SOME RESULTS ON NON-CLUB ISOMORPHIC ARONSZAJN TREES

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Abstract. Let $\lambda$ be a regular cardinal satisfying $\lambda^{<\lambda} = \lambda$ and $\diamond(S_{\lambda}^{<\lambda})$. Then there exists a family of $2^{\lambda^+}$ many completely club rigid special $\lambda^+$-Aronszajn trees which are pairwise far.

In this article we will be concerned with building Aronszajn trees which are not club isomorphic and have strong rigidity properties. This topic goes back to Gaifman-Specker [4], who proved that if $\lambda$ is a regular cardinal satisfying $\lambda^{<\lambda} = \lambda$, then there exists a family of $2^{\lambda^+}$ many normal $\lambda$-complete $\lambda^+$-Aronszajn trees which are pairwise non-isomorphic. Abraham [1] and Todorcevic [7] constructed in ZFC $\omega_1$-Aronszajn trees which are rigid, that is, have no automorphisms other than the identity. Later the focus shifted from isomorphisms between trees to club isomorphisms. Abraham-Shelah [2] proved that under PFA, any two normal $\omega_1$-Aronszajn trees are club isomorphic. Krueger [6] provided a generalization of this result to higher cardinals. Abraham-Shelah [2] also showed that the weak diamond principle on $\omega_1$ implies the existence of a family of $2^{\omega_1}$ many normal club rigid $\omega_1$-Aronszajn trees which are pairwise not club embeddable into each other.

Building off of this work, we will use the diamond principle to construct a family of pairwise non-club isomorphic Aronszajn trees. Specifically, assume that $\lambda$ is a regular cardinal satisfying $\lambda^{<\lambda} = \lambda$ and the diamond principle $\diamond(S_{\lambda}^{<\lambda})$ holds, where $S_{\lambda}^{<\lambda} := \{\alpha < \lambda^+: \text{cf}(\alpha) = \lambda\}$. Then there exists a family $\{T_\alpha : \alpha < 2^{\lambda^+}\}$ of normal $\lambda$-complete special $\lambda^+$-Aronszajn trees such that for each $\alpha < 2^{\lambda^+}$, the only club embedding from a downwards closed normal subtree of $T_\alpha$ into $T_\alpha$ is the identity, and for all $\alpha < \beta < 2^{\lambda^+}$, $T_\alpha$ and $T_\beta$ do not contain club isomorphic downwards closed normal subtrees. We also discuss some related results, such as obtaining a large family of Suslin trees with similar properties and generalizing the Abraham-Shelah result on weak diamond to higher cardinals.

1. Background

We assume that the reader is familiar with the basic definitions and facts about trees, including Aronszajn and Suslin trees. Let $T$ be a tree. We write $\text{ht}_T(x)$ for the height of a node $x$ in the tree $T$, and for any ordinal $\delta$, $T(\delta)$ is the set of nodes of $T$ with height $\delta$. For a set of ordinals $X$, let $T \restriction X$ be the set of nodes $x \in T$ such that $\text{ht}_T(x) \in X$. For a regular uncountable cardinal $\kappa$, a tree $T$ of height
κ is a κ-tree if its levels have size less than κ. A tree T is normal if it satisfies the properties that every element of T has incomparable nodes above it, different elements of T of the same limit height do not have the same set of predecessors, and every element of T has nodes above it at any larger height below the height of the tree. A branch of T is a maximal chain. For an infinite cardinal λ, T is λ-complete if every chain of T whose order type is less than λ has an upper bound.

For any x ∈ T, let x ↓ := {y ∈ T : y <_T x}. For a limit ordinal α, let T(α) ↓ denote the collection of all sets x ↓, where x ∈ T(α). In other words, T(α) ↓ is the set of all cofinal branches of T ↾ α which have upper bounds in T. A subtree of a tree T is any subset U of T considered as a tree with the inherited order <_T ∩ (U × U).

A set X ⊆ T is downwards closed if for all x ∈ X, {y ∈ T : y <_T x} ⊆ X. For any set Y ⊆ T, the downward closure of Y is the set {z ∈ T : ∃y ∈ Y z ≤_T y}. For any x ∈ T, let T_x be the set of all nodes of T which are comparable to x.

Trees T and U of height a regular uncountable cardinal κ are club isomorphic if there exists a club C ⊆ κ such that T ↾ C and U ↾ C are club isomorphic, and an isomorphism between T ↾ C and U ↾ C is called a club isomorphism of T and U. Similarly, T is club embeddable into U if there exists an injective and order preserving function f : T ↾ C → U ↾ C for some club C ⊆ κ, and in that case f is a club embedding. The trees T and U are near if there exist downwards closed normal subtrees of T and U of height κ which are club isomorphic, and otherwise T and U are far. A tree T of height κ is club rigid if the only club isomorphism of T with itself is the identity. A tree T is completely club rigid if the only club isomorphism between downwards closed normal subtrees of T is the identity. A tree T being completely club rigid is equivalent to the statement that whenever F : T ↾ C → T ↾ C is injective, order preserving, and height preserving, where T′ is a downwards closed normal subtree of T and C ⊆ κ is club, then F is the identity.

For a regular uncountable cardinal κ, a κ-tree is κ-Aronszajn if it has no cofinal branch, and is κ-Suslin if it has no chains or antichains of size κ. In this article we will be exclusively concerned with λ^+-trees, where λ is a regular cardinal. A λ^+-tree T is special if it is a union of λ many antichains, or equivalently, there exists a map of T into λ which is injective on chains. Every special tree is Aronszajn and not Suslin.

For a regular cardinal λ, the linear order Q_λ is defined as the set of all functions f : λ → 2 such that the set {α < λ : f(α) = 1} is non-empty and has size less than λ, ordered lexicographically. If λ^{<λ} = λ, then Q_λ has size λ. Two important facts about Q_λ which we will use are: between any two elements of Q_λ there exists an increasing sequence of order type λ, and any increasing sequence of Q_λ with order type less than λ has a least upper bound in Q_λ ([5, Lemma 3.4]). A tree T of height λ^+ is Q_λ-embeddable if there exists an order preserving map from T into Q_λ. In particular, if λ^{<λ} = λ, then being Q_λ-embeddable implies being special.

