Applications of Martin’s Axiom

1. Products of c.c.c. Spaces

We consider the question of when the product of two c.c.c. topological spaces is c.c.c.. The next lemma shows the question is the same for partial orders, topological spaces, and compact Hausdorff spaces. Recall that for any partial order \( P = (P, \leq) \) there is an associated topological space \( X_P \) on \( P \), where a neighborhood base at \( p \in P \) consists of the single open set \( N_p = \{ q \in P : q \leq p \} \). Clearly this space is not (except for very trivial partial orders) even \( T_1 \).

**Lemma 1.1.** (ZFC) Let \( \kappa \) be an infinite regular cardinal. Then the following are equivalent.  

1. There are \( \kappa \)-c.c. partial orders \( P, Q \) such that \( P \times Q \) is not \( \kappa \)-c.c.
2. There are \( \kappa \)-c.c. topological spaces \( X, Y \) such that the product \( X \times Y \) is not \( \kappa \)-c.c.
3. There are \( \kappa \)-c.c. compact Hausdorff spaces \( X, Y \), such that the product \( X \times Y \) is not \( \kappa \)-c.c.
4. There are complete Boolean algebras \( B, C \) such that their product \( B \times C \) is not \( \kappa \)-c.c.

**Proof.** Note first that for any partial order \( P \), \( P \) is \( \kappa \)-c.c. iff \( X_P \) is \( \kappa \)-c.c. Also \( X_{P \times Q} = X_P \times X_Q \). Suppose first that \( P, Q \) are \( \kappa \)-c.c. partial orders, but \( P \times Q \) is not \( \kappa \)-c.c. Then \( X_P, X_Q \) are \( \kappa \)-c.c. topological spaces. Since \( P \times Q \) is not \( \kappa \)-c.c., neither is \( X_{P \times Q} = X_P \times X_Q \). So (1) implies (2).

To see (2) implies (1), let \( X, Y \) be two topological spaces which are \( \kappa \)-c.c., but \( X \times Y \) is not \( \kappa \)-c.c. Let \( P \) be the collection of non-empty open sets in \( X \) ordered by \( U \leq V \) iff \( U \subseteq V \). Likewise define \( Q \). Clearly \( P, Q \) are partial orders. \( P \) and \( Q \) are \( \kappa \)-c.c. partial orders since a \( \kappa \) antichain in \( P \) would be a \( \kappa \) collection of pairwise disjoint open sets in \( X \). Let \( \{ W_\alpha \}_{\alpha < \kappa} \) be a \( \kappa \) family of pairwise disjoint open sets in \( X \times Y \). Without loss of generality we may assume that \( W_\alpha = U_\alpha \times V_\alpha \) is a product of basic open sets. Let \( p_\alpha = (U_\alpha, V_\alpha) \in P \times Q \). The \( p_\alpha \) form an antichain in \( P \times Q \) since if \( (U, V) \leq (U_\alpha, V_\alpha), (U_\beta, V_\beta) \) then \( U \subseteq U_\alpha \cup U_\beta, V \subseteq V_\alpha \cup V_\beta \), so \( U \times V \subseteq W_\alpha \cup W_\beta \). So, \( P \times Q \) is not \( \kappa \)-c.c.

(4) implies (1) trivially, since Boolean algebras are partial orders. For the other direction, let \( P, Q \) be as in (1). Let \( B \) be the regular open algebra of \( P \), and likewise define \( C \). Thus, \( B \) and \( C \) are complete Boolean algebras. Since every element of \( B \) (i.e., every regular open subset of \( X_P \)) contains a basic open subset of \( X_P \), it is clear that \( B \) is \( \kappa \)-c.c., and likewise for \( Q \). For \( p \in P \) let \( B_p = \text{intcl}(N_p) \in B \) and likewise define \( C_q \in C \) for \( q \in Q \). Let \( \{(p_\alpha, q_\beta)\}_{\alpha < \kappa} \) be an antichain in \( P \times Q \). Let \( B_\alpha = B_{p_\alpha} \), and likewise for \( C_\alpha \). Then the \( (B_\alpha, C_\alpha) \) form an antichain in \( B \times C \). For suppose \( (B, C) \leq (B_\alpha, C_\alpha), (B_\beta, C_\beta) \). We may assume \( B = B_p, C = C_q \). So, \( B_p \subseteq B_{p_\alpha} \cap B_{p_\beta} \). This says \( N_{p_\alpha} \) and \( N_{p_\beta} \) are dense below \( p \). Thus, there is an \( r \in P \) with \( r \leq p_\alpha, p_\beta \). Similarly, \( q_\alpha \) and \( q_\beta \) are compatible. Hence \( (p_\alpha, q_\alpha) \) and \( (p_\beta, q_\beta) \) are compatible, a contradiction.

