ON SIEGEL MODULAR FORMS OF DEGREE TWO.

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Introduction. Let \( H_n \) be the Siegel upper half plane of degree \( n \) and \( \Gamma_n(\ell) \) the principal congruence subgroup of \( \text{Sp}(n, \mathbb{Z}) \) of level \( \ell \). Let \( A(\Gamma_n(\ell))_k \) be the space of modular forms of weight \( k \) with respect to \( \Gamma_n(\ell) \) and put

\[
A(\Gamma_n(\ell)) = \bigoplus_{k \geq 0} A(\Gamma_n(\ell))_k.
\]

Then \( A(\Gamma_n(\ell)) \) is a positively graded, integral domain and finitely generated over \( \mathbb{C} \), and the projective variety \( \mathcal{S}(\Gamma_n(\ell)) \) associated with this graded ring is the Satake compactification of the quotient \( \Gamma_n(\ell)/H_n \). In [9] Igusa showed that the blowing up \( \mathcal{M}(\Gamma_n(\ell)) \) of \( \mathcal{S}(\Gamma_n(\ell)) \) with respect to the sheaf of ideals defined by all cusp forms is non-singular for \( n = 2 \) or 3 and \( \ell \geq 3 \).

We shall examine the condition under which multiple forms on \( \Gamma_2(\ell)/H_2 \) can be extended to \( \mathcal{M}(\Gamma_2(\ell)) \) (Sec. 2). It follows immediately from this study that the variety \( \mathcal{M}(\Gamma_2(\ell)) \) is of general type for \( \ell \geq 4 \).

We can construct a line bundle \( L \) on \( \mathcal{M}(\Gamma_2(\ell)) \) which corresponds to modular forms of weight one with respect to \( \Gamma_2(\ell) \) for \( \ell \geq 3 \). It is a natural problem to establish the explicit Riemann-Roch theorem for this line bundle \( L \). In Sec. 3 we shall calculate the related intersection numbers. The result is given as follows;

(i) \( c(L)^3 = 2^{-6}3^{-2}5^{-1}10 \prod_{p / \ell} (1-p^{-2})(1-p^{-4}) \),

(ii) \( c(L)^2 c(D) = 0 \),

(iii) \( c(L)c(D)^2 = -2^{-3}3^{-1}5^2 \prod (1-p^{-2})(1-p^{-4}) \),

(iv) \( c(D)^3 = -11 \cdot 2^{-2}3^{-1}5 \prod (1-p^{-2})(1-p^{-4}) \),

(v) \( c_2 c(D) = 2^{-3}5 \prod (1-p^{-2})(1-p^{-4}) \),

(vi) \( c_2 c(L) = 4c(L)^3 \),

where \( D \) is a divisor determined by the complement \( \mathcal{M}(\Gamma_2(\ell)) - \Gamma_2(\ell)/H_2 \).

It follows from the results in Sec. 2 that the canonical bundle \( K \) of
\( S(\Gamma_2(\ell)) \) is given by \( 3L - [D] \). Therefore by the Riemann-Roch theorem and the vanishing theorem of Kodaira's type, we obtain the following dimension formula for the vector space \( S(\Gamma_2(\ell))_k \) of cusp forms of weight \( k \geq 4 \): (Sec. 4)

\[
\dim S(\Gamma_2(\ell))_k = \ell^{10} \cdot 2^{-10} 3^{-5} 5^{-1} (2k-2)(2k-3)(2k-4)(1-\ell^{-2})(1-\ell^{-4}) \\
- 2^{-6} 3^{-2} (2k-3) \ell^6 (1-\ell^{-2})(1-\ell^{-4}) \\
+ 2^{-53} (1-\ell^{-2})(1-\ell^{-4}).
\]

This formula was also obtained by Y. Morita (under a slightly stronger restriction on the weight \( k \)) by using the Selberg trace formula ([11]).

1. The principal congruence subgroup \( \Gamma_n(\ell) \) of level \( \ell \) is defined by

\[
\Gamma_n(\ell) = \{ M \in \text{Sp}(n, \mathbb{Z}); M = \ell \mod \ell \},
\]

and the index is given by

\[
[\Gamma_n(1); \Gamma_n(\ell)] = \prod_{p|\ell} p^{(2n-1)\ell p-1} (1-\ell^{-2k}).
\]

The boundary of the Satake compactification \( \tilde{\mathcal{S}}(\Gamma_n(\ell)) \) of the quotient \( \Gamma_n(\ell)/\mathcal{H}_n \) is a disjoint union of quasi-projective varieties, each of which is a conjugate of the image of \( \Gamma_m(\ell)/\mathcal{H}_m \) under the dual \( \Phi^* \) of the Siegel \( \Phi \)-operator for some \( m < n \).