Let κ be a regular uncountable cardinal and S ⊆ κ stationary. The diamond principle ♦(S) states that there exists a sequence ⟨s_α : α ∈ S⟩, where each s_α ⊆ α, such that for any set X ⊆ κ there are stationarily many α ∈ S such that s_α = X ∩ α. It is common to replace subsets of κ in the above with related objects. For example, as we will use in this article, if n and m_1, . . . , m_n are positive integers, then ♦(S) is equivalent to the existence of a diamond sequence which guesses a finite sequence.
This section assume that a large family of pairwise far Aronszajn trees in Section 3. For the remainder of Section 3.

\[ \alpha \prec \lambda \]

\[ X \]

\[ \text{Definition 2.1.} \] Let \( \alpha \leq \lambda^+ \). A standard \( \alpha \)-tree is pair \((T, \pi)\) satisfying:

1. \( T \) is a tree of height \( \alpha \) and \( \pi : T \to \mathbb{Q}_\lambda \) is order preserving;
2. for all \( \beta < \alpha \), \( T(\beta) = I_\beta \);
3. every node \( x \) of \( T \) such that \( \text{ht}_T(x) + 1 < \alpha \) has \( \lambda \) many immediate successors;
4. if \( x \) and \( y \) are distinct nodes of \( T \) of the same limit height then they do not have the same set of predecessors;
5. for all \( \beta < \alpha \) of cofinality less than \( \lambda \), every cofinal branch \( b \) of \( T \) of \( \beta \) has an upper bound \( y \) and \( \pi(y) = \sup\{\pi(x) : x <_T y\} \);
6. for all \( x \in T \), \( \text{ht}_T(x) < \beta < \alpha \), and \( q \) with \( \pi(x) \leq \mathbb{Q}_{\lambda} q \):
   a. if \( \beta \) is either a successor ordinal or an ordinal of cofinality \( \lambda \), then there exists \( y \above x \) with height \( \beta \) such that \( \pi(y) = q \);
   b. if \( \beta \) is a limit ordinal of cofinality less than \( \lambda \), then there exists \( y \above x \) with height \( \beta \) such that \( \pi(y) \leq \mathbb{Q}_{\lambda} q \).

Observe that the underlying set of any standard \( \alpha \)-tree is equal to \( \lambda \cdot \alpha \).

\[ \text{Definition 2.2.} \] Let \((T, \pi)\) be a standard \( \alpha \)-tree and \((U, \tau)\) a standard \( \beta \)-tree, where \( \alpha < \beta \leq \lambda^+ \). Then \((U, \tau)\) extends \((T, \pi)\) if \( U \) is an end-extension of \( T \) and \( \tau \upharpoonright T = \pi \).

The next lemma is easy to check.

\[ \text{Lemma 2.3.} \] Let \( \alpha \leq \lambda^+ \) be a limit ordinal. Assume that \( \langle (T_\beta, \pi_\beta) : \beta < \alpha \rangle \) is a sequence such that each \( (T_\beta, \pi_\beta) \) is a standard \( \beta \)-tree and for all \( \beta < \gamma < \alpha \), \( (T_\gamma, \pi_\gamma) \) extends \( (T_\beta, \pi_\beta) \). Define \( T_\alpha := \bigcup\{T_\beta : \beta < \alpha\} \) and \( \pi_\alpha := \bigcup\{\pi_\beta : \beta < \alpha\} \). Then \( (T_\alpha, \pi_\alpha) \) is an \( \alpha \)-tree which extends \( (T_\beta, \pi_\beta) \) for all \( \beta < \alpha \).

\[ \text{Lemma 2.4.} \] Suppose that \( (T, \pi) \) is a standard \( \alpha \)-tree, where \( \alpha < \lambda^+ \) is a limit ordinal, and \( B \) is a set of cofinal branches of \( T \) of size less than \( \lambda \). Let \( z \in T \) and \( q \in \mathbb{Q}_\lambda \) with \( \pi(z) \leq \mathbb{Q}_{\lambda} q \). Then there exists a cofinal branch \( b \) of \( T \) satisfying that \( z \in b \), for all \( y \in b \), \( \pi(y) < \mathbb{Q}_{\lambda} q \), and \( b \notin B \).

\[ \text{Proof.} \] Fix an increasing and continuous sequence of ordinals \( \langle \alpha_i : i < \text{cf}(\alpha) \rangle \) which is cofinal in \( \alpha \) such that \( \alpha_0 = \text{ht}_T(z) + 1 \). Fix an increasing sequence \( \langle q_i : i < \text{cf}(\alpha) \rangle \) of elements of \( \mathbb{Q}_\lambda \) in between \( \pi(z) \) and \( q \).

We recursively build an increasing sequence \( \langle x_i : i < \text{cf}(\alpha) \rangle \) of nodes of \( T \) above \( z \) as follows. We will maintain that \( \alpha_i \) and \( \pi(x_i) \leq \mathbb{Q}_{\lambda} q_i \). To define \( x_0 \), consider the immediate successors of \( z \). By Definition 2.1(6a) and the fact that there are \( \lambda \) many elements of \( \mathbb{Q}_\lambda \) between \( \pi(z) \) and \( q_0 \), it follows that there are \( \lambda \) many immediate successors \( y \) of \( z \) such that \( \pi(y) < \mathbb{Q}_{\lambda} q_0 \). Since \( |B| < \lambda \), we can choose \( x_0 \) as such a \( y \) which does not belong to any branch in \( B \).
If $x_i$ is defined for some $i < \text{cf}(\alpha)$, apply Definition 2.1(6) to choose $x_{i+1}$ above $x_i$ of height $\alpha_{i+1}$ such that $\pi(x_{i+1}) \leq \beta, q_{i+1}$. If $x_j$ is defined for all $j < \delta$, where $\delta < \text{cf}(\alpha)$ is a limit ordinal, then since $\text{cf}(\delta) < \lambda$, we can let $x_\delta$ be the unique node of $T$ with height $\alpha_{\delta}$ which is above $x_j$ for all $j < \delta$. Then $\pi(x_\delta) = \sup\{\pi(x_j) : j < \delta\} \leq \sup\{q_j : j < \delta\} \leq q_\delta$. This completes the construction of the sequence. Let $b$ be the downward closure of $\{x_i : i < \text{cf}(\alpha)\}$. \hfill $\Box$

**Lemma 2.5.** Suppose that $\alpha < \lambda^+$ is either a successor ordinal or a limit ordinal of cofinality less than $\lambda$. Let $(T, \pi)$ be a standard $\alpha$-tree. Then there exists a standard $(\alpha+1)$-tree $(T^+, \pi^+)$ which extends $(T, \pi)$.

**Proof.** First, assume that $\alpha = \beta + 1$ is a successor ordinal, so that $T(\beta)$ is the top level of $T$. Partition $I_\alpha$ into $\lambda$ many subsets of size $\lambda$. Define $T^+$ so that the map sending a node of $T(\beta)$ to the set of immediate successors of that node is a bijection between $T(\beta)$ and the partition. Define $\pi^+$ which extends $\pi$ so that for each $x \in T(\beta)$, $\pi^+$ restricted to the set of immediate successors of $x$ in $T^+$ is a bijection between the set of such successors and the set $\{q \in \mathbb{Q}_\lambda : \pi(x) < \beta, q\}$.

