More Forcing Constructions

We use forcing to establish the consistency of some combinatorial principles which are of use in many constructions. First we introduce Jensen’s diamond principle.

Definition 0.1. ♦ is the statement: there are $A_\alpha \subseteq \alpha$ for $\alpha < \omega_1$ such that for all $A \subseteq \omega_1$, $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary.

Clearly ♦ implies CH as every subset of $\omega$ must appear among the $A_\alpha$. This already shows that ♦ is not provable in ZFC. In fact it is known that GCH does not imply ♦ (Jensen). ♦ is thus a powerful strengthening of CH, and is useful in places where CH alone is not sufficient. First we show that we may force to make ♦ true.

Lemma 0.2. Let $M$ be a transitive model of ZFC. Then there is a generic extension $M[G]$ in which ♦ holds.

Proof. Let $\mathbb{P}$ consist of all countable sequences $p = \{A_\beta\}_{\beta < \alpha}$ where $\alpha < \omega_1$ and $A_\beta \subseteq \beta$. We order $\mathbb{P}$ by extension, so $p \leq q$ iff $\text{dom}(p) \supseteq \text{dom}(q)$ and $p|\text{dom}(q) = q$. Let $G$ be $M$ generic for $\mathbb{P}$. Clearly $\mathbb{P}$ is countably closed, so $\omega_1^{M[G]} = \omega_1^M$. The generic can be identified with a sequence $\{A_\beta\}_{\beta < \omega_1}$. We show that this sequence witnesses ♦ in $M[G]$. Fix $A \subseteq \omega_1$, $A \in M[G]$, and a $C \subseteq \omega_1$, $C \in M[G]$ with (C is c.u.b.)$^M[G]$. Let $A = \tau_\sigma$, $C = \sigma_\beta$. Let $p_0 \in G$, $p_0 \Vdash (\sigma$ is c.u.b.). Let $G$ be the canonical name for the generic. We claim that $p_0 \Vdash \exists \alpha((\alpha \in \sigma) \land (\tau \cap \alpha = G(\alpha)))$.

Suppose not, and let $p_1 \leq p_0$, $p_1 \in G$ with $p_1 \Vdash \forall \alpha \lnot((\alpha \in \sigma) \land (\tau \cap \alpha = G(\alpha)))$.

Working in $M$, we construct a sequence of conditions $q_n = p_1 \geq q_1 \geq \ldots$ as follows. Assume $q_n$ has been defined. Let $q_{n+1} \leq q_n$ be such that

1. $\text{dom}(q_{n+1}) > \text{dom}(q_n)$ and there is a $\beta_n \in (\text{dom}(q_n), \text{dom}(q_{n+1}))$ such that $q_{n+1} \Vdash \beta_n \in \sigma$.

2. For every $\alpha < \text{dom}(q_n)$, either $q_{n+1} \Vdash \alpha \in \tau$ or $q_{n+1} \Vdash \alpha \notin \tau$.

There is no problem getting $q_{n+1}$ as $p_0 \Vdash (\sigma$ is unbounded $)$ and $\mathbb{P}$ is countably closed. Let $q = \bigcup q_n$. Let $\alpha = \text{dom}(q) = \bigcup \text{dom}(q_n)$. From (1) and the fact that $p_0 \Vdash (\sigma$ is closed $)$ it follows that $q \Vdash \beta_n \in \sigma$. From (2), there is a $B \subseteq \alpha$, $B \in M$, with $q \Vdash \tau \cap \alpha = B$. Let $q' = q \cup \{\alpha, B\}$. Then $q' \Vdash ((\alpha \in \sigma) \land (\tau \cap \alpha = G(\alpha)))$.