Let \( \mathcal{M}(\Gamma_n(\ell)) \rightarrow \mathcal{S}(\Gamma_n(\ell)) \) be the monoidal transform of \( \mathcal{S}(\Gamma_n(\ell)) \) along its boundary.

**Theorem.** ([9]). The monoidal transform \( \mathcal{M}(\Gamma_n(\ell)) \) is non-singular for \( n = 2 \) or \( 3 \) and \( \ell \geq 3 \).

Now the local parameters for \( n = 2 \) and \( \ell \geq 3 \) are given explicitly as follows. Let \( \omega \) be a point of \( \mathcal{M}(\Gamma_2(\ell)) \) such that its projection is the image point of a point \( t_0 \) of \( \mathcal{H}_1 \). Then take a sequence of points in \( \Gamma_2(\ell)/\mathcal{H}_2 \) which converges to \( \omega \), and take representatives of these points in \( \mathcal{H}_2 \) to obtain a sequence of points with \( (t, z, \omega) = (t' \ z) \), say, as a typical term. By taking a subsequence if necessary, we can assume that \( (t, z) \) converges to \( (t_0, z_0) \) and \( \text{Im } \omega \rightarrow \infty \). Let \( \xi = e(\omega/\ell) \); then \( \xi \rightarrow \xi_0 = 0 \), where \( e(x) \) stands for \( e^{2\pi i x} \). If we denote the local parameters at \( t_0, z_0, \) and \( \xi_0 \) by \( t - t_0, z - z_0, \) and \( \xi \) respectively, then \( (t - t_0, z - z_0, \xi) \) is a local coordinate system of \( \mathcal{M}(\Gamma_2(\ell)) \) at \( \omega \). ([9])

2. Let \( m \) be a vector in \( \mathbb{Z}^{2n} \) and \( m', m'' \) be vectors in \( \mathbb{Z}^n \) determined by the first and the last \( n \) components of \( m \). Now if \( t \) is a point in \( \mathcal{H}_n \) and \( z \) is a
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point in $\mathbb{C}^n$, the following series

$$\theta_m(\tau, z) = \sum_{p \in \mathbb{Z}^n} e\left(\frac{1}{2}(p + m'/2)\tau(p + m'/2) + (p + m'/2)(z + m''/2)\right)$$

converges absolutely and uniformly in every compact subset of $H_n \times \mathbb{C}^n$. Therefore for a fixed $m$, it represents an analytic function of the two variables $\tau$ and $z$, which is called the theta-function of characteristic $m$. If we put $z = 0$, we get an analytic function $\theta_m(\tau) = \theta_m(\tau, 0)$ on $H_n$, which is called the theta-constant. There are ten theta-constants which are not identically zero for $n = 2$. We denote by $\theta(\tau)$ the product of all such functions.

**Proposition.** ([8]). Let $\psi(\tau) = \theta(\tau)^2$, then it is a unique cusp form of weight ten with respect to $\Gamma_2(1)$.

The modular form in the above proposition has the following Fourier-Jacobi expansion;

$$\psi(\tau) = \left[ -6(\theta_{00}\theta_{10}\theta_{01})(t)^6\theta_{11}(t, z)^2 + \cdots \right] e(w),$$

where the unwritten part is a convergent power series in $t$, $z$, and $e(w)$.

Let $\tau = (t, z, w)$ be the coordinate of $H_2$ and $d\tau = dt \wedge dz \wedge dw$. Using the above cusp form $\psi(\tau)$, we set

$$\varphi = \psi(\tau)^6(d\tau)^{10};$$

then it is $\Gamma_2(1)$-invariant 10-pole 3-form on $H_2$. Therefore it is, in particular, $\Gamma_2(\ell)$-invariant, so it can be regarded as a 10-pole 3-form on $\Gamma_2(\ell) \setminus H_2 \subset \mathcal{M}(\Gamma_2(\ell))$. Now we examine the condition under which $\varphi$ can be extended to the whole of $\mathcal{M}(\Gamma_2(\ell))$.