Secondly, assume that $\alpha$ is a limit ordinal of cofinality less than $\lambda$. Since $\lambda^{<\lambda} = \lambda$, $T$ has exactly $\lambda$ many cofinal branches. Define $T^+$ which end-extends $T$ by adding a top level consisting of $I(\alpha)$, where we put exactly one node above each cofinal branch of $T$. For each $z \in T^+(\alpha)$ above a branch $b$, define $\pi^+(z) := \sup\{\pi(y) : y \in b\}$. Most of the requirements of $(T^+, \pi^+)$ being an $(\alpha+1)$-tree are obvious, but let us check $6b$ of Definition 2.1 carefully. Let $x \in T$ and $q \in \mathbb{Q}_\lambda$ with $\pi(x) < \beta, q$. Applying Lemma 2.4 with $B = \emptyset$, fix a cofinal branch $b$ of $T$ such that $x \in b$ and for all $x' \in b$, $\pi(x') < \beta, q$. Then the upper bound $y$ of $b$ on level $\alpha$ of $T^+$ is above $x$ and satisfies that $\pi(y) = \sup\{\pi(x') : x' \in b\} \leq \beta, q$. \hfill $\Box$

**Notation 2.6.** Let $\alpha < \lambda^+$. If $(T, \pi)$ is a standard $\alpha$-tree, define $X_{T, \pi} := \{(x, q) : x \in T, \pi(x) < \beta, q\}$.

Note that since $\lambda^{<\lambda} = \lambda$, $X_{T, \pi}$ has size $\lambda$.

**Definition 2.7.** Let $(T, \pi)$ be a standard $\alpha$-tree, where $\alpha \in S^\lambda_{\alpha^+}$. For a set $B$ of cofinal branches of $T$, a $\mathbb{Q}_\lambda$-tethering of $B$ is any bijection $h : X_{T, \pi} \rightarrow B$ satisfying that for all $(x, q) \in X_{T, \pi}$, $x \in h(x, q)$ and for all $y \in h(x, q)$, $\pi(y) < \beta, q$. If there exists a $\mathbb{Q}_\lambda$-tethering for $B$, then $B$ is $\mathbb{Q}_\lambda$-tethered.

**Definition 2.8.** Let $\alpha \leq \lambda^+$ and suppose that $(T, \pi)$ is a standard $\alpha$-tree. A downwards closed subtree $T'$ of $T$ is said to be avoidable if for all $x \in T$ and $q \in \mathbb{Q}_\lambda$ with $\pi(x) < \beta, q$, there exists some $y \in T \setminus T'$ above $x$ such that $\pi(y) < \beta, q$.

**Lemma 2.9.** Let $(T, \pi)$ be a standard $\alpha$-tree, where $\alpha \in S^\lambda_{\alpha^+}$, and suppose that $T'$ is a downwards closed subtree of $T$ which is avoidable. Then there exist disjoint $\mathbb{Q}_\lambda$-tethered sets of cofinal branches $B_0$ and $B_1$ of $T$ satisfying that every branch in $B_0$ or $B_1$ contains a node which is not in $T'$.

**Proof.** Let $\langle(x_i, q_i) : i < \lambda\rangle$ enumerate $X_{T, \pi}$. We will define $B_0 = \{b_{0,i} : i < \lambda\}$ and $B_1 = \{b_{1,i} : i < \lambda\}$ by recursion.

Let $i < \lambda$ and assume that $b_{0,j}$ and $b_{1,j}$ are defined for all $j < i$. Consider $(x_i, q_i)$. Then $\pi(x_i) < \beta, q_i$. Since $T'$ is avoidable, we can fix $z$ above $x_i$ in $T \setminus T'$ such that $\pi(z) < \beta, q_i$. Applying Lemma 2.4, fix a cofinal branch $b_{0,i}$ of $T$ satisfying that $z \in b_{0,i}$, for all $y \in b_{0,i}$, $\pi(y) < \beta, q_i$, and $b_{0,i}$ is not a member of $\{b_{n,j} : j < i, n < 2\}$.
Applying Lemma 2.4 again, fix a cofinal branch \( b_1 \) of \( T \) satisfying that \( z \in b_{1,i} \) for all \( y \in b_{1,i}, \pi(y) <_{\mathcal{Q}_\lambda} q_i \), and \( b_{1,i} \) is not a member of \( \{b_{n,j} : j < i, n < 2\} \cup \{b_{0,i}\} \).

This completes the construction. The maps given by \( h_0(x,q_i) := b_{0,i} \) and \( h_1(x,q_i) := b_{1,i} \) for all \( i < \lambda \) are \( \mathcal{Q}_\lambda \)-tetherings of \( B_0 \) and \( B_1 \) respectively.

\[ \square \]

**Lemma 2.10.** Let \((T, \pi)\) be a standard \( \alpha \)-tree, where \( \alpha \in S^+_\lambda \), and \( T' \) a downwards closed subtree of \( T \). Suppose that \( f : T' \upharpoonright c \to T \upharpoonright c \) is injective, order preserving, and height preserving, where \( c \subseteq \alpha \) is a club. Assume that \( x \in T' \upharpoonright c, f(x) \neq x \), and \( T'_x \) is not avoidable. Fix \( x^* \in T'_x \) and \( q \in \mathcal{Q}_\lambda \) with \( \pi(x^*) <_{\mathcal{Q}_\lambda} q \) such that for all \( y > x^* \) in \( T \), if \( \pi(y) <_{\mathcal{Q}_\lambda} q \) then \( y \in T'_x \). Fix \( q^* \) with \( \pi(x^*) <_{\mathcal{Q}_\lambda} q^* <_{\mathcal{Q}_\lambda} q \). Then there exists a set \( B \) of cofinal branches of \( T \) and a \( \mathcal{Q}_\lambda \)-tethering \( h \) of \( B \) such that \( b := h(x^*, q^*) \) satisfies that \( b \subseteq T'_x \), but the downward closure of \( f[b \cap (T \upharpoonright c)] \) is not in \( B \).

**Proof.** Let \( \langle (x_i,q_i) : i < \lambda \rangle \) be an enumeration of \( X_{T,\pi} \) such that \( x_0 = x^* \) and \( q_0 = q^* \). We define \( B = \{b_i : i < \lambda \} \) by recursion.