This contradicts the definition of $p_1$. \[\Box\]

The forcing used in lemma 0.2 is isomorphic to a dense subset of $\text{FN}(\omega_1, 2, \omega_1)$ ($\text{FN}(\omega_1, 2, \omega_1)$ is isomorphic to $\mathbb{P} = \text{FN}(A, 2, \omega_1)$, where $A = \{\langle \beta, \alpha \rangle : \beta < \alpha < \omega_1\}$.

The forcing used in the lemma is a dense subset of $\mathbb{P}$.) Thus we get:

Corollary 0.3. Let $M$ be a transitive model of ZFC and $G$ be $M$-generic for $\text{FN}(\omega_1, 2, \omega_1)^M$. Then ♦ holds in $M[G]$.

Exercise 1. Show that if $M$ is a transitive model of ZFC and $G$ is $M$-generic for $\text{coll}(\omega_1, \kappa)$, for some $\kappa \geq \omega_1$, then ♦ holds in $M[G]$. [hint: First identify $\text{coll}(\omega_1, \kappa) = \text{FN}(\omega_1, \kappa, \omega_1)$ with $\mathbb{P} = \text{FN}(A, \kappa, \omega_1)$ where $A = \{\langle \alpha, \beta \rangle : \alpha < \beta < \omega_1\}$.

A generic $G$ for $\mathbb{P}$ induces a $G^*: \omega_1^2 \to 2$ by $G^*(\alpha, \beta) = 0$ iff $G(\alpha, \beta)$ is even. Show that $G^*$ gives a ♦ sequence in $M[G]$.

We can generalize ♦ to higher cardinals.
Definition 0.4. Let $\kappa$ be a regular cardinal. Then $\diamondsuit_\kappa$ is the statement that there is a sequence $A_\alpha \subseteq \alpha$, $\alpha < \kappa$, such that for all $A \subseteq \kappa$, $\{\alpha < \kappa : A \cap \alpha = A_\alpha\}$ is stationary.

The proof of lemma 0.2 generalizes to give:

Lemma 0.5. Let $M$ be a transitive model of ZFC and $\kappa$ a regular cardinal of $M$. Then there is a generic extension $M[G]$ which preserves all cardinals and cofinalities $\leq \kappa$ in which $\diamondsuit_\kappa$ holds.\[\]

Proof. Let $P$ consist of all functions $p$ with $\text{dom}(p)$ an ordinal less than $\kappa$, and with $p(\alpha) \subseteq \alpha$ for all $\alpha \in \text{dom}(p)$. Clearly $P$ is $\kappa$ closed (since $\kappa$ is regular), and so $P$ preserves all cardinalities and cofinalities $\leq \kappa$. Let $G$ be $M$ generic for $P$. The proof that $\diamondsuit_\kappa$ holds in $M[G]$ is essentially identical to the proof of lemma 0.2. Fix $A = \tau_G \subseteq \kappa$, and $C = \sigma_G \subseteq \omega_1$ with $(C$ is c.u.b.)$^M[G]$ . Let $p_0 \in G$, $p_0 \vdash (\sigma$ is c.u.b.$)$. Suppose $p_1 \leq p_0$ with $p_1 \vdash \forall \alpha \neg((\alpha \in \sigma) \land (\tau \cap \alpha = G(\alpha)))$.\[\]

We construct the sequence $q_0 = p_1 \geq q_1 \geq q_2 \geq \ldots$ as before. We use now $P$ being $< \kappa$ closed to get (2). As before, let $q = \bigcup_n q_n$ and let $q^* = q \cup \{\langle B, \alpha \rangle\}$ where $\alpha = \text{dom}(q) = \bigcup_n \text{dom}(q_n)$ and $B \in M$, $q \vdash \tau \cap \alpha = B$. This is a contradiction exactly as above. □

The forcing of lemma 0.5 is again equivalent to $\text{FN}(\kappa, 2, \kappa)$, thus we have:

Corollary 0.6. Let $M$ be a transitive model of ZFC and $\kappa$ a regular cardinal of $M$. Let $G$ be $M$-generic for $\text{FN}(\kappa, 2, \kappa)^M$. Then $\diamondsuit_\kappa$ holds in $M[G]$.