(3) trivially implies (2). To finish we show that (1) implies (3). Given \( P \) and \( Q \), let \( B \) and \( C \) again be their completions. Let \( X' \) be the Stone space of the Boolean algebra \( B \), that is, \( X' \) is the collection of ultrafilters on \( B \). \( X' \) is a topological space with basic open sets of the form \( N_V = \{ U \in X' : V \subseteq U \} \) for \( V \) a regular open set in
$X'$. $X'$ is clearly Hausdorff, and it is a standard fact (see the following exercise) that $X'$ is compact. $X'$ is also $\kappa$-c.c., for if $N_{V_i}$ were pairwise disjoint open sets in $X'$, then the $V_i$ would be pairwise disjoint open sets in $B$, contradicting $B$ being $\kappa$-c.c. Likewise define $Y'$. To see $X' \times Y'$ is not $\kappa$-c.c., let $(p, q_n)$ again be a $\kappa$ size antichain in $P \times Q$. Let $B_\alpha$, $C_\alpha$ be as before, and consider the basic open sets $N_{B_\alpha} \times N_{C_\alpha}$ in $X' \times Y'$. These are pairwise disjoint since if $N_{B_\alpha} \cap N_{B_\beta} \neq \emptyset$ then $B_\alpha \cap B_\beta \neq \emptyset$, and then (as argued above) $p_\alpha$ and $p_\beta$ are compatible in $P$ (and likewise for $Q$).

We showed before that if there is a Suslin tree $T$, then $T$ gives a partial order which is $\kappa$-c.c. (i.e., $T$ with the reverse of the tree ordering) but for which $T \times T$ is not $\kappa$-c.c. We now show that CH also implies the non-productivity of $\kappa$-c.c.

**Theorem 1.2** (Galvin, Laver). Assume $\text{ZFC} + \text{CH}$. Then there are two $\kappa$-c.c. partial orders $P$, $Q$ whose product $P \times Q$ is not $\kappa$-c.c.

**Corollary 1.3.** Assuming CH, there are two $\kappa$-c.c. compact Hausdorff spaces whose product is not $\kappa$-c.c.

**Proof.** Suppose $f : (\omega_1)^2 \to \{0, 1\}$ is a partition of the pairs of countable ordinals. For $i \in [0, 1]$ we define the partial order $P_i$ to consist of all finite $p \subseteq \omega_1$ such that $f''p = \{i\}$ (that is, $P_i$ consists of finite sets which are homogeneous for color $i$).

We order $P_i$ by: $p \preceq q$ if $p \supseteq q$. No matter how we choose $f$, the product $P_0 \times P_1$ will not be $\kappa$-c.c., since the elements $(\alpha, \alpha)$ for $\alpha < \omega_1$ form an antichain. It remains to show that we can choose $f$ so that $P_0$ and $P_1$ are $\kappa$-c.c.

Let (using CH) $\{F^0_\alpha\}_{\alpha < \omega_1}$ enumerate all $\omega$ sequences $F_\alpha = (F^0_\alpha, F^1_\alpha, \ldots, F^n_\alpha, \ldots)$ of pairwise disjoint finite subsets of $\omega_1$. Suppose we have defined $f(\alpha, \beta)$ when $\max(\alpha, \beta) < \gamma$. We define $f(\alpha, \gamma)$ for $\alpha < \gamma$ as follows. Let $S_\gamma, n < \omega$, enumerate all objects of the form $S_\gamma = (F^k_{\alpha_n}, X_n, i_n)$ such that

1. $\alpha_n < \gamma$, $i \in [0, 1]$, and $X_n$ is a finite subset of $\gamma$.
2. $X_n \cap \bigcup l F^k_{\alpha_n} = \emptyset$ and there are infinitely many $k \in \omega$ such that $f''(F^k_{\alpha_n} \cup X_n) = \{i\}$.