Begin by choosing \( b_0 \) to be some cofinal branch of \( T \) which contains \( x_0 \) and satisfies that for all \( y \in b_0, \pi(y) <_{\mathcal{Q}_\lambda} q_0 \); this is possible by Lemma 2.4. Observe that \( b_0 \subseteq T'_x \). Let \( b_0 \) be the downward closure in \( T \) of \( f[b_0 \cap (T \upharpoonright c)] \). Note that \( b_0 \neq b_0 \) since \( x_0 \in b_0, f(x) \in b_0 \), and \( b_0 \) cannot contain two distinct elements from the same level of \( T \).

Let \( 0 < i \) and suppose that \( b_0 \) is defined for all \( j < i \). Applying Lemma 2.4, fix a cofinal branch \( b_i \) of \( T \) satisfying that \( x_i \in b_i \), for all \( y \in b_0, \pi(y) <_{\mathcal{Q}_\lambda} q_i \), and \( b_i \) is not a member of the set \( \{b_j : j < i\} \cup \{b_i\} \). This completes the construction of \( B \). Now define \( h(x_i,q_i) := b_i \) for all \( i < \lambda \).

\[ \square \]

**Lemma 2.11.** Suppose that \((T, \pi)\) is a standard \( \alpha \)-tree, where \( \alpha \in S^+_\lambda \), and \( B \) is a set of cofinal branches of \((T, \pi)\) and \( h \) a \( \mathcal{Q}_\lambda \)-tethering of \( B \). Then there exists a standard \((\alpha + 1)\)-tree \((T', \pi')\) which extends \((T, \pi)\) satisfying that \( T'^{\alpha} \downarrow = B \) and for all \((x, q) \in X_{T,\pi}, \), if \( z \) is the upper bound of \( h(x,q) \) in \( T'^{\alpha} \) then \( \pi'^{\alpha}(z) = q \).

**Proof.** Extend \( T \) to \( T' \) by letting the top level of \( T' \) be \( I_\alpha \) and placing one node above each branch in \( B \). If \( z \) is an upper bound of \( b = h(x,q) \), then for all \( y \in b, \pi(y) <_{\mathcal{Q}_\lambda} q \). Thus, it makes sense to define \( \pi'^{\alpha}(z) := q \). Now easily \((T', \pi')\) is as required.

\[ \square \]

**Lemma 2.12.** Suppose that \((T, \pi)\) is a standard \( \alpha \)-tree, where \( \alpha \in S^+_\lambda \), and \( T' \) is a downwards closed subtree of \( T \) which is avoidable. Then there exists a standard \((\alpha + 1)\)-tree \((T'^{\alpha}, \pi'^{\alpha})\) which extends \((T, \pi)\) such that every member of \( T'^{\alpha}(\alpha) \downarrow \) contains an element which is not in \( T' \).

**Proof.** Immediate from Lemma 2.9, just using one of the two sets of branches mentioned there, and Lemma 2.11.

\[ \square \]

**Lemma 2.13.** Suppose that \((T, \pi)\) is a standard \( \alpha \)-tree, where \( \alpha \in S^+_{\lambda} \). Then there exist standard \((\alpha + 1)\)-trees \((T^0, \pi^0)\) and \((T^1, \pi^1)\) which extend \((T, \pi)\) and satisfy that \( T^{\alpha} \downarrow \) and \( T^{\alpha} \downarrow \) are disjoint.

**Proof.** Immediate from Lemma 2.9, letting \( T' = \emptyset \), and Lemma 2.11.

\[ \square \]

**Lemma 2.14.** Let \((T, \pi)\) be a standard \( \alpha \)-tree, where \( \alpha \in S^+_{\lambda} \), and \( T' \) a downwards closed subtree of \( T \). Suppose that \( f : T' \upharpoonright c \to T \upharpoonright c \) is injective, order preserving, and height preserving, where \( c \subseteq \alpha \) is a club. Assume that \( x \in T' \upharpoonright c, f(x) \neq x \), and
$T'$ is not avoidable. Fix $x^* \in T'$ and $q \in \mathbb{Q}_\lambda$ with $\pi(x^*) <_{\mathbb{Q}_\lambda} q$ such that for all $y$ above $x^*$ in $T$, if $\pi(y) <_{\mathbb{Q}_\lambda} q$ then $y \in T'$. Then there exists a standard $(\alpha + 1)$-tree $(T', \pi')$ which extends $(T, \pi)$, there exists $b \in T'(\alpha)^\downarrow$ for which $x$ and $x^*$ are in $b$ and $b \subseteq T'$, and there exists $z \in T'(\alpha)$ which is an upper bound of $b$ such that $\pi^+ (z) <_{\mathbb{Q}_\lambda} q$, but the downward closure of $f[b \cap (T \upharpoonright c)]$ is not in $T'(\alpha)^\downarrow$.

**Proof.** Immediate from Lemmas 2.10 and 2.11. 

**Lemma 2.15.** Suppose that $(T, \pi)$ and $(U, \tau)$ are standard $\alpha$-trees, where $\alpha \in S^\lambda_\omega$, $c \subseteq \alpha$ is a club, $T'$ is a downwards closed subtree of $T$ which is not avoidable, and $f : T' \upharpoonright c \to U \upharpoonright c$ is injective and order preserving. Let $x \in T$ and $q \in \mathbb{Q}_\lambda$ be such that $\pi(x) <_{\mathbb{Q}_\lambda} q$ and for all $y$ above $x$ in $T$, if $\pi(y) <_{\mathbb{Q}_\lambda} q$ then $y \in T'$.

Then there exist standard $(\alpha + 1)$-trees $(T', \pi')$ and $(U', \tau')$ which extend $(T, \pi)$ and $(U, \tau)$ respectively, there exists $b \in T'(\alpha)^\downarrow$ which contains $x$, and there exists $z \in T'(\alpha)$ which is an upper bound of $b$ such that $\pi^+ (z) <_{\mathbb{Q}_\lambda} q$, satisfying that $b \subseteq T'$ but $f[b \cap (T \upharpoonright c)]$ does not have an upper bound in $U'$.

**Proof.** Using Lemma 2.13, fix a standard $(\alpha + 1)$-tree $(T', \pi')$ which extends $(T, \pi)$. Using Lemma 2.13 again, fix standard $(\alpha + 1)$-trees $(T^0, \pi^0)$ and $(T^1, \tau^1)$ which extend $(U, \tau)$ such that the sets $U^0(\alpha)^\downarrow$ and $U^1(\alpha)^\downarrow$ are disjoint.