There are two natural weakenings of $\diamondsuit$ which turn out to be equivalent to $\diamondsuit$ by a theorem of Kunen. One weakening is to replace “stationary” in the definition of $\diamondsuit$ with “non-empty.” The second is to allow countably many sets at each stage $\alpha < \omega_1$ to do the guessing. The following theorem gives the equivalence.

Theorem 0.7. (Kunen) The following are equivalent in ZFC.

(1) $\diamondsuit$

(2) There is a sequence $\{A_\alpha\}_{\alpha < \omega_1}$, where each $A_\alpha$ is a countable collection of subsets of $\alpha$, such that for every $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 : A \cap \alpha \in A_\alpha\}$ is stationary.

(3) There is a sequence $\{A_\alpha\}_{\alpha < \omega_1}$, where each $A_\alpha$ is a countable collection of subsets of $\alpha$, such that for every $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 : A \cap \alpha \in A_\alpha\}$ is non-empty.

Proof. First we show that (2) implies (1). Let $\pi : \omega \times \omega_1 \rightarrow \omega_1$ be a bijection. Let $D \subseteq \omega_1$ be c.u.b. such that for $\alpha \in D$, $\pi| \omega \times \alpha$ is a bijection between $\omega \times \alpha$ and $\alpha$. Let $\{A_\alpha\}_{\alpha < \omega_1} = \{A_\alpha^m\}_{\alpha < \omega_1}$ witness (2). We define $B_\alpha^{m,n}$ to be a $\diamondsuit$ sequence. If not, then for each $n$ let $E_n \subseteq \omega_1$, and $C_n$ be c.u.b. witnessing the failure of $B_\alpha^{m,n}$ to be a $\diamondsuit$ sequence. Define $E$ to code the $E_n$ by: $\alpha \in E$ iff $\pi^{-1}(\alpha) = (k, \beta)$ and $\beta \in E_k$.

By (2), let $\alpha \in D \cap \bigcap_n C_n$ such that $E \cap \alpha \in A_\alpha$. Say $E \cap \alpha = A^*_\alpha$. But then $B_\alpha^{m,n} = \{\beta < \alpha : \pi(n, \beta) \in A^*_\alpha\} = \{\beta < \alpha : \pi(n, \beta) \in E \cap \alpha\} = E_n \cap \alpha$, a contradiction to the definition of $E_n$ and $\alpha \in C_n$.

We next show (3) implies (2). Let $\{A_\alpha\}_{\alpha < \omega_1}$ be as in (3). Without loss of generality we may assume the $A_\alpha$ are increasing. We define a new sequence $\{A'_\alpha\}_{\alpha < \omega_1}$ as follows. We view subsets of $\omega$ as coding bounded subsets of $\omega_1$ in some manner.
(e.g., take a bijection between $2^\omega$ and $\omega_1^{<\omega}$). For successor ordinals let $A'_\alpha = A_\alpha$. For $\alpha$ limit let $A'_\alpha \supseteq A_\alpha$ be such that:

(i) For all $n \in \omega$ and all $A \in A_{\alpha+n}$, $\{ \beta < \alpha : 2 \cdot \beta \in A \} \in A'_\alpha$.

(ii) Suppose $A \in A_\alpha$, $\beta < \alpha$ is a limit, and $x = \{ n \in \omega : \beta + 2n + 1 \in A \}$ codes a subset $A_\beta$ of $\alpha$. Then $A_\beta \in A'_\alpha$.

