[20] T. Hina and K. Masumoto, On representations of p-adic split and non split symplectic groups, and their character relation, J. Fac. Sci. Univ. Tokyo. Sect. IA Math.

[21] T. Ibukiyama, On symplectic Euler factors of genus two, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30** (1984) (Resumé was published in Proc. Japan Acad. 57, Ser. A No. 5 (1981), 271–275).

[22] ——, On the graded ring of Siegel modular forms of genus two belonging to certain level two congruence subgroups, preprint.

[23] ——, On automorphic forms of $Sp(2, \boldsymbol{R})$ and its compact form $Sp(2)$, Séminaire de Théorie des nombres de Paris, 1982–83 125–134, Birkhäuser Boston, Inc.

[24] ——, On relations of dimensions between automorphic forms of $Sp(2, \boldsymbol{R})$ and its compact twist $Sp(2)$ (I), (this volume),

[25] ——, On construction of half integral weight Siegel modular forms of $Sp(2, \boldsymbol{R})$ from its compact twist $Sp(2)$ to appear in J. reine angew. math.

[26] T. Ibukiyama, T. Katsura and F. Oort, Supersingular curves of genus two and class numbers, to appear in Compositio math.

[27] J. Igusa, On Siegel modular forms of genus two (I), Amer. J. Math., **84** (1962), 175–200; (II) Amer. J. Math., **86** (1964), 392–412.

[28] Y. Ihara, On certain arithmetical Dirichlet series, J. Math. Soc. Japan, **16** (1964), 214–225.

[29] H. Jacquet and R. P. Langlands, Automorphic forms on $GL(2)$, Lecture Notes in Math., Springer (1972).

[30] H. Jacquet, I. Piatetski-Shapiro and J. A. Shalika, Automorphic forms on $GL(3)$, Ann. of Math., **109** (1979) 169–258.

[31] T. Katsura and F. Oort, Families of supersingular abelian surfaces, preprint.

[32] M. Kneser, Strong Approximation, Proc. Symp. Pure Math., **IX** (1966), 187–196.

[33] R. P. Langlands, Dimension of spaces of automorphic forms, Amer. J. Math., **85** (1963), 99–125; Proc. Symp. Pure Math., **IX** (1966) 253–257.

[34] ——, Problems in the theory of automorphic forms, Lecture Notes in Math. vol. **170**, Springer, (1970), 18–86.

[35] ——, Automorphic representations, Shimura varieties, and motives, Proc. Symp. Pure Math., **XXXIII**, part 2 (1979), 205–246.

[36] ——, Stable conjugacy; definitions and lemmas, Canad. J. Math., **31** (1979), 700–725.

[37] ——, Les début d'une formule des trace stables, Publications Mathématiques de l'université Paris VII (1983).

[38] H. Maaß, Über ein Analogon zur Vermutung von Saito-Kurokawa, Invent. math., **60** (1980), 85–104.

[39] Y. Morita, An explicit formula for the dimensions of spaces of Siegel modular forms of degree two, J. Fac. Sci. Univ. Tokyo Sec. IA Math., **21** (1974), 167–248.

[40] J. Rogawski, Representations of $GL(n)$ and division algebra over a p-adic field, preprint (1981).

[41] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with application to Dirichlet series, J. Indian Math.

Soc., **20** (1956), 47–87.
[42] J-P. Serre, Nombres de points des courbes algébrique sur F_q, Séminaire de théorie des nombres, Bordeaux, Année 1982–1983, exposé no. 22.
[43] H. Shimizu, On zeta functions of quaternion algebras, Ann. of Math., **81** (1965), 166–193.
[44] G. Shimura, Arithmetic of alternating forms and quaternion hermitian forms, J. Math. Soc. Japan, **15** (1963), 33–65.
[45] T. Shintani, On zeta functions associated with the vector space of quadratic forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **22** (1975), 25–65.
[46] J. Tits, Reductive groups over local fields, Proc. Symp. Pure Math., **XXXIII**, part 1, (1979), 29–70.
[47] M. F. Vigneras, Représentation des algèbres central simples sur un corps local non archimedean, preprint (1982).
[48] G. Warner, Harmonic analysis on semi-simple Lie groups (I), (II), Springer (1972).
[49] A. Weil, Adeles and algebraic groups, Lecture Notes, Institute for advanced study, Princeton (1959–60).
[50] ——, Classical groups, Princeton Univ. Press 1939.
[51] H. Yoshida, Siegel modular forms and the arithmetic of quadratic forms, Invent. math., **60** (1980), 193–248.

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Advanced Studies in Pure Mathematics 7, 1985
Automorphic Forms and Number Theory
pp. 103–111

Moonshine for $PSL_2(F_7)$

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0. In [1], Conway and Norton assigned a Thompson series of the form

$$q^{-1}+\sum_{n=1}^{\infty} H_n(m)q^n, \quad q=e^{2\pi i z}$$

to each element m of the Fischer-Griess group F_1 where H_n are characters of F_1, and they conjectured among others that Thompson series are generators of the modular function fields of genus zero for some modular groups which contain $\Gamma_0(N)$ for some N. In [6], Queen studied moonshine for other simple groups, for example, Thompson's group F_3.

In this paper, we consider these phenomena for $PSL_2(F_7)$ and its relation to Conway-Norton's monstrous moonshine.

Let $G=PSL_2(F_7)$. G acts on $F_7\cup\{\infty\}$ as linear fractional transformations, so G can be considered as the subgroup of S_8. Then, each element of G is written by products of cycles and these are of the following forms:

$$1^8,\ 1\cdot 7,\ 1^2\cdot 3^2,\ 2^4,\ 4^2.$$

For each product of cycles of length n_i, $m=(n_1)(n_2)\cdots(n_s)$, $n_1\geq\cdots\geq n_s\geq 1$ $\sum_{i=1}^{s} n_i=8$, in G, we associate following modular forms:

$$\eta_{1,m}(z)=\prod_{i=1}^{s}\eta(3n_iz),$$

$$\eta_{2,m}(z)=\prod_{i=1}^{s}\eta(n_iz)^3,$$

where $\eta(z)$ is the Dedekind η-function. Then $\eta_{1,m}(z)$ (resp. $\eta_{2,m}(z)$) is a cusp form of weight $s/2$ (resp. $3s/2$) on $\Gamma_0(9n_1n_s)$ (resp. $\Gamma_0(n_1n_s)$) with some character and is known to be a common eigenfunction of all Hecke operators (cf. [4]).

We shall prove

Received December 19, 1983.