As $(T', \pi')$ is a standard $(\alpha + 1)$-tree, we can fix $z \in T'(\alpha)$ above $x$ such that $\pi(z) <_{\mathbb{Q}_\lambda} q$. Let $b := \{y \in T : y <_T z\}$. Note that for all $y \in b$ above $x$, $\pi(y) <_{\mathbb{Q}_\lambda} \pi(z) = q$, and hence $y \in T'$ by the choice of $x$ and $q$. Since $T'$ is downwards closed, $b \subseteq T'$.

Let $b_f$ be the downward closure of the chain $f[b \cap (T \upharpoonright c)]$ in $U$. Then $b_f$ is a cofinal branch of $U$. Since the sets $U^0(\alpha)^\downarrow$ and $U^1(\alpha)^\downarrow$ are disjoint, $b_f$ does not belong to one of them. Fix $j < 2$ such that $b_f \notin U^j(\alpha)^\downarrow$. Then $b_f$ has no upper bound in $U^j(\alpha)$. Define $(U^+, \pi^+)$ as $(U^j, \pi^j)$. Then $f[b \cap (T \upharpoonright c)]$ does not have an upper bound in $U^+$.

**3. A Pairwise Far Family from Diamond**

We now use the results of the preceding section to prove our main theorem.

**Theorem 3.1.** Let $\lambda$ be a regular cardinal such that $\lambda^{< \lambda} = \lambda$. Assume $\diamondsuit(S^\lambda_\omega)$. Then there exists a pairwise far family of $2^{\lambda^+}$ many normal $\lambda$-complete special $\lambda^+$-Aronszajn trees which are completely club rigid.

Using $\diamondsuit(S^\lambda_\omega)$, fix a sequence $\langle (x_\alpha, s_\alpha, t_\alpha, f_\alpha) : \alpha \in S^\lambda_\omega \rangle$, where each $x_\alpha \subseteq \alpha$ and $s_\alpha, t_\alpha, f_\alpha \subseteq \alpha^2$, such that for any sets $X \subseteq \lambda^+$ and $G, H, F \subseteq (\lambda^+)^2$, there are stationarily many $\alpha \in S^\lambda_\omega$ such that $X \cap \alpha = x_\alpha$, $G \cap \alpha^2 = s_\alpha$, $H \cap \alpha^2 = t_\alpha$, and $F \cap \alpha^2 = f_\alpha$.

We will recursively construct a family $\{ (T_s, \pi_s) : s \in \leq^{< \lambda^+} 2 \}$ satisfying that each $(T_s, \pi_s)$ is a standard dom$(s)$-tree and if $s \subseteq t$ then $(T_t, \pi_t)$ extends $(T_s, \pi_s)$. After this construction is complete, we will define a large subset of $\lambda^+$ which will index the required pairwise far family.

As the base case, let $T_0 := \emptyset$ and $\pi_0 := \emptyset$. For each $s \in \{0, 1\}$, let $T_s$ be the tree of height one whose single level equals $I_0$ and let $\pi_s$ be any bijection from $\alpha$ onto $\mathbb{Q}_\lambda$.

Suppose that $\alpha \subseteq \lambda^+$ is a limit ordinal and $(T_s, \pi_s)$ is defined as required for all $s \subseteq \alpha^* \alpha^* 2$. For any $s \in \alpha^*$, define $T_s := \bigcup \{ T_{s|\gamma} : \gamma < \alpha \}$ and $\pi_s := \bigcup \{ \pi_{s|\gamma} : \gamma < \alpha \}$. The validity of this step follows from Lemma 2.3.
Now assume that $\alpha + 1$ is a successor ordinal and $(T_s, \pi_s)$ is defined as required for all $s \leq \alpha$. We split the definition into two cases. In the first case, assume that $\alpha$ is either a successor ordinal or a limit ordinal of cofinality less than $\lambda$. Fix $s \in \alpha^+$, and we will define $(T_t, \pi_t)$ for $t \in \{s^-, 0, s^+\}$. Using Lemma 2.5, fix an $(\alpha + 1)$-tree $(T_s^+, \pi_s^+)$ which extends $(T_s, \pi_s)$. Now define $(T_s^+, \pi_s^+)$ for all $s \in \alpha^2$ except for a few exceptional functions $s$ which we will discuss momentarily.

In the second case, assume that $\alpha \in S^+_{\lambda_\alpha}$. Let us define $T_s^-$ and $T_s^-$ for all $s \in \alpha^2$ except for a few exceptional functions $s$ which we will discuss momentarily. For all $s$ other than the possible exceptions, apply Lemma 2.13 to fix an $(\alpha + 1)$-tree $(T_s^+, \pi_s^+)$ which extends $(T_s, \pi_s)$. Now define $(T_s^-, \pi_s^-)$ and $(T_s^-, \pi_s^-)$ as $(T_s^+, \pi_s^+)$. Now we discuss the possible exceptions by considering the objects $s_0, t_0$, and $f_{s_0}$. We assume that $\alpha = \lambda \cdot \alpha$; if not, then there are no exceptions and we are done. There are four mutually exclusive cases we will consider. If none of these cases occurs, then there are no exceptions and we are done.

Case 1: The objects $s_0$ and $t_0$ are in $\alpha^2$, $s_0 = t_0$, and there exists a downwards closed subtree $T' \in T$ of $T_{s_0}$ and a club $c \subseteq \alpha$ such that $f_{s_0} : T' \upharpoonright c \rightarrow T_{s_0} \upharpoonright c$ is injective, order preserving, height preserving, and not the identity. Note that $T'$ is the downward closure of $T' \upharpoonright c$ in $T_{s_0}$, and hence it is uniquely determined from $f_{s_0}$. Let $s := s_0 = t_0$ and $f := f_{s_0}$.

Subcase 1a: There exists $x \in T' \upharpoonright c$ such that $f(x) \neq x$ and $T_s'$ is avoidable. Fix such an $x$. Since $T_s'$ is avoidable and downwards closed, applying Lemma 2.12 we can fix a standard $(\alpha + 1)$-tree $(T_s^+, \pi_s^+)$ which extends $(T_s, \pi_s)$ such that every member of $T_s^+(\alpha) \downarrow$ contains an element which is not in $T_s'$. Now define $(T_{s_0}^-, \pi_{s_0}^-)$ and $(T_{s_0}^-, \pi_{s_0}^-)$ as $(T_s^+, \pi_s^+)$. Let $t := s_0 = t_0$ and $f := f_{s_0}$.