By diagonalizing, there is a sequence $\{X_n\}_{n \in \omega}$ with each $X_n$ of the form $F^k_{\alpha_n}$ for some $n = n(l)$, $k = k(l)$, such that the $Y_i$ are pairwise disjoint, for each $n$ there are infinitely many $l$ such that $n(l) = n$, and $f''(Y_i \cup X_n(l)) = \{i_n(l)\}$). For $\alpha \in Y_i$, define $f(\alpha, \gamma) = i_n(l)$. The definition of $f$ gives us the following property.

(*) for any $\alpha < \gamma$, finite $X \subseteq \gamma$, and $i \in [0, 1]$, if $\sup(\cup F^0_\alpha) < \gamma$, $X \cap (\cup F^0_\alpha) = \emptyset$ and there are infinitely many $n$ such that $f''(F^0_{\alpha_n} \cup X)^2 = \{i\}$, then there are infinitely many $n$ such that $f''(F^0_{\alpha_n} \cup X)^2 = \{i\}$ and $f''(F^0_{\alpha_n} \cup \gamma)^2 = \{i\}$.

To see this works, suppose that $\{p_\alpha\}_{\alpha < \omega_1}$ was an $\omega_1$-antichain for $P_0$ (the other case being similar). Thinning to $\Delta$-system, we may assume that $p_\alpha = \tau \cup G_\alpha$, where $G_\alpha \cap G_\beta = \emptyset$ for $\alpha \neq \beta$. To get a contradiction it suffices to find $\alpha \neq \beta$ such that $f''(G_\alpha \cup G_\beta) = \{0\}$. Consider the first $\omega$ many elements, and fix $\alpha < \omega_1$ such that $F^0_\alpha = (G_0, G_1, \ldots, G_n, \ldots)$. Let $\delta > \alpha$ and $\delta > \sup \bigcup G_\alpha$. Fix one of the $G_\beta$ with $G_\beta \cap \delta = \emptyset$ (which we can do as the $G_\beta$ are pairwise disjoint).

Say $G_\beta = \{\gamma_1, \ldots, \gamma_m\}$. Using (*) $m$ times (at step $p$ we use $X = \{\gamma_1, \ldots, \gamma_{p-1}\}$, $\gamma = \gamma_p$) we thin the $G_n$ sequence to an infinite subsequence $(G_0', G_1', \ldots)$ such that $f''(G_0' \cup G_1')^2 = \{0\}$. This contradicts the $p_\alpha$ being an antichain.

We now show that $\text{MA} + \neg \text{CH}$ implies that the product of two $\kappa$-c.c. partial orders is $\kappa$-c.c.
Definition 1.4. A partial order $\mathbb{P}$ is strongly c.c.c. if whenever $A \subseteq P$ with $|A| = \omega_1$, then there is a $B \subseteq A$ with $|B| = \omega_1$ such that for any finite $\{p_1, \ldots, p_n\} \subseteq B$ there is a $p \in P$ with $p \leq p_1, \ldots, p_n$.

Lemma 1.5. Assume ZFC+$\neg$MA+$\neg$CH. Then every c.c.c. partial order is strongly c.c.c.

Proof. Let $P$ be c.c.c., and $A = \{p_\alpha\}_{\alpha < \omega_1}$. First we claim that there is a $p \in A$ such that any $q \leq p$ is compatible with $p_\alpha$ for uncountably many $\alpha$. For if not, then we could get $q_{\alpha_0} \leq p_{\alpha_0}$ (where $\alpha_0 = 0$) and $\alpha_1$ such that $q_{\alpha_0}$ is incompatible with all $p_\beta$ for $\beta \geq \alpha_1$. We could then get $q_{\alpha_1} \leq p_{\alpha_1}$ and $\alpha_2$ so that $q_{\alpha_1}$ is incompatible with all $p_\beta$ for $\beta \geq \alpha_2$. Continuing we define an $\omega_1$ antichain $\{q_\alpha\}_{\alpha < \omega_1}$.

We may assume $p_0$ is such that any $q \leq p_0$ is compatible with $\omega_1$ many $p_\alpha$.