The differential $d\tau$ is expressed as

$$d\tau = \frac{1}{2\pi i} \ell dt \wedge dz \wedge \xi^{-1} d\xi,$$

with respect to the local coordinate system $(t - t_0, z - z_0, \xi)$. Now $\varphi$ has the following expansion;

$$\varphi = \text{const. } [(\theta_{00}^2\theta_{10}^2\theta_{01})(t)^{18}\theta_{11}^6(t, z)^6 + \cdots ]\xi^{3\ell - 10}(dt \wedge dz \wedge d\xi)^{10},$$

where the unwritten part is a convergent power series in $t$, $z$, and $\xi$. Therefore $\varphi$ is holomorphic with respect to $(t - t_0, z - z_0, \xi)$ if and only if $3\ell - 10 > 0$. Therefore if $\ell \geq 4$, $\varphi$ can be extended to $\mathcal{M}(\Gamma_2(\ell))$ by the continuation theorem as a holomorphic 10-pole 3-form.
By a well-known asymptotic behaviour of the dimensions of the vector spaces of modular forms with respect to $\text{Sp}(2,\mathbb{Z})$, we obtain:

**Theorem***. The non-singular model $\mathfrak{M}(\Gamma_2(\ell))$ is of general type for $\ell \geq 4$. In particular, in this case, it is non-rational.

3. From now on we fix a level $\ell \geq 3$. Throughout this section we shall denote by $Y$ the Satake compactification of the quotient $\Gamma_2(\ell)/\mathbb{H}_2$, and by $\pi:X \to Y$ the Igusa’s desingularization. We denote by $D$ and $B$ the complements of $\Gamma_2(\ell)/\mathbb{H}_2$ in $X$ and $Y$ respectively. Then $D$ and $B$ are decomposed into the same number of irreducible components,

$$D = \sum D_\ell, \quad B = \sum B_\ell,$$

where the number $\mu(\ell)$ of irreducible components is given by (2)

$$\mu(\ell) = \frac{1}{2} \ell^4 \prod_{p | \ell} (1 - p^{-4}).$$

Each $B_\ell$ is isomorphic to the standard compactification of $\Gamma_1(\ell)/\mathbb{H}_1$, namely it is set-theoretically the union of $\Gamma_1(\ell)/\mathbb{H}_1$ and cusps $P_\ell$,

$$B_\ell = (\Gamma_1(\ell)/\mathbb{H}_1) \cup P_1 \cup \cdots \cup P_{v(\ell)},$$

where the number $v(\ell)$ of cusps is given by

$$v(\ell) = \frac{1}{2} \ell^2 \prod_{p | \ell} (1 - p^{-2}).$$

The restriction of $\pi$ to $D_\ell$, which we also denote by $\pi$, gives rise to a projection $D_\ell \to B_\ell$. By this projection, $D_\ell$ is the elliptic modular surface of level $\ell$ in the sense of Shioda [13]. That is, its general fibers are elliptic curves with level $\ell$ structures and it has singular fibers over the cusps of $B_\ell$. The singular fibers consist of $\ell$ lines with multiplicity one and with self-intersection number $-2$, and $\ell$ lines intersect like edges of an $\ell$-gon. [9] (For terminology see [10].)

The group $\Gamma_2(1)/\Gamma_2(\ell)$ operates on $X$ as a group of automorphisms and $D_\ell$’s are mapped isomorphically to each other by this group.

**Lemma 1.** The Euler number $e(D_\ell)$ of $D_\ell$ is given by

$$e(D_\ell) = \ell v(\ell)$$

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*I was informed by Prof. Igusa that this result is already known among some of the specialists, but there is no statements with complete proofs in the literature.*
Proof. By the theory of elliptic surface, $e(D_i)$ is equal to the sum of the Euler number of singular fibers of $D_i$, so that we have,

$$e(D_i) = v(\ell)(1 - 1 + \ell) = lv(\ell).$$

Q.E.D.

**Lemma 2.** ([13]). Let $K(D_i)$ be the canonical bundle of $D_i$ and let $\pi: D_i \to B_i$ be the natural projection. Then we have

$$K(D_i) = \pi^* M_i,$$

where $M_i$ is a line bundle on $B_i$ which corresponds to cusp forms of weight three with respect to $\Gamma_1(\ell)$. Moreover the degree of $M_i$ is given by

$$\deg(M_i) = 2^{-3}(\ell - 4)\prod(1 - p^{-2}).$$

As in Section 2, we denote by $\theta(\tau)$ the product of all even theta constants of degree two. We know that it is a cusp form of weight five with respect to $\Gamma_2(2)$ and its square is a cusp form of weight ten with respect to $\Gamma_2(1)$. We have the following.