Subcase 1b: For all $x \in T' \upharpoonright c$ with $f(x) \neq x$, $T_s'$ is not avoidable. Since $f$ is not the identity, we can fix $x \in T_s \upharpoonright c$ such that $f(x) \neq x$. Fix $x^* \in T_s'$ and $q \in Q_\alpha$ with $\pi(x^*) <_{Q_\alpha} q$ such that for all $y$ above $x^*$ in $T_s$, if $\pi(y) <_{Q_\alpha} q$ then $y \in T_s'$. By Lemma 2.14 there exists a standard $(\alpha + 1)$-tree $(T_s^+, \pi_s^+)$ such that for some $b \in T_s^+(\alpha) \downarrow$, $x$ and $x^*$ are in $b$, $b \leq T_s'$, the upper bound $z$ of $b$ in $T_s^+$ satisfies that $\pi(z) <_{Q_\alpha} q$, and the downward closure of $f[b \cap (T \upharpoonright c)]$ is not in $T_s^+(\alpha) \downarrow$. Now define $(T_{s_0}^-, \pi_{s_0}^-)$ and $(T_{s_0}^-, \pi_{s_0}^-)$ as $(T_s^+, \pi_s^+)$. Let $s := s_0 = t_0 = t_0$, and $f := f_{s_0}$.

Case 2: The objects $s_0$ and $t_0$ are in $\alpha^2$, $s_0 \neq t_0$, and there exists a downwards closed subtree $T' \in T$ of $T_{s_0}$ and a club $c \subseteq \alpha$ such that $f_{s_0} : T' \upharpoonright c \rightarrow T_{s_0} \upharpoonright c$ is injective and order preserving. Note again that $T'$ is determined from $f_{s_0}$. Let $s := s_0$, $t := t_0$, and $f := f_{s_0}$.

Subcase 2a: $T'$ is avoidable. Since $T'$ is avoidable and downwards closed, applying Lemma 2.12 there exists a standard $(\alpha + 1)$-tree $(T_s^+, \pi_s^+)$ which extends $(T_s, \pi_s)$ such that every member of $T_s^+(\alpha) \downarrow$ contains an element which is not in $T'$. By Lemma 2.13, fix a standard $(\alpha + 1)$-tree $(T_s^+, \pi_s^+)$ which extends $(T_t, \pi_t)$. Now define $(T_{s_0}^-, \pi_{s_0}^-)$ and $(T_{s_0}^-, \pi_{s_0}^-)$ as $(T_s^+, \pi_s^+)$ and $(T_{t_0}^-, \pi_{t_0}^-)$. Let $s := s_0 = t_0 = t_0$, and $f := f_{s_0}$.

Subcase 2b: $T'$ is not avoidable. Since $T'$ is not avoidable, fix $x \in T_s$ and $q \in Q_\alpha$ such that $\pi_s(x) <_{Q_\alpha} q$ and for all $y$ above $x$ in $T_s$ with $\pi_s(y) <_{Q_\alpha} q$, $y \in T'$.
Applying Lemma 2.15, fix standard \((\alpha + 1)\)-trees \((T^+_\alpha, \pi^+_\alpha)\) and \((T^-_\alpha, \pi^-_\alpha)\) which extend \((T_\alpha, \pi_\alpha)\) and \((T_\alpha, \pi_\alpha)\) respectively, \(b \in T^+_\alpha(\alpha) \downarrow\) which contains \(x\), and \(z \in T^+_s(\alpha)\) which is an upper bound of \(b\) with \(\pi^+_s(z) <_{Q_\lambda} q\), such that \(b \subseteq T^\prime\) but \(f[b \cap (T^ \setminus c)]\) does not have an upper bound in \(T^+_s\). Now define \((T_s^0, \pi_s^0)\) and \((T_s^{-1}, \pi_s^{-1})\) as \((T^+_\alpha, \pi^+_\alpha)\) and \((T^-_{\alpha-1}, \pi^-_{\alpha-1})\) as \((T^+_\alpha, \pi^+_\alpha)\).

This completes the construction. Note that for all \(s \in \lambda^+\), \(T_s\) is a normal \(\lambda^\prime\)-complete special \(\lambda^+\)-Aronszajn tree.

We now define a large subcollection of \(\{T_s : s \in \lambda^+\} \|\) which will witness Theorem 3.1.  

**Definition 3.2.** For each \(X \subseteq \lambda^+\), define a function \(h_X : \lambda^+ \to 2\) by letting \(h_X(\alpha) = 1\) iff \(X \cap \alpha = x_\alpha\).

Observe that if \(X\) and \(Y\) are distinct subsets of \(\lambda^+\), then for any \(\alpha < \lambda^+\) greater than the least member of the symmetric difference of \(X\) and \(Y\), if \(h_X(\alpha) = 1\) then \(h_Y(\alpha) = 0\). Also there exists an \(\alpha\) such that \(h_X(\alpha) = 1\) by the diamond property.

**Notation 3.3.** For every set \(X \subseteq \lambda^+\), let \(T_X := T_{h_X}\) and \(\pi_X := \pi_{h_X}\).

We complete the proof by showing that for every set \(X \subseteq \lambda^+\), \(T_X\) is completely club rigid, and for all distinct subsets \(X\) and \(Y\) of \(\lambda^+\), \(T_X\) and \(T_Y\) are far.

Let \(X \subseteq \lambda^+\). Suppose for a contradiction that there exists a downwards closed normal subtree \(T_X^\prime\) of \(T_X\), a club \(C \subseteq \lambda^+\), and an injective, order preserving, and height preserving function \(F : T_X^\prime \setminus C \to T_X \setminus C\) which is not the identity.

By an elementary argument, we can find a club \(D \subseteq \lambda^+\) satisfying that for all \(\alpha \in D:\)

- \(\lambda \cdot \alpha = \alpha\);
- \(C \cap \alpha\) is club in \(\alpha\);
- \(F \upharpoonright \alpha = F \cap \alpha^2\);
- \(F \upharpoonright \alpha\) is not the identity;
- for all \(x \in T_X \setminus \alpha\) and \(q \in Q_\lambda\), if there exists some \(y\) above \(x\) in \(T_X \setminus T_X^\prime\) such that \(\pi_X(y) <_{Q_\lambda} q\), then there exists such \(y\) in \(T_X \setminus \alpha\).

Note that to obtain the last statement, we use the fact that \(\lambda^\lambda = \lambda\) and so \(Q_\lambda\) has size \(\lambda\).

By the diamond property, fix a stationary set \(S \subseteq S^\lambda_{\alpha}\) satisfying that for all \(\alpha \in S:\)

- \(x_\alpha = X \cap \alpha\);
- \(s_\alpha = h_X \cap \alpha^2\);
- \(t_\alpha = h_X \cap \alpha^2\);
- \(f_\alpha = F \cap \alpha^2\).

Observe that since \(h_X\) maps into 2, \(h_X \cap \alpha^2 = h_X \upharpoonright \alpha\).