Let $Q$ be the partial order consisting of all finite subsets $u = \{q_1, \ldots, q_n\} \subseteq P$ which contain $p_0$ and which are compatible, that is, there is a $q \in P$ with $q \leq q_1, \ldots, q_n$. We order $Q$ by $u \leq v$ iff $u \supseteq v$. We claim that $Q$ is c.c.c. For suppose $\{u_\alpha\}_{\alpha < \omega_1}$ were an uncountable antichain in $Q$. For each $\alpha < \omega_1$, let $r_\alpha \in P$ with $r_\alpha$ extending all of the elements of $u_\alpha$. Since $P$ is c.c.c., there are $\alpha \neq \beta$ such that $r_\alpha \parallel r_\beta$. Then $r$ extends all elements of $u_\alpha$ and $u_\beta$, so $u_\alpha \cup u_\beta \in Q$, a contradiction. Thus, $Q$ is c.c.c.

For each $\alpha < \omega_1$, let $D_\alpha \subseteq Q$ be those $u \in Q$ which contain a $p_\beta$ for some $\beta > \alpha$. Then $D_\alpha$ is dense in $Q$. For given $u \in Q$, let $q \in P$ extend all of the elements of $u$. Since $p_0 \in u$, $q$ is compatible with $\omega_1$ many of the $p_\beta$. Let $\beta > \alpha$ with $p_\beta \parallel q$. Then $u \cup \{p_\beta\} \in D_\alpha$.

By MA (and the fact that $\omega_1 < 2^{\omega}$ from $\neg$CH) there is a filter $G$ on $Q$ meeting all of the $D_\alpha$. Let $B = \cup G$, so $B \subseteq A$. Since $G$ meets all of the $D_\alpha$, $|G| = \omega_1$. Since $G$ is a filter, for any finite $\{u_1, \ldots, u_n\} \subseteq G$, there is a $p \in G$ extending all of the elements in $\bigcup_{i=1}^n u_i$. It follows that finite number of elements from $B$ have a common extension in $P$. □

Lemma 1.6. If $P$ is strongly c.c.c.and $Q$ is c.c.c., then $P \times Q$ is c.c.c.

Proof. Suppose $(p_\alpha, q_\alpha)$ were an uncountable antichain in $P \times Q$. Since $P$ is strongly c.c.c., we may thin the sequence so that for any finite subset $u$ of $\{p_\alpha\}_{\alpha < \omega_1}$ the elements of $u$ have a common extension (actually, all we need is that any two are compatible). Since $Q$ is c.c.c., get now $\alpha < \beta$ such that $q_\alpha \parallel q_\beta$. Then $p_\alpha \parallel p_\beta$ and $q_\alpha \parallel q_\beta$, so $(p_\alpha, q_\alpha) \parallel (p_\beta, q_\beta)$, a contradiction. □

Theorem 1.7. Assume ZFC+$\neg$MA+$\neg$CH. Then the product of two c.c.c. partial orders is c.c.c. Likewise the product of two c.c.c. topological spaces is c.c.c.

Proof. Immediate from lemmas 1.5 and 1.6. □

As a consequence, we have the following topological result.

Theorem 1.8. Assume ZFC+$\neg$MA+$\neg$CH. Then if $(X_\alpha, \tau_\alpha)_{\alpha \in I}$ are c.c.c. topological spaces, then their product $X = \prod_{\alpha \in I} X_\alpha$ is also c.c.c.

Proof. Suppose $\{V_\alpha\}_{\alpha < \omega_1}$ is an uncountable antichain (i.e., pairwise disjoint open sets) in $X$. Without loss of generality we may assume the $V_\alpha$ are basic open, say $V_\alpha = U_{\beta_1} \times \cdots \times U_{\beta_m}$, where $U_{\beta_i}$ is open in $X_{\beta_i}$ and $n$ depends on $\alpha$. We call $\{\beta_1, \ldots, \beta_m\}$ the support of $V_\alpha$. We may thin the antichain so that the supports form a $\Delta$-system with say root $r = \{\beta_1, \ldots, \beta_m\}$. Then clearly $\{W_\alpha\}$ also must form
an antichain, where \( W_\alpha \) is the product of the \( U_{\beta} \) for \( \beta_\alpha \in r \). This is a contradiction, as theorem 1.7 gives that \( X_{\beta_1} \times \cdots \times X_{\beta_n} \) is c.c.c.