**Theorem.** ([5]). Let $\Delta$ be the set of diagonal elements in $H_2$. Then the zero set of $\theta(\tau)$ is precisely the union of all $\Gamma_2(1)$-conjugates of $\Delta$.

Let $E$ be the closure of $\Gamma_2(\ell) \setminus \Gamma_2(1)\Delta$ in $X$, and decompose $E$ into irreducible components;

$$E = \sum E_\alpha.$$

**Lemma 3** Under the decomposition $E = \sum E_\alpha$, the number $\lambda(\ell)$ of irreducible components is given by

$$\lambda(\ell) = \frac{1}{2} \ell^2 \prod(1 + p^{-2}).$$

Proof. Let

$$G = \{ M \in \Gamma_2(1); M \Delta = \Delta \}$$

and

$$G' = \left\{ \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix} \in \Gamma_2(1) \right\}.$$
It is easy to see that $G' \cong \Gamma_1(1) \times \Gamma_1(1)$, $G' \subset G$ with $[G; G'] = 2$, and $G = G' \cup G' V$, where

$$V = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$ 

Since $G \cap \Gamma_2(\ell) = G' \cap \Gamma_2(\ell) \cong \Gamma_1(\ell) \times \Gamma_1(\ell)$, we have

$$\lambda(\ell) = [\Gamma_2(1); \Gamma_2(\ell) G]$$

$$= [\Gamma_2(1); \Gamma_2(\ell) [G; \Gamma_2(\ell)]^{-1}$$

$$= \frac{1}{2} [\Gamma_2(1); \Gamma_2(\ell)] [G'; G' \cap \Gamma_2(\ell)]^{-1}$$

$$= \frac{1}{2} \frac{10\Pi(1 - p^{-2})}{p^4 \Pi(1 - p^{-4})} [\frac{3\Pi(1 - p^{-2})}{p^2 \Pi(1 - p^{-2})}]^{-2}$$

$$= \frac{1}{2} \frac{p^4 \Pi(1 - p^{-2})}{p^2 \Pi(1 - p^{-2})}.$$ 

Q.E.D.

As we have remarked before, the group $\Gamma_2(1)/\Gamma_2(\ell)$ operates on $X$ as a group of automorphisms and by this action the sets of components $\{D_j\}$ and $\{E_a\}$ are homogeneous. Therefore, to study the intersection properties among them, it suffices to see at special places. Let $D_1$ be the component of $D$ at the infinity in the sense that $\text{Im } w \to \infty$, where $\tau = \left( \frac{t}{z} \frac{\bar{z}}{w} \right)$ is the coordinates of $H_2$.

With the same notations as in the proof of lemma 2, let $E_1$ be the closure of $\Gamma_2(\ell) \cap G \Delta$ in $X$. Obviously the quotient $\Gamma_2(\ell) \cap G \Delta$ is isomorphic to $(\Gamma_1(\ell) \setminus H_1) \times (\Gamma_1(\ell) \setminus H_1)$. We remark that, if $w$ is the coordinate of $H_1$, we can take $e(w/\ell)$ as the local coordinate of the cusp at the infinity in the standard compactification $(\Gamma_1(\ell) \setminus H_1)^*$. This is the same as that of $X$ which determines the divisor $D_1$. Therefore it follows from the form of the local coordinate system of $X$ at $D_1$ (Sec. 1), that $D_1$ and $E_1$ intersect transversally with multiplicity one. The intersection $D_1 \times E_1$ is isomorphic to the standard compactification of $\Gamma_1(\ell) \setminus H_1$. More precisely, on $D_1$ it consists of origins of general fibers of $\pi$, and on $E_1$ it is isomorphic to the product $(\Gamma_1(\ell) \setminus H_1)^* \times P$, where $P$ is the cusp at the infinity in the standard compactification of $\Gamma_1(\ell) \setminus H_1$. Therefore $E_1$, hence each $E_a$, is isomorphic to $(\Gamma_1(\ell) \setminus H_1)^* \times (\Gamma_1(\ell) \setminus H_1)^*$. 