To see this works, fix $A \subseteq \omega_1$ and a c.u.b. $C \subseteq \omega_1$. We may assume $C$ consists of limit ordinals. We must show that for some $\alpha \in C$ that $A \cap \alpha \in A'_\alpha$. Define $B \subseteq \omega_1$ as follows. For each limit $\beta$ let $x_\beta \subseteq \omega$ code $A \cap NC_\beta$, where $NC_\beta$ is the least element of $C$ greater than $\beta$. Let $B = (2 \cdot \gamma : \gamma \in A) \cup \{ \beta + 2n + 1 : \beta \text{ is limit } \land n \in x_\beta \}$.

By (3), fix now $\alpha < \omega_1$ such that $B \cap \alpha \in A_\alpha$. If $\alpha \in C$ then $A \cap \alpha \in A'_\alpha$ by (i) (with $n = 0$). Let $\gamma$ be the largest element of $C$ less than $\alpha$. If $\alpha < \gamma + \omega$, then by (i) we still have $A \cap \gamma \in A'_\alpha$. If $\alpha \geq \gamma + \omega$, then from (ii) we have that $A \cap \delta \in A'_\delta$, where $\delta = NC_\gamma$.

Thus, in all cases we have an $\alpha \in C$ with $A \cap \alpha \in A'_\alpha$. □

Another generalization of $\Diamond$ is the following.

**Definition 0.8.** Let $\kappa$ be a regular cardinal and $S \subseteq \kappa$ stationary. Then $\Diamond_S^S$ is the statement that there is a sequence $\{ A_\alpha \}_{\alpha \in S}$ with $A_\alpha \subseteq \alpha$, such that for all $A \subseteq \kappa$, the set $\{ \alpha \in S : A \cap \alpha = S_\alpha \}$ is stationary.

**Lemma 0.9.** Let $M$ be a transitive model of ZFC, and $\kappa$ a regular cardinal of $M$. Let $\lambda > (2^{<\kappa})^M$ be a cardinal of $M$. Let $G$ be $M$-generic for $\text{FN}(\lambda, 2, \kappa)$. Then in $M[G]$ we have that for every stationary $S \subseteq \kappa$, $\Diamond_S^S$ holds.

**Proof.** The forcing $\mathbb{P} = \text{FN}(\lambda, 2, \kappa)$ is $< \kappa$ closed in $M$, so all cardinalities and cofinalities $\leq \kappa$ are preserved. Also, $\mathbb{P}$ is $(2^{<\kappa})^+ \text{ c.c.}$ in $M$.

First we show that for every $S \in M$ which is stationary in $M[G]$ that $\Diamond_S^S$ holds in $M[G]$. Fix such an $S \in M$. Since $\lambda \equiv \kappa^2 \oplus \lambda$, $\text{FN}(\lambda, 2, \kappa) \equiv \text{FN}(\kappa^2, 2, \kappa) \times \text{FN}(\lambda, 2, \kappa)$. Let $P$ be the forcing of lemma 0.5, that is, conditions $p$ are functions with $\text{dom}(p) < \kappa$ and for all $\alpha \in \text{dom}(p)$, $p(\alpha) \subseteq \alpha$. Since $\mathbb{P} \times \text{FN}(\lambda, 2, \kappa)$ is dense in $\text{FN}(\kappa^2, 2, \kappa) \times \text{FN}(\lambda, 2, \kappa)$, we may view $G$ as a product $G = G_1 \times G_2$ where $G_1 \subseteq \mathbb{P}$, $G_2 \subseteq \text{FN}(\lambda, 2, \kappa)$ (more precisely, $M[G] = M[G_1][G_2]$ where $(G_1, G_2)$ is generic for the product).