2. Almost Disjoint Sets

If \( x, y \subseteq \omega \), we say \( x \) and \( y \) are almost disjoint if \( x \cap y \) is finite.

Lemma 2.1. There is a \( 2^\omega \) sequence \( \{ x_\alpha \}_{\alpha < 2^\omega} \) of infinite subsets of \( \omega \) such that if \( \alpha \neq \beta \) then \( x_\alpha \) and \( x_\beta \) are pairwise disjoint.

Proof. Let \( \{ y_\alpha \}_{\alpha < 2^\omega} \) be distinct infinite subsets of \( \omega \). Let \( f: \omega^{<\omega} \to \omega \) be one-to-one. Define \( x_\alpha = \{ f(y_\alpha \cap n) : n \in \omega \} \subseteq \omega \). Let \( n \in y_\alpha - y_\beta \), say then for any \( m > \max\{ f(y_\alpha \cap 1), \ldots, f(y_\alpha \cap n) \} \), if \( m \in x_\alpha \) then \( m \notin x_\beta \).

Theorem 2.2. Assume ZFC + MA. Then \( \forall \kappa < 2^\omega \left( 2^\kappa = 2^\omega \right) \). In particular, \( 2^\omega \) is regular.

Proof. We use the almost disjoint sets forcing. Fix a sequence \( \{ x_\alpha \}_{\alpha < 2^\omega} \) of infinite, almost disjoint subsets of \( \omega \). Fix \( \kappa < 2^\omega \), and we use this sequence to code subsets of \( \kappa \). Let \( A \subseteq \kappa \). We claim that there is an \( x \subseteq \omega \) such that for all \( \alpha < \kappa \) if \( \alpha \in A \) then \( x \cap x_\alpha \) is finite, and if \( \alpha \notin A \) then \( x \cap x_\alpha \) is infinite. This clearly suffices to show that \( 2^\kappa \leq 2^\omega \).

To see the claim, consider the partial order \( \mathbb{P} \) which consists of all pairs \( p = (s, F) \) where \( s \in 2^{<\omega} \) and \( F \subseteq A \) is finite. We regard \( s \) as the characteristic function of the subset we are trying to build. If \( p' = (s', F') \), then we define \( p' \leq p \) iff \( s' \) extends \( s, F \subseteq F' \), and for all \( n \in \text{dom}(s') - \text{dom}(s) \), \( s'(n) = 1 \) then for all \( \alpha \in F \) we have \( x_\alpha(n) = 0 \).

To see \( \mathbb{P} \) is c.c.c. simply note that \( (p, F) \) is compatible with \( (p', F') \) if \( s = s' \), and there are only countably many choices for \( s \).

For \( \alpha \in A \), let \( D_\alpha = \{ (s, F) : x_\alpha \in F \} \). Clearly \( D_\alpha \) is dense as we may extend any \( (s, F) \) to \( (s, F \cup \{ x_\alpha \}) \). For \( \alpha < \kappa, \alpha \notin A, \) and \( k \in \omega \), let \( D_{\alpha, k} = \{ (s, F) : \exists m \geq k \ (m \in x_\alpha \land s(m) = 1) \} \). To see this is dense, note that \( x_\alpha \) is almost disjoint from all the \( x_\beta \) for \( \beta \in F \). Hence there is an \( m \geq k \) such that \( m \in x_\alpha \setminus \bigcup_{\beta \in F} x_\beta \). Extend \( s \) to \( s' \) where \( s'(m) = 1 \) and \( s'(l) = 0 \) for all other \( l \in \text{dom}(s') - \text{dom}(s) \). Then \( (s', F) \leq (s, F) \) and \( (s', F) \in D_{\alpha, k} \).