Fix \(\alpha \in S \cap D\) and we will get a contradiction. Note that we are in Case 1 of the construction as witnessed by \(T^\prime := T_X \upharpoonright \alpha\) and \(c := C \cap \alpha\). Let \(s := s_\alpha = t_\alpha = h_X \upharpoonright \alpha\) and \(f := f_\alpha\). Since \(X \cap \alpha = x_\alpha, h_X(\alpha) = 1\). Hence, \(T_X \upharpoonright (\alpha + 1) = T_{h_X(\alpha+1)} = T_{s^{-1}}\). Also, since \(\lambda \cdot \alpha = \alpha\), the underlying set of \(T_s\) is \(\alpha\), so \(f = F \upharpoonright \alpha = F \upharpoonright T_s\).

We claim that for all \(x \in T^\prime \setminus c\) with \(f(x) \neq x\), \(T^\prime_x\) is not avoidable. Suppose for a contradiction that this is not the case. Then we are in Subcase 1a of the construction. By definition, for some \(x \in T^\prime \setminus c\) such that \(f(x) \neq x\) and \(T^\prime_x\) is avoidable, we have that every member of \(T_{s^{-1}(\alpha)} \downarrow\) contains an element which is
not in \( T'_x \). Since \( T'_X \) is normal, we can find \( z \in T'_X \) of height \( \alpha \) above \( x \). Then \( b := \{ y : y < T_x \} \) is in \( T_X(\alpha) \downarrow = T_{x-1}(\alpha) \downarrow \). But since \( T'_X \) is downwards closed and contains \( z, b \subseteq T'_x \), which is a contradiction.

So indeed for all \( x \in T' \setminus c \) with \( f(x) \neq x \), \( T'_x \) is not avoidable. Hence, we are in Subcase 1b of the construction. By definition, (a) there exists \( x \in T_s \in c \) such that \( f(x) \neq x \), (b) there exists \( x^* \in T'_s \) and \( q \in Q_{\lambda} \) with \( \pi(x^*) < q \), \( q \) such that for all \( y \) above \( x^* \) in \( T_s \), if \( \pi(y) < q \), then \( y \in T'_s \), and (c) there exists \( b \in T_{x-1}(\alpha) \downarrow \) and an upper bound \( z \) of \( b \) in \( T_{x-1}(\alpha) \) such that \( \pi_{x-1}(z) < q \), \( q \), the nodes \( x \) and \( x^* \) are in \( b, b \subseteq T' \), but the downward closure of \( f[b \cap (T \setminus c)] \) is not in \( T_{x-1}(\alpha) \downarrow \). In particular, the downward closure of \( f[b \cap (T \setminus c)] \) does not have an upper bound in \( T_X \).

We claim that \( z \in T'_X \). If not, then \( z \in T_X \setminus T'_X \) and \( \pi_X(z) = \pi_{x-1}(z) < q \). As \( \alpha \in D \), by the last property in the description of \( D \) it follows that there is some \( y \) above \( x^* \) in \( (T_X \uparrow \alpha) \setminus T'_X = T_s \setminus T' \) such that \( \pi_X(y) = \pi_s(y) < q \). But this contradicts the choice of \( x^* \) and \( q \).

Since \( F \uparrow \alpha = f, F(z) \) is an upper bound of the downward closure of \( f[b \cap (T \setminus c)] \) in \( T_X \), which is a contradiction.

Now let \( X \) and \( Y \) be distinct subsets of \( \lambda^+ \), and we will prove that \( T_X \) and \( T_Y \) are far. Suppose for a contradiction that there exist downwards closed normal subtrees \( T'_X \) and \( T'_Y \) of \( T_X \) and \( T_Y \) respectively, a club \( C \subseteq \lambda^+ \), and a club isomorphism \( F : T'_X \setminus C \to T'_Y \setminus C \).

By an elementary argument, we can find a club \( D \subseteq \lambda^+ \) satisfying that for all \( \alpha \in D \):

- \( \lambda \cdot \alpha = \alpha \);
- \( C \cap \alpha \) is club in \( \alpha \);
- \( F \uparrow \alpha = F \cap \alpha^2 \);
- \( \alpha \) is greater than the least member of the symmetric difference of \( X \) and \( Y \);
- \( h_X \uparrow \alpha \neq h_Y \uparrow \alpha \);
- for all \( x \in T_X \cup \alpha \) and \( q \in Q_{\lambda} \), if there exists some \( y \) above \( x \) in \( T_X \setminus T'_X \) such that \( \pi_X(y) < q \), then there exists such a \( y \) in \( T_X \uparrow \alpha \).

Note that to obtain the last statement, we use the fact that \( \lambda^{<\lambda} = \lambda \) and so \( Q_{\lambda} \) has size \( \lambda \).

By the diamond property, fix a stationary set \( S \subseteq S_{\lambda^+} \) satisfying that for all \( \alpha \in S \):

- \( x_\alpha = X \cap \alpha \);
- \( s_\alpha = h_X \cap \alpha^2 \);
- \( t_\alpha = h_Y \cap \alpha^2 \);
- \( f_\alpha = F \cap \alpha^2 \).

Observe that since \( h_X \) and \( h_Y \) map into 2, \( h_X \cap \alpha^2 = h_X \uparrow \alpha \) and \( h_Y \cap \alpha^2 = h_Y \uparrow \alpha \).

Fix \( \alpha \in S \cap D \) and we will get a contradiction. Since \( X \cap \alpha = x_\alpha \), \( h_X(\alpha) = 1 \). As \( \alpha \) is greater than the least member of the symmetric difference of \( X \) and \( Y \), it follows that \( h_Y(\alpha) = 0 \). Note that we are in Case 2 of the construction as witnessed by \( T' := T'_X \uparrow \alpha \) and \( c := C \cap \alpha \). Let \( f := f_\alpha \) and \( t := t_\alpha = h_Y \uparrow \alpha \). Then \( T_X \uparrow (\alpha + 1) = T_{h_X(\alpha+1)} = T_{x-1} \) and \( T_Y \uparrow (\alpha + 1) = T_{h_Y(\alpha+1)} = T_{t+0} \). Also, since \( \lambda \cdot \alpha = \alpha \), the underlying set of the tree \( T_s \) is \( \alpha \). So \( f = F \uparrow \alpha = F \uparrow T_s \).
We claim that \( T' \) is not avoidable. Suppose for a contradiction that \( T' \) is avoidable. Then we are in Subcase 2a. By definition, every member of \( T_{s-1}(\alpha) \) contains an element which is not in \( T' \). Fix a node \( z \in T_X \) of height \( \alpha \) and let \( b := \{ y \in T : y <_{T_X} z \} \). Then \( b \) has an upper bound in \( T_Y(\alpha) = T_{s-1}(\alpha) \). By assumption, \( b \) contains an element which is not in \( T' \). But \( z \in T_X \) and \( T_X \) is downwards closed, so \( b \subseteq T' \) which is a contradiction.