If $D_i$ intersects with $E_1$, the intersection $D_i \times E_1$ takes form on $E_1$ of either $(\Gamma_1(\ell) \times H_1)^* \times \{\text{cusp}\}$ or $\{\text{cusp}\} \times (\Gamma_1(\ell) \setminus H_1)^*$. There are $2\nu(\ell) D_i$'s which in-
intersect with $E_1$, and they are given by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & 0 & b \\
0 & 0 & 1 & 0 \\
0 & c & 0 & d
\end{pmatrix}
\]

$D_1$, or

\[
\begin{pmatrix}
0 & a & 0 & b \\
1 & 0 & 0 & 0 \\
0 & c & 0 & d \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

where \((a \ b \ c \ d)\) runs over a complete set of representatives of $\Gamma_1(1)/\Gamma_1(\ell)\Gamma_1\infty$ with

\[
\Gamma_1\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL(2,\mathbb{Z}) \right\}.
\]

On the other hand, if $E_\alpha$ intersects with $D_1$, the intersection $E_\alpha \cdot D_1$ is a image of a section of $\pi$ which consists of points of order $\ell$ of general fibers of $\pi$. There are $\ell^2$ such sections so that there are the same number $E_\alpha$'s which intersect with $D_1$, and they are given by

\[
\begin{pmatrix}
1 & 0 & 0 & b \\
a & 1 & b & 0 \\
0 & 0 & 1 & -a \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

$E_1$, $0 \leq a \leq \ell$, $0 \leq b \leq \ell$.

Now the partial derivative $\partial/\partial z$ does not vanish on $\Delta$, where $\tau = \left( \begin{smallmatrix} t \\ z \\ w \end{smallmatrix} \right)$. ([5]) So that if $\alpha \neq \beta$, $E_\alpha$ does not intersect with $E_\beta$ in $\Gamma_2(\ell)\backslash H_2$. On the other hand it is easy to see that if $\alpha = \beta$, then $E_\alpha \cap E_\beta \cap D_1 = \emptyset$ for every $i$. We summarize the results.

**Lemma 4.** The divisor $E$ is a disjoint union of non-singular surfaces each of which is isomorphic to the product $R_1 \times R_2$, where $R_i$ is the standard compactification of $\Gamma_1(\ell)\backslash H_1$.

**Lemma 5.** Let $E_\alpha \cong R_1 \times R_2$, and let $p_i$ be the $i$-th projection of $R_1 \times R_2$. Let $L_i$ be a line bundle on $R_i$ which corresponds to modular forms of weight one with respect to $\Gamma_1(\ell)$. Then the normal bundle $N(E_\alpha)$ of $E_\alpha$ in $X$ is given by

\[
N(E_\alpha) = -(p_1^*L_1 + p_1^*L_2).
\]

**Proof.** Since the $E_\alpha$'s are conjugate under the group $\Gamma_2(1)$, we may assume $E_\alpha$ is the closure $E_1$ of $\Gamma_2(\ell) \cap G \backslash \Delta$ in $X$, where $G$ and $\Delta$ are the same as before.
Take an element

\[
M = \begin{pmatrix}
a_1 & 0 & b_1 & 0 \\
0 & a_2 & 0 & b_2 \\
c_1 & 0 & d_1 & 0 \\
0 & c_2 & 0 & d_2
\end{pmatrix}
\]

in \( \Gamma_2(\ell) \cap G \), and if we set \( \tau = \begin{pmatrix} t \\ z \\ w \end{pmatrix} \) and \( M\tau = \tau' = \begin{pmatrix} t' \\ z' \\ w' \end{pmatrix} \), then

\[
z' = z / \{(c_1t + d_1)(c_2w + d_2) - c_1c_2z^2\}
\]

Therefore we have

\[
\lim_{z \to 0} z' / z = (c_1t + d_1)^{-1}(c_2w + d_2)^{-1}.
\]

Since the local coordinate of \((\Gamma_1(\ell) \setminus H_1)^*\) at a cusp is the same as that of \(E_1\) induced from \(X\), we obtain the lemma. Q.E.D.

On the Satake compactification \(Y\), we have a natural ample line bundle \(M\) which corresponds to modular forms of weight one. We set

\[
L = \pi^*M.
\]

Since the graded ring \(A(\Gamma_2(\ell))\) is normal, it follows from the definition of the Satake compactification that the 0-th cohomology group \(H^0(Y, \mathcal{O}(kM))\) is canonically isomorphic to the vector space \(A(\Gamma_2(\ell))_k\) of modular forms of weight \(k\) with respect to \(\Gamma_2(\ell)\). Since \(Y\) is a normal variety,

\[
H^0(X, \mathcal{O}(kL)) = H^0(Y, \mathcal{O}(kM)),
\]

hence we have

\[
H^0(X, \mathcal{O}(kL)) = A(\Gamma_2(\ell))_k.
\]

**Lemma 6.** The restriction \(L|_{E_\alpha}\) of \(L\) to \(E_\alpha\) is expressed as

\[
P_1^*L_1 + P_2^*L_2.
\]

where the notations are the same as in Lemma 5.