Replacing $M$ by $M[G_2]$, it is enough to show that if $M$ is a transitive model of ZFC, $\kappa$ is regular in $M$, $G$ is $M$-generic for $\mathbb{P}$, and $S \in M$ is stationary in $M[G]$, then $M[G]$ satisfies $\Diamond_S^S$. To see this, let $A = \tau_G \subseteq \kappa$ and $C = \sigma_G$ be a.u.b. in $M[G]$. Suppose $p \forces \neg \exists \alpha \in S \cap G \cdot G(\alpha) = \tau \cap \alpha$. In $M[G]$ define

$$D = \{ \alpha < \kappa : \forall \beta < \alpha \exists \gamma, \delta < \alpha \cdot (\gamma > \beta \land G(\gamma) \prod \{ \gamma \in \sigma \}) \land (G(\delta) \prod \{ \beta \in \tau \sqcup \delta \prod \{ \beta \notin \tau \}) \}.$$ 

Easily $M[G]$ satisfies that $D$ is a a.u.b. subset of $\kappa$ (recall $\kappa$ is regular in $M[G]$). Since $S$ is stationary in $M[G]$, let $\alpha \in S \cap D$, with $\alpha > \text{dom}(p)$. Let $q = G(\alpha)$. Thus, $\alpha \leq p$. From $\alpha \in D$ we get that $q \not\forces (\dot{\alpha} \in \sigma)$. Also, there is a $B \in M$ such that $q \not\forces \tau \cap \alpha = B$. Let $q' = q \cup (\{ \alpha, B \})$. Then $q' \leq p$ and $q' \not\forces (\dot{\alpha} \in (S \cap \sigma) \cap G(\dot{\alpha}) = \tau \cap \dot{\alpha})$, a contradiction.

Returning to the proof of the theorem, suppose now $S \in M[G]$ is stationary in $M[G]$, where $G$ denotes our $\text{FN}(\lambda, 2, \kappa)$ generic. Assume for convenience that $\lambda$ is regular in $M$. For any $\rho < \lambda$, we may write $\text{FN}(\lambda, 2, \kappa) \equiv \text{FN}(\rho, 2, \lambda) \times \text{FN}(\lambda - \rho, 2, \kappa)$, and view $G$ as a product $G^-_\rho \times G^+_\rho$ accordingly. It suffices to show that
for some $\rho < \lambda$ that $S \in M[G^{-\rho}]$. For then we may apply the previous paragraph (using $M[G^{-\rho}]$ as our ground model) to conclude that $\mathcal{D}^S$ holds in $M[G]$. In fact, we show that any $S \subseteq \kappa, S \in M[G]$ lies in some $M[G^{-\rho}]$. Since $\mathcal{F} = \text{FN}(\lambda, 2, \kappa)$ is $\lambda$ c.c., there is a nice name for $S$ in $M$ of size $\kappa < \lambda$ (using $\lambda$ regular in $M$). Let $\tau$ be such a nice name, that is, $|\tau|^M < \lambda$. Clearly then, $S = \tau_G = \tau_{G^{-\rho}}$ for large enough $\rho$ (i.e., $\rho$ greater than the domains of all conditions in $\omega^\lambda(\tau)$). Thus, $S \in M[G^{-\rho}]$.

If $\lambda$ is not regular in $M$, the argument of the previous paragraph still goes through writing instead $\text{FN}(\lambda, 2, \kappa) = \text{FN}(T, 2, \kappa) \times \text{FN}(\lambda - T, 2, \kappa)$ for some $T \subseteq \lambda$ of size $\leq 2^{<\kappa} < \lambda$ (but $T$ may now be cofinal in $\lambda$).

1. Suslin’s Hypothesis

We say a linear order $(X, \prec)$ is dense if whenever $x < y$ then there is a $z \in X$ with $x < z < y$. We say $(X, \prec)$ is without endpoints if there is no least or maximal element of $X$. We say $(X, \prec)$ is separable if there is a countable set $D \subseteq X$ such that every non-empty interval $(x, y)$ contains a point of $D$. We say $(X, \prec)$ is c.c.c. if there is no uncountable family of pairwise disjoint open sets (equivalently, intervals). The linear order is separable or c.c.c. iff $X$ viewed as a topological space (with the order topology) is separable or c.c.c. We say $(X, \prec)$ is complete if every non-empty bounded set in $X$ has a least upper bound and a greatest lower bound.

