1. Suppose that $T : X \rightarrow X$ is a continuous map of a compact metric space $X$ into itself. Suppose also that $X_1$ and $X_2$ are two closed forward-invariant ($T(X_i) \subset X_i$) subsets of $X$ such that $X_1 \cup X_2 = X$. Show that for every continuous function $\phi : X \rightarrow \mathbb{R}$,

$$P(\phi) = \max\{P(\phi|_{X_1}, T|_{X_1}), P(\phi|_{X_2}, T|_{X_2})\}.$$ 

2. Prove that each Bernoulli measure (on a full shift with a finite alphabet) is a Gibbs state of a continuous function.

3. Suppose that $\mu$ is a Gibbs measure for a continuous function $\phi : X \rightarrow \mathbb{R}$. Suppose also that $\psi = \phi + c + u - u \circ T$, where $c \in \mathbb{R}$ is a constant and $u : X \rightarrow \mathbb{R}$ is a continuous function. Prove that $\mu$ is a Gibbs state for $\psi$.

4. Give an example of a continuous transformation of a compact metric space with at least two measures of maximal entropy.

5. Let $T_\alpha(x) = x + \alpha (\text{mod} 1)$ be the rotation of the circle about an irrational angle $\alpha$. Show that for every real-valued continuous function $\phi$ on the circle, $P(\phi) = \int \phi dl$, where $l$ denotes the Lebesgue measure on the circle.

6. The formula $h_{\text{top}}(T^n) = n h_{\text{top}}(T)$, may suggest that $h_{\text{top}}(T \circ S) = h_{\text{top}}(T) + h_{\text{top}}(S)$. Show that this is not in general true even if $T$ and $S$ commute.

7. Suppose that $T : X \rightarrow X$ is a homeomorphism of a compact metric space $X$. Define $\Omega(T)$ to be the set of all points $x \in X$ such that for every open neighbourhood $V$ of $x$ there exists $n \geq 1$ such that $T^n(V) \cap V \neq \emptyset$. Show that $T(\Omega(T)) = \Omega(T)$, that $\Omega(T)$ is a closed subset of $X$ and that $h_{\text{top}}(T) = h_{\text{top}}(T|_{\Omega(T)})$.

8. Let $\sigma : \{1, 2, \ldots, d\}^\infty$ be the full (one-sided) shift. Fix real numbers $a_1, a_2, \ldots, a_d$. Define the function $\phi : \{1, 2, \ldots, d\}^\infty \rightarrow \mathbb{R}$ by the formula that

$$\phi(\omega_0 \omega_1 \ldots) = a_{\omega_0}.$$ 

Prove that 

$$P(\phi) = \log \sum_{j=1}^d e^{a_j}.$$ 

9. Fix a function $\phi : X \rightarrow \mathbb{R}$. Show that the function $t \mapsto P(t\phi)$, $t \in \mathbb{R}$, is Lipschitz continuous. Assuming that $\phi$ is everywhere positive, prove that the function $t \mapsto P(t\phi)$, $t \in \mathbb{R}$, is increasing.

10. Suppose that $\phi : X \rightarrow \mathbb{R}$ is a continuous function such that $h_{\text{top}}(T) > \sup(\phi) - \inf(\phi)$. Show that all equilibria of the function $\phi$ are of positive entropy.