The proof is straightforward, so we omit the proof.

**Lemma 7.** Let \([E]\) and \([D]\) be line bundles which are determined by the
divisors $E$ and $D$. Then the line bundle $10L$ has the following expression;

$$10L = 2\left[ E \right] + \ell \left[ D \right].$$

Proof. As we have observed, we have the cusp form $\theta^2$ of weight ten with respect to $\Gamma_2(1)$, and it is naturally interpreted as a section of the line bundle $10L$ on $X$. Since the divisor of zeroes of $\theta^2$ is $2E + \ell D$, we have $10L = 2[E] + \ell [D]$. Q.E.D.

We shall always identify a cohomology class in $H^6(X, \mathbb{Z})$ with its value at the fundamental cycle $X$.

Theorem 1. Let $c(E)$ be the Chern class of the line bundle $[E]$. Then we have

$$c(E)^3 = 2^{-6}3^{-2}\ell^{10}\Pi(1 - p^{-2})(1 - p^{-4}).$$

Proof. Since $E = \sum E_a$ is a disjoint union,

$$c(E)^3 = \sum c(E_a)^3.$$  

As in Lemma 5, let $E_a = R_1 \times R_2$ and let $L_i$ be a line bundle on $R_i$ which corresponds modular forms of weight one with respect to $\Gamma_1(\ell)$. Then we have

$$c(E_a)^3 = c(N(E_a))^2$$

$$= \left[ -c(p_1*L_1 + p_2*L_2) \right]^2$$

$$= 2c(p_1*L_1)c(p_2*L_2)$$

$$= 2\left[ 2^{-3}3^{-1}\ell^3\Pi(1 - p^{-2}) \right]^2$$

$$= 2^{-5}3^{-2}\ell^6\Pi(1 - p^{-2}),$$

hence

$$c(E)^3 = \lambda(\ell)c(E_a)^3$$

$$= 2^{-6}3^{-2}\ell^{10}\Pi(1 - p^{-2})(1 - p^{-4}).$$

Q.E.D.

Theorem 2. Let $c(D)$ be the Chern class of the line bundle $[D]$. Then we have

$$c(E)^2c(D) = -2^{-4}3^{-1}\ell^2\Pi(1 - p^{-2})(1 - p^{-4}).$$
Proof. Since the sum $\sum E_\alpha$ is disjoint,
$$c(E)^2c(D) = \sum c(E_\alpha)^2c(D).$$

On the other hand, by the intersection properties among $E_\alpha$ and $D_i$'s, we have
$$c(E_\alpha)^2c(D) = c(N(E_\alpha))c(D|E_\alpha)$$
$$= -2\nu(\ell)2^{-3}3^{-1}\beta|| (1 - p^{-2}),$$

hence
$$c(E)^2c(D) = \lambda(\ell)c(E_\alpha)^2c(D)$$
$$= -2^{-4}3^{-1}\beta|| (1 - p^{-2})(1 - p^{-4}).$$

Q.E.D.

**Theorem 3.** We have
$$c(E)c(D)^2 = 2^{-2}\ell\beta|| (1 - p^{-2})(1 - p^{-4}).$$

Proof. By the observation at the beginning of this section, we have
$$c(E_\alpha)c(D)^2 = \sum c(E_\alpha)c(D_i)c(D_i)$$
$$= 2\nu(\ell)^2,$$

hence
$$c(E)c(D)^2 = \lambda(\ell)2\nu(\ell)^2$$
$$= 2^{-2}\ell\beta|| (1 - p^{-2})(1 - p^{-4}).$$

**Theorem 4.** Let $c(L)$ be the Chern class of the line bundle $L$. Then we have
$$c(L)^2c(D) = 0.$$

Proof. Let $\pi : D_i \to B_i$ be the natural projection. Then the restriction $L|D_i$ of $L$ to $D_i$ is isomorphic to $\pi^*L_i'$, where $L_i'$ is a line bundle on $B_i$ which corresponds to modular forms of weight one with respect to $\Gamma_i(\ell')$. Therefore
we have
\[ c(L)^2 c(D) = \sum c(L)^2 c(D_i) \]
\[ = \sum c(L|D_i)^2 \]
\[ = \mu(\ell) c(\pi^* L_i)^2 \]
\[ = 0. \]

\text{Q.E.D.}

\text{Corollary. With the same notations as above, we have}

(i) \( c(D)^3 = -11 \cdot 2^{-2} 3^{-1} \ell'^4 \Pi(1 - p^{-2})(1 - p^{-4}) \),

(ii) \( c(L)^3 = 2^{-6} 3^{-2} 5^{-1} \ell'^{10} \Pi(1 - p^{-2})(1 - p^{-4}) \),

(iii) \( c(L) c(D)^2 = -2^{-3} 3^{-1} \ell'^8 \Pi(1 - p^{-2})(1 - p^{-4}) \).

These are direct numerical calculations based on Theorem 1, 2, 3 and 4 and Lemma 7, so we omit the proof.

\text{Theorem 5. Let} \( c_2 \) \text{be the second Chern class of the tangent bundle} \( T(X) \) \text{of} \( X \). \text{Then we have}

\[ c_2 c(D) = 2^{-3} \ell(\ell - 2) \Pi(1 - p^{-2})(1 - p^{-4}) . \]

\text{Proof. We have an exact sequence of vector bundles on} \( D_i \);

\[ 0 \to T(D_i) \to T(X)|D_i \to N(D_i) \to 0, \]

where \( T(D_i) \) is the tangent bundle of \( D_i \) and \( N(D_i) \) is the normal bundle of \( D_i \) in \( X \). Therefore we have

\[ c_2(T(X)|D_i) = c_2(T(D_i)) + c_1(T(D_i)) c(N(D_i)). \]

Since the second Chern class of a surface is its Euler number,

\[ c_2(T(D_i)) = \ell \nu(\ell). \]

Since

\[ c(L)c(D)^2 = \sum c(L)c(D_i)^2 = \mu(\ell) c(L|D_i)c(N(D_i)), \]

it follows from the corollary to Theorem 4 that

\[ c(L|D_i)c(N(D_i)) = -2^{-2} 3^{-1} \ell'^4 \Pi(1 - p^{-2}). \]
Now we remark that for any line bundle $N$ on $B_t$, the intersection number of $\pi^*N$ with a fixed line bundle on $D_i$ is proportional with the degree of $N$.

As we have observed in the proof of theorem 4, the line bundle $L|D_t$ is given by

$$L|D_t = \pi^*L_i'',$$

where $L_i'$ is a line bundle on $B_i$ which corresponds to modular forms of weight one.

On the other hand, the canonical bundle $K(D_t)$ of $D_t$ is given in Lemma 2, so that we have

$$c(K(D_t))c(N(D_t)) = (\deg M_i/\deg L_i')(-2^{-2}3^{-1}\ell^4\Pi(1 - p^{-2}))$$

$$= (-2^{-2}\ell^4 + \ell^3)\Pi(1 - p^{-2}).$$

Hence we have

$$c_2c(D) = \sum c_2(T(X)|D_t)$$

$$= \mu(\ell)[c_2(T(D_t)) - c(K(D_t))c(N(D_t))]$$

$$= \mu(\ell)[\ell\nu(\ell) + (2^{-2}\ell^4 - \ell^3)\Pi(1 - p^{-2})]$$

$$= \frac{1}{2}(\ell - 2)\ell^7\Pi(1 - p^{-2})(1 - p^{-4}).$$

Q.E.D.

**Theorem 6.** We have

$$c_2c(E) = 2^{-4}3^{-2}\ell^8(\ell - 3)(\ell - 6)\Pi(1 - p^{-2})(1 - p^{-4}).$$

**Proof.** As in the proof of Theorem 5, we have an exact sequence of vector bundles on $E_a$;

$$0 \longrightarrow T(E_a) \longrightarrow T(X)|E_a \longrightarrow N(E_a) \longrightarrow 0,$$

therefore

$$c_2(T(X)|E_a) = c_2(T(E_a)) + c_1(T(E_a))c(N(E_a)).$$

The Euler number $c_2(T(E_a))$ of $E_a$ is given by

$$c_2(T(E_a)) = e(R_1) \times e(R_2)$$

$$= \left[2^{-2}3^{-1}\ell^4(\ell - 6)\Pi(1 - p^{-2})\right]^2,$$
where $E_a \cong R_1 \times R_2$ and $e(R_i)$ is the Euler number of $R_i$. On the other hand, if $K_i$ is the canonical bundle of $R_i$ and if $p_i$ is the $i$-th projection of $R_1 \times R_2$, then the canonical bundle $K(E_a)$ of $E_a$ is given by