By MA, let \( G \) be a filter on \( \mathbb{P} \) meeting all of the \( D_\alpha \) for \( \alpha \in A \) and all of the \( D_{\alpha, k} \) for \( \alpha \notin A \). Let \( x = \bigcup\{ s : \exists F \ (s, F) \in G \} \) be the real determined from \( G \). Suppose first that \( \alpha \in A \). Since \( D_\alpha \) is dense, let \( (s, F) \in G \cap D_\alpha \). Suppose \( (s', F') \in G \) as well. Since \( G \) is a filter, there is some \( (s'', F'') \in G \) with \( (s'', F'') \leq (s, F), (s', F') \). Thus, for all \( n \in \text{dom}(s'') - \text{dom}(s) \) we have \( s''(n) = 1 \to (n \notin x_\alpha) \). Hence \( x \cap x_\alpha \subseteq \{ n : s(n) = 1 \} \), so \( x \) is almost disjoint from \( x_\alpha \). Suppose now that \( \alpha \notin A \). For any \( k \in \omega \) there is an \( (s, F) \in G \cap D_{\alpha, k} \). Thus there is an \( m \in x \cap x_\alpha \) with \( m \geq k \). This shows \( x \cap x_\alpha \) is infinite.

3. Measure and Category

We first show that MA + \( \neg \text{CH} \) implies the \( < 2^\omega \) additivity of measure and category. First we consider category. Recall a subset of a Polish space is meager if it is contained in the countable union of closed nowhere dense sets. A set \( A \) is comeager if its complement is meager, equivalently if \( A \) contains a countable intersection of dense open sets. The Baire category theorem (for complete metric spaces) says that every comeager set is dense, in particular non-empty. It is immediate that
a countable union of meager sets is meager (equivalently, a countable intersection of comeager sets is comeager). The following result shows that with MA we may extend this to $<2^\omega$ size unions.

**Theorem 3.1.** Assume MA. Then the union of $<2^\omega$ meager sets in a Polish space is meager.

**Proof.** Let $\kappa < 2^\omega$ and fix dense open sets $U_\alpha$, $\alpha < \kappa$. We show that there are dense open sets $V_\alpha$ such that $\bigcap_{\alpha} V_\alpha \subseteq \bigcap_{\alpha < \kappa} U_\alpha$. Fix a base $\{B_\alpha\}_{\alpha<\omega}$ for the Polish space. Let $F$ consist of all finite sequences $p = \langle(W_0,F_0),\ldots,(W_n,F_n)\rangle$ where each $F_i \subseteq \kappa$ is finite, $W_i$ is a finite union of basic open sets, and $W_\alpha \subseteq \bigcap_{\beta < \alpha} U_\alpha$. If $p' = \langle(W'_0,F'_0),\ldots,(W'_n,F'_n)\rangle$ then we define $p' \leq p$ iff $m \geq n$ and for all $i \leq n$ we have $F'_i \supseteq F_i$ and $W'_i \supseteq W_i$.

$F$ is c.c.c. since if $m = n$ and $W'_i = W'_i$, then $p$ is compatible with $p'$. For each $\alpha < \kappa$, let $D_\alpha$ be those $p$ such that $\alpha \in F_i$ for some $i$. Clearly each $D_\alpha$ is dense. For each $i$ and basic open set $B_i$, let $E_{i,j}$ be those $p$ such that $W_\alpha \cap B_j \neq \emptyset$. Easily each $E_{i,j}$ is also dense. From MA, let $G$ be a filter on $F$ meeting all of these dense sets. Define $V_i$ to be the union of all the $W_i$ such that $(W_i,F_i)$ is the $i^{th}$ coordinate of a $p \in G$. Since $G$ meets each $E_{i,j}$, each $V_i$ is dense open. Let $x \in \bigcap_{i} V_i$. Fix $\alpha < \kappa$ and we show that $x \in W_\alpha$. Let $p \in G \cap D_\alpha$, say $p = \langle(W_0,F_0),\ldots,(W_n,F_n)\rangle$ with $\alpha \in F_i$. If $q = \langle(W'_0,F'_0),\ldots,(W'_m,F'_m)\rangle$ is also in $G$, then if $m \geq n$ we have $W'_i \subseteq W_i$ as well, since $p$ is compatible with $q$ (any common extension will have $i^{th}$ coordinate $(W,F)$ where $\alpha \in F$ and $W \supseteq W_i \cup W'_i$). Thus, $x \in V_i \subseteq U_\alpha$.

We now prove the analogous result for measure.

**Theorem 3.2.** Assume MA. Let $\mu$ be a $\sigma$-finite Borel measure on a separable metric space $X$. Then the union of $<2^\omega$ subsets of $X$ of $\mu$ measure 0 has $\mu$ measure 0.