Recall that for topological spaces in general, $2^{\omega^\beta}$ countable $\Rightarrow$ separable $\Rightarrow$ c.c.c. A countable product of $2^{\omega^\alpha}$ countable spaces is second countable, but an $\omega$ product of $\mathbb{R}_{\text{std}}$ is not $2^{\omega}$ countable (or even first countable). A $\leq 2^\omega$ length product of separable spaces is separable, but a $(2^\omega)^+$ product of $\mathbb{R}_{\text{std}}$ is not separable. Finally, an arbitrary product of second countable spaces is c.c.c. (we’ll discuss products of c.c.c. more later).

The following lemma is an important but elementary fact from analysis. It is just asserting that $\mathbb{R}$ is the unique completion of $\mathbb{Q}$. The proof is left to the exercises.

**Lemma 1.1.** $(\mathbb{R}, \prec_{\text{std}})$ is the unique, up to isomorphism of linear orders, linear order $(X, \prec)$ satisfying:

1. $(X, \prec)$ is dense and without endpoints.
2. $(X, \prec)$ is complete.
3. $(X, \prec)$ is separable.

**Exercise 2.** (Cantor) Show that any two countable dense linear orders without endpoints are isomorphic. [hint: Use a “back and forth” argument. Construct $f = \bigcup f_n$ in $\omega$ stages, where each $f_n$ will be an isomorphism from a subset of the first order of size $n$ to a subset of the second order. At even stages $n = 2m$, arrange that the $m^{\text{th}}$ element of the first order is in the domain of $f_n$. At odd stages $n = 2m + 1$, arrange that the $m^{\text{th}}$ element of the second order is in the range of $f_n$.]

**Exercise 3.** Prove lemma 1.1. [hint: Start with exercise 2. Then extend the $f$ of that exercise to the completions of the countable dense sets, using the fact that both are complete.]

Note that the requirement that $(X, \prec)$ not have endpoints is rather trivial: if it has them then we can simply remove them without effecting the other properties of lemma 1.1.
A natural question, raised by Suslin, is whether the characterization of the real line, lemma 1.1, continues to hold if we weaken the requirement of separability to that of being c.c.c. The statement that it does is called Suslin’s hypothesis, SH.

**Definition 1.2.** Suslin’s hypothesis, SH is the statement that $(\mathbb{R}, \prec_{\text{std}})$ is the unique, up to isomorphism of linear orders, linear order $(X, \prec)$ which is dense, without endpoints, complete, and c.c.c.

A counterexample to SH is called a Suslin line. That is, a Suslin line is a linear order $(X, \prec)$ which is dense, without endpoints, complete, and c.c.c., but not separable. Thus, SH is the statement that there are no Suslin lines.

The following lemma says that the existence of a Suslin line is equivalent to the existence of a linear order $(X, \prec)$ which is c.c.c. but not separable, as the other properties can be easily arranged.

**Lemma 1.3.** If there is a linear order $(X, \prec)$ which is c.c.c. but not separable, then there is a Suslin line.

**Proof.** Let $(X, \prec)$ be a c.c.c. but not separable linear order. Define an equivalence relation on $X$ by $x \sim y$ iff $(x, y)$ is separable. Each equivalence class $[x]$ is an interval of $X$. Distinct equivalence classes correspond to disjoint intervals. The classes then inherit an order from $X$, namely $[x] < [y]$ iff $x < y$ (equivalently, all the points of $[x]$ are less than any point of $[y]$). Let $\tilde{X}$ be the set of equivalence classes with this induced order. We claim that $\tilde{X}$ is dense in itself, and c.c.c. but not separable. Granting this, we can then finish by removing the endpoints of $\tilde{X}$, if any, then taking the completion. It is easy to check that the completion is still c.c.c. and not separable (see the following exercise). First note that every equivalence class $I = [x]$ is separable. To see this, let $(x_n, y_n)$ be a maximal family of pairwise disjoint intervals contained in $I$ (which must be countable as $X$ is c.c.c.). Let $D_n$ be dense in $(x_n, y_n)$, and let $D = \bigcup_n D_n$. Then $D$ together with the first and last elements of $I$ (if any) is dense in $I$.