$$K(E_a) = p_1^*K_1 + p_2^*K_2.$$ 

Therefore we have

$$c(K(E_a))c(N(E_a)) = -c(p_1^*K_1 + p_2^*K_2)c(p_1^*L_1 + p_2^*L_2)$$

$$= -2c(p_1^*K_1)c(p_2^*L_2)$$

$$= -2(2^{-3}-2 \ell^3 - 2^{-1} \ell^2)2^{-3}2^{-3-}1\ell^2(1 - p^{-2})$$

$$= -2^{-4}3^{-2} \ell^5(\ell - 6)(1 - p^{-2}).$$ 

Hence we obtain

$$c_2c(E) = \sum c_2(T(X)|E_a)$$

$$= \lambda(\ell)[2^{-4}3^{-2} \ell^4(\ell - 6)^2 + 2^{-4}3^{-2} \ell^5(\ell - 6)^2] \Pi(1 - p^{-2})^2$$

$$= 2^{-4}3^{-2} \ell^5(\ell - 6)(\ell - 3)\Pi(1 - p^{-2})(1 - p^{-4}).$$ 

Q.E.D.

**Corollary.** We have

$$c_2c(L) = 4c(L)^3.$$ 

**Proof.** Since $10L = 2[E] + \ell[D]$, we have

$$c_2c(L) = 5^{-1}c_2c(E) + 10^{-1}c_2c(D)$$

$$= 2^{-4}3^{-2}5^{-1} \ell^{10} \Pi(1 - p^{-2})(1 - p^{-4}).$$ 

Q.E.D.

4. As an application of the results in Section 3, we shall calculate the dimension of the vector space $S(T_2(\ell))$ of cusp forms of weight $k$ with respect to $T_2(\ell)$. Let $L, M, X, Y$ be the same as in Section 3. In Section 3 we have observed the following isomorphism

$$H^0(X, \mathcal{O}(kL)) \cong A(\Gamma_2(\ell)).$$ 

As for the space $S(T_2(\ell))$ of cusp forms, it is easy to verify the isomorphism:

$$H^0(X, \mathcal{O}(kL - [D])) \cong S(T_2(\ell)).$$
Now from the consideration in Section 2, it follows that the canonical bundle $K$ of the Igusa’s non-singular model is given by

$$K = 3L - [D],$$

so that the first Chern class $c_1$ is given by

$$c_1 = -c(K) = -3c(L) + c(D).$$

If we apply the Riemann-Roch-Hirzebruch theorem to the line bundle $L_k = kL - [D]$ on $X$, [6] we obtain

$$
\sum (-1)^p \dim H^p(X, \mathcal{O}(L_k)) \\
= 6^{-1}c(L_k)^3 + 4^{-1}c(L_k)^2c_1 + 12^{-1}c(L_k)(c_1^2 + c_2) + 24^{-1}c_1c_2 \\
= 2^{-2}3^{-1}(k-1)(k-2)(k-3)c(L)^3 + (2^{-2}3^{-1}k - 2^{-2})c(L)c(D)^2 \\
- 2^{-3}3^{-1}c(D)c_2.
$$

To estimate the higher cohomology groups, we need the following vanishing theorem.

**Theorem.** ([4], [12]). Let $Z$ be a normal projective variety, let $\pi: Z' \rightarrow Z$ be a resolution and let $K'$ be the canonical bundle of $Z'$. If $B$ is an ample line bundle on $Z$, then

$$H^p(Z', \mathcal{O}(\pi^*B + K')) = 0,$$

for $p > 0$.

In our case, the Satake compactification is normal, the line bundle $M$ is ample and the canonical bundle $K$ of the Igusa’s non-singular model is given by

$$K = 3\pi^*M - [D] = 3L - [D],$$

therefore it follows from the above theorem that

$$H^p(X, \mathcal{O}(kL - [D])) = 0,$$

for $k > 4$ and $p > 0$.

So we obtain the following.

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*I was informed this theorem by Prof. Freitag.*
THEOREM. Let \( S(\Gamma_2(l))_k \) be the space of cusp forms of weight \( k \) with respect to \( \Gamma_2(l) \). Then we have the following dimension formula for \( l > 3 \) and \( k > 4 \):

\[
\dim S(\Gamma_2(l))_k = \dim H^0(X, \mathcal{O}(kL - \lfloor D \rfloor)) = 2^{-10}3^{-35}^{-1}(2k-2)(2k-3)(2k-4) l^{10} \prod (1-p^{-2})(1-p^{-4})
\]

\[
-2^{-6}3^{-2}(2k-3)l^8 \prod (1-p^{-2})(1-p^{-4})
\]

\[
+2^{-5}3^{-1}l(l-2)(1-p^{-2})(1-p^{-4}).
\]

REFERENCES.