**Proof.** We may assume that $\mu$ is finite, that is, a probability measure. Recall that any Borel probability measure on a metric space is regular, that is, for every Borel set $B \subseteq X$ and every $\epsilon > 0$, there is a closed set $F$ and an open set $U$ with $F \subseteq B \subseteq U$ and $\mu(U - F) < \epsilon$. Recall also that a set $A \subseteq X$ is defined to have measure 0 if it is contained in a Borel set of measure 0, and a set is measurable if is equal to a Borel set modulo a set of measure 0. The measurable sets form a $\sigma$-algebra containing the Borel sets, and $\mu$ extends to a countably additive measure on this algebra.

Fix $\kappa < 2^\omega$, and suppose $A_\alpha$, for $\alpha < \kappa$, are measure zero sets. Fix $\epsilon > 0$, and we show that there is an open set $U$ containing $A = \bigcup_{\alpha<\kappa} A_\alpha$ of measure $< \epsilon$. Let $F$ consist of all open sets in $X$ of measure $< \epsilon$. Define $p \leq q$ iff $p \supseteq q$. We show that $F$ is c.c.c. Suppose $\{p_\alpha\}_{\alpha<\omega}$ were an uncountable antichain in $F$. By thinning, we may assume that for some $\epsilon_1 < \epsilon$ that $\mu(p_\alpha) < \epsilon_1$ for all $\alpha$. For each $\alpha$ there is a finite union of basic open sets (with respect to a fixed countable base for $X$) $U_\alpha \subseteq p_\alpha$ such that $\mu(U_\alpha - U_\alpha) < (\epsilon - \epsilon_1)/2$. Fix $\alpha \neq \beta$ such that $U_\alpha = U_\beta$. Then $\mu(p_\alpha - p_\beta) \leq \mu(U_\alpha) + \mu(p_\alpha - U_\alpha) + \mu(p_\beta - U_\alpha) < \epsilon$.

For each $\alpha < \kappa$, let $D_\alpha$ be those $p_\alpha$ containing $A_\alpha$. Easily each $D_\alpha$ is dense. By MA, let $G$ be a filter on $F$ meeting all of the $D_\alpha$. Let $V$ be the union of all the open sets $p \in G$. Clearly $V$ contains each $A_\alpha$. We claim that $\mu(V) < \epsilon$. Since $G$ is a filter, any finite union of open sets $p \in G$ has measure $< \epsilon$. By countable additivity, any countable union of sets $p \in G$ has measure $\leq \epsilon$. However, in any
second countable space the union of a family of open sets is given by a union of a countable subfamily. Hence, $\mu(V) \leq \epsilon$. \hfill \Box

As a corollary we get a little more:

**Corollary 3.3.** Assume MA. Let $\mu$ be a $\sigma$-finite Borel measure on a separable metric space $X$. Then the algebra $\mathcal{M}$ of measurable sets is closed under $\kappa < 2^\omega$ unions and intersections for all $\kappa < 2^\omega$. Also $\mu$ is $\kappa$-additive for all $\kappa < 2^\omega$.

**Proof.** We prove this by induction on $\kappa < 2^\omega$. Suppose $\{A_\alpha\}_{\alpha < \kappa}$ are given with $A_\alpha \in \mathcal{M}$. We show that $A = \bigcup_{\alpha < \kappa} A_\alpha$ is in $\mathcal{M}$. Let $B_\alpha = A_\alpha - \bigcup_{\beta < \alpha} A_\beta$ be the disjointification of the $A_\alpha$. By induction, each $B_\alpha$ is in $\mathcal{M}$. From $\sigma$-finiteness, only countably many of the $B_\alpha$ can have non-zero $\mu$ measure. The union of the rest has $\mu$ measure 0 by theorem 3.2. Thus, $A$ is the union of countably many sets in $\mathcal{M}$ together with a measure 0 set (which is by definition measurable). Thus $A \in \mathcal{M}$. This shows $\mathcal{M}$ is $\kappa < 2^\omega$-additive. If the $A_\alpha$ are pairwise disjoint, then again only countably many of them can have non-zero $\mu$ measure, and the union of the rest has measure 0. The countable additivity of $\mu$ then gives that $\mu(A) = \sum \mu(A_\alpha)$. \hfill \Box

4. Random and Generic Reals

We introduce two new forcings for adding a real, one corresponding to measure and the other to category, $\mathbb{P}_m$ and $\mathbb{P}_c$. The category version turns out to be isomorphic (actually, have an isomorphic completion) to ordinary Cohen forcing, so this is not really a new forcing but just a different point of view. In the measure case, $\mathbb{P}_m$ will be a new forcing, which we call the random real forcing.