To see $\tilde{X}$ is dense in itself, suppose $[x] < [y]$. If $([x], [y]) = \emptyset$, then $(x, y) \subseteq ([x] \cup [y])$, and thus $D \cup E$ is dense in $(x, y)$ where $D$ is dense in $[x]$ and $E$ is dense in $[y]$. Thus, $x \sim y$, so $[x] = [y]$, a contradiction.

To see $\tilde{X}$ is not separable, if $\{[d_n]\}_{n \in \omega}$ were dense in $\tilde{X}$, then let for each $n D_n \subseteq [d_n]$ be dense in $[d_n]$. Then $D = \bigcup_n D_n$ is dense in $X$, a contradiction.

To see $\tilde{X}$ is c.c.c., suppose $([x_n], [y_n])$ is an antichain in $\tilde{X}$. Then $(x_n, y_n)$ is an antichain in $X$, so must be countable. $\square$

**Exercise 4.** Let $(X, \prec)$ be a linear order. The completion $\tilde{X}$ of $X$ can be defined by adding points $\hat{x}$ for all cuts (bounded above, downward closed sets) $S \subseteq X$ which do not have a least upper bound in $X$. The point $\hat{x}$ is greater than any element of $S$ but less than any element of $X$ which is greater than all the elements of $X$. Show that $\hat{x}$ is in $\tilde{X}$. Show that if $\hat{x} \in \tilde{X} - X$, then $X$ has no largest element below $\hat{x}$, and $X$ has no least element above $\hat{x}$. Show that $X$ is separable iff $\tilde{X}$ is separable. [if $D$ is countable dense in $X$, show that $D \cup E$ is dense in $X$ where $D = \hat{D} \cap X$ and $E$ is chosen so that for all nonempty intervals $(d_1, d_2)$ where $d_1$, $d_2 \in \hat{D}$ we have $E \cap (d_1, d_2) \neq \emptyset$.] Show that $X$ is c.c.c. iff $\tilde{X}$ is c.c.c. [if $(\hat{x}, \hat{y})$ is a non-empty interval in $\tilde{X}$, show that there are $x, y \in X$ with $\hat{x} \leq x < y \leq \hat{y}$ and $(x, y)$ non-empty in $X$.]
Suslin’s hypothesis we formulated by Suslin around 1920. It was reformulated in terms of a certain kind of tree, called Suslin trees, by Kurepa in the 30’s. As we will see, SH is independent of ZFC, by results from the late 60’s. Jech and Tennenbaum showed the consistency of $\text{ZFC} + \neg\text{SH}$, and Solovay and Tennenbaum the consistency of $\text{ZFC} + \text{SH}$. Jensen showed that Suslin trees exist in $L$, that is, $\neg\text{SH}$ holds in $L$ (we discuss these points in more detail below).

2. Various Trees

As we mentioned, Kurepa introduced the notion of a Suslin tree, and showed $\neg\text{SH}$ is equivalent to the existence of a Suslin tree. In other words, the existence of a Suslin line is equivalent to the existence of a Suslin tree. We prove this in this section, and introduce a few other types of trees of interest.

**Definition 2.1.** A tree is a partially ordered (i.e., transitive, irreflexive) set $(T, <_T)$ with the property that for every $x \in T$, $\{y \in T : y <_T x\}$ is well-ordered. For $x \in T$ we write $|x|_T$ (or just $|x|$ if $T$ is understood) to denote the order-type of $\{y \in T : y <_T x\}$. We call this the rank or height of $x$ in $T$. The height of $T$ is defined by $|T| = \sup \{|x|_T + 1 : x \in T\}$. By the $\alpha$th level of $T$ we mean the set of $x \in T$ of height $\alpha$. A chain of $T$ is a subset which is linearly ordered by $<_T$. A branch $b$ of $T$ means a chain which is closed downwards (i.e., if $x \in b$ and $y <_T x$, then $y \in b$). An antichain $A$ of $T$ is a subset of $T$ of pairwise incomparable elements.

We are interested in trees of size $\kappa$, for various cardinals $\kappa$, which satisfy a certain non-triviality condition:

**Definition 2.2.** Let $\kappa \in \text{CARD}$. A $\kappa$-tree is a tree $T$ of height $\kappa$ such that all levels of the tree have size $< \kappa$. That is, $\forall \alpha < \kappa \ | \{x \in T : |x|_T = \alpha\}| < \kappa$.

We now introduce several particular trees of interest.

**Definition 2.3.** Let $\kappa$ be a cardinal. A $\kappa$ Aronszajn tree is a $\kappa$ tree with no branch of size $\kappa$. An Aronszajn tree refers to an $\omega_1$ Aronszajn tree.

Requiring more we get the notion of a Suslin tree.

**Definition 2.4.** Let $\kappa$ be a cardinal. A $\kappa$ Suslin tree is a $\kappa$ tree with no chains of size $\kappa$ and no antichains of size $\kappa$. A Suslin tree refers to an $\omega_1$ Suslin tree.

In other words, a $\kappa$ Suslin tree is a $\kappa$ Aronszajn tree with no antichains of size $\kappa$.

**Definition 2.5.** Let $\kappa$ be a cardinal. A $\kappa$ Kurepa tree is a $\kappa$ tree with $\geq \kappa^+$ many branches of length $\kappa$. A Kurepa tree refers to an $\omega_1$ Kurepa tree.

It is not immediately clear if any of these kinds of trees exist. We will see that Aronszajn trees exist in ZFC, but the existence of Suslin and Kurepa trees is independent of ZFC.

If desired, a $\kappa$ tree can, with perhaps a fairly trivial modification, be viewed as a subtree of $(\kappa^-)^\kappa$, where $\kappa^- = \sup \{\lambda \in \text{CARD} : \lambda < \kappa\}$. The modifications required are: we assume the tree has a single root, that is, a single element of height 0 (if not, we add one), and if $x \neq y \in T$ have limit height, then $\{z : z <_T x\} \neq \{z : z <_T y\}$ (this can be arranged by adding extra elements at limit levels, one for each branch of that height, that sit immediately below the old points of that limit height). Given these adjustments, it is now straightforward to define an isomorphism $\pi$ between
T and a subtree of \((\kappa^-)^\kappa\). If \(x \in T\) has height \(\alpha\), then \(\pi(x)\) will be a sequence with domain \(\alpha\) (define \(\pi(x)\) by induction on \(|x|_T\), at limit stages take unions of the \(\pi(y)\) for \(y <_T x\), and at successor steps use the fact that any \(x \in T\) has \(< \kappa\) many immediate extensions).

**Definition 2.6.** A tree \(T\) is branching if every \(x \in T\) has at least two distinct immediate extensions. \(T\) is said to be pruned if it has a single root and for every \(x \in T\) and every \(\alpha < |x|_T\) (with \(\alpha < |T|\)) there is a \(y \in T\) of height \(\alpha\) with \(x <_T y\).

In other words, a pruned tree has the property that every element of the tree has arbitrarily high extensions in the tree.

If \(T\) is a \(\kappa\) tree and \(\kappa\) is regular, then there is a canonical subtree \(T'\) of \(T\) which is a pruned \(\kappa\) tree. Let \(T'\) be those \(x \in T\) which have \(\kappa\) many extensions in \(T\) (which implies \(x\) has extensions of arbitrary height). It