So indeed, \( T' \) is not avoidable and we are in Subcase 2b. By the definition of \( T_{s-l} \) and \( T_{l-0} \) in this case, (a) there exists \( x \in T_s \) and \( q \in Q_X \) such that \( \pi_x(y) <_{Q_X} q \) and for all \( y \) above \( x \) in \( T_s \) with \( \pi_y(y) <_{Q_X} q, y \in T' \), and (b) there exists \( b \in T_{s-l}(\alpha) \) which contains \( x \), has some upper bound \( z \in T_{s-1}(\alpha) \) such that \( \pi_{s-1}(z) <_{Q_X} q, b \subseteq T' \), and \( f[b \cap (T_s \cup c)] \) does not have an upper bound in \( T_{l-0} \).

We claim that \( z \in T_X \). If not, then \( z \in T_X \setminus T_X' \) and \( \pi_X(z) = \pi_{s-1}(z) <_{Q_X} q \). As \( \alpha \in D \), by the last property in the description of \( D \) it follows that there is some \( y \) above \( x \) in \( (T_X \upharpoonright \alpha) \setminus T_X' = T_s \setminus T' \) such that \( \pi_X(y) = \pi_y(y) <_{Q_X} q \). But this contradicts the choice of \( x \) and \( q \).

Now \( z \) is an upper bound of \( b \) in \( T_{s-l} \). As \( \alpha \in C, z \in T_X \setminus C. \) Since \( F \upharpoonright \alpha = f, F(z) \) is an upper bound of \( f[b \cap (T_s \cup c)] \) in \( T_Y(\alpha) = T_{l-0}(\alpha) \). So \( f[b \cap (T_s \cup c)] \) has an upper bound in \( T_{l-0} \), which contradicts the choice of \( b \).

4. Additional Results

In this section we briefly discuss two additional results which are related to Theorem 3.1.

There is an alternative version of Theorem 3.1 which involves Suslin trees in place of special Aronszajn trees.

**Theorem 4.1** ([3]). *Let \( \lambda \) be a regular cardinal such that \( \lambda^{<\lambda} = \lambda \). Assume \( \diamond(S^\lambda_\alpha^+) \). Then there exists a pairwise far family of \( 2^{\lambda^\omega} \) many normal \( \lambda \)-complete \( \lambda^+ \)-Suslin trees which are completely club rigid.*

This theorem can be proven by a modification of the proof of Theorem 3.1. Let us point out the main changes which are needed and refer the interested reader to [3] for the details. First, the lemmas of Section 2 can be adjusted to the Suslin tree setting with the following differences. Any references in the lemmas to order preserving maps of trees into \( Q_X \) are removed. The idea of a \( Q_X \)-tethered set of branches is replaced by a *normal* set of branches \( B \), which means for all \( x \in T \) there exists \( b \in B \) with \( x \in B \). We add a lemma which asserts the existence of a normal set of branches such that each branch in the collection meets a given maximal antichain of \( T \). The idea of an avoidable subtree is replaced with the simpler property of a subtree \( T' \) satisfying that for all \( x \in T \), there exists \( y \) above \( x \) in \( T \setminus T' \).

We then construct a family of Suslin trees with the following changes to Section 3. The diamond sequence is expanded to include a parameter \( y_\alpha \) for guessing maximal antichains. For \( \alpha \in S^\lambda_\alpha^+ \), we adjust the case division as follows. Add a new case in which \( y_\alpha \) is a maximal antichain of \( T_{s_\alpha} \), and then extend \( T_{s_\alpha} \) so that every branch in \( T_{s_\alpha-0}(\alpha) \) and \( T_{s_\alpha-1}(\alpha) \) meets \( y_\alpha \). This case will allow for the sealing off of potential antichains as in the classical construction of a Suslin tree from diamond. When \( y_\alpha \) is not a maximal antichain of \( T_{s_\alpha} \), we use Subcases 1b and 2b as before with the obvious modifications. Subcases 1a and 2a are no longer necessary by the following lemma.
Lemma 4.2. Let $\kappa$ be a regular uncountable cardinal and suppose that $T$ is a $\kappa$-Suslin tree. Then for any downwards closed subtree $T'$ of $T$ of size $\kappa$, there exists $x \in T'$ such that $T_x \subseteq T'$.

Proof. Suppose for a contradiction that $T'$ is a counterexample to the lemma. We build a sequence $\langle x_i, y_i : i < \kappa \rangle$ by recursion. Let $i < \kappa$ and suppose that for all $j < i$, $x_j$ and $y_j$ are defined. Choose some $x_i \in T'$ with height greater than the heights of $x_j$ and $y_j$ for all $j < i$. Since $T_{x_i}$ is not subset of $T'$ and $T'$ is downwards closed, we can find $y_i$ above $x_i$ in $T \setminus T'$. Now check that $\{y_i : i < \kappa\}$ is an antichain. \hfill $\square$

Now we discuss the second result. Abraham-Shelah [2] proved that assuming $2^\omega < 2^{\omega_1}$, there exists a family of $2^{\omega_1}$ many normal special $\omega_1$-Aronszajn trees which are pairwise non-club embeddable into each other. It is not known whether this theorem generalizes to cardinals greater than $\omega_1$. For example, it is an open question whether GCH implies the existence of two countably complete normal special $\omega_2$-Aronszajn trees which are not club isomorphic ([8]).

On the other hand, $2^\omega < 2^{\omega_1}$ is equivalent to the weak diamond principle on $\omega_1$, and weak diamond can be relativized to stationary sets. In order to provide a generalization of the Abraham-Shelah result to a cardinal $\lambda > \omega$ in place of $\omega_1$, it is natural to make the stronger assumption that the weak diamond principle holds on the critical cofinality, rather than assuming just $2^\lambda < 2^{\lambda^+}$. Thus, the next theorem appears to be the appropriate generalization of their result to higher cardinals.

Theorem 4.3. Let $\lambda$ be a regular cardinal such that $\lambda^{<\lambda} = \lambda$ and assume that the weak diamond principle holds on $S_\lambda^{\lambda^+}$. Then there exists a family of $2^{\lambda^+}$ many $\lambda$-complete normal club rigid special $\lambda^+$-Aronszajn trees which are pairwise not club embeddable into each other.

The proof of this theorem is essentially the same as the proof of the case $\lambda = \omega$ from [2], taking into account the standard methods for building higher Aronszajn trees. We refer the interested reader to [2] or [3] for the details.

References

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