Let $X$ be a Polish space (e.g., $X = \mathbb{R}$), and $\mathcal{I}$ an ideal on $X$. For example, $\mathcal{I}$ could be the ideal $\mathcal{I}_m$ of Lebesgue measure 0 sets, or the ideal $\mathcal{I}_c$ of meager sets. We define an equivalence relation $\sim_I$ of the Borel subsets of $X$ by $A \sim_I B$ iff $A \Delta B \in \mathcal{I}$. For $B \subseteq X$ a Borel set, we let $[B]_\mathcal{I}$, or just $[B]$ if $\mathcal{I}$ is understood, denote the equivalence class of $B$. We let $\mathbb{P}_{\mathcal{I}}$ be the partial order whose elements are the equivalence classes $[B]$ (a Borel subset of $X$) with $B \notin \mathcal{I}$, ordered by $[A] \leq [B]$ iff $A - B \in \mathcal{I}$. This is easily well-defined and a partial order. In fact, $\mathbb{P}_{\mathcal{I}}$ is actually a Boolean algebra under the obvious operations (for example $[A] + [B] = [A \cup B]$, $-[A] = [X - A]$) provided we add $0 = [\emptyset]$ back in.

We say $\mathcal{I}$ is c.c.c. if the Boolean algebra $\mathbb{P}_{\mathcal{I}}$ is c.c.c. In other words, there is no $\omega_1$ sequence $A_\alpha$ of Borel subsets of $X$ such that for $\alpha \neq \beta$ we have $A_\alpha \Delta A_\beta \in \mathcal{I}$. For a general ideal, this will not be a complete Boolean algebra, but we have the following.

**Lemma 4.1.** If $\mathcal{I}$ is c.c.c. and countably-additive then $\mathbb{P}_{\mathcal{I}}$ is a complete Boolean algebra.

**Proof.** Consider a collection $\{[A_\alpha]\}_{\alpha < \kappa}$ of elements of the Boolean algebra. We show that the least upper bound $\sum [A_\alpha]$ exists. Let $\mathcal{B}$ be a maximal collection subject to being an antichain in $\mathbb{P}_{\mathcal{I}}$ and for each $b \in \mathcal{B}$ there is an $\alpha$ such that $b \subseteq [A_\alpha]$. Since $\mathbb{P}_{\mathcal{I}}$ is c.c.c. we may write $\mathcal{B} = \{[B_\alpha]\}$. Let $\mathcal{B} = \bigcup_n B_n$ (we have implicitly chosen representatives for the equivalence classes). We first claim that $[B] = \sum [B_n]$. Clearly $[B_n] \leq [B]$ for each $n$. On the other hand, if $[B']$ is also such that $[B_n] \leq [B']$ for each $n$, then by countable additivity of $\mathcal{I}$ we get that $B = \bigcup_n B_n \leq B'$ mod $\mathcal{I}$. Hence $[B]$ is the least upper bound of the $[B_n]$. 


Next we show that $[B]$ is the least upper bound of the $[A_\alpha]$. Since for each $n$, $[B_n] \preceq [A_\alpha]$ for some $\alpha$, clearly any upper bound for the $[A_\alpha]$ is an upper bound for the $[B_n]$. So, it suffices to show that $B$ is an upper bound for the $[A_\alpha]$. Fix $\alpha$, and we show that $[A_\alpha] \preceq [B]$. If not, then $A_\alpha - B \notin \mathcal{I}$. Then $[A_\alpha - B]$ could be added to the $[B_n]$, contradicting the maximality of $B$. \hfill \Box

**Exercise 1.** Show that for any Polish space $X$, the category forcing $\mathbb{P}_c$ is c.c.c. show that for any $\sigma$-finite Borel measure $\mu$ on a separable metric space $X$, the corresponding measure forcing $\mathbb{P}_m$ is c.c.c.

**Corollary 4.2.** $\mathbb{P}_c$ and $\mathbb{P}_m$ are complete Boolean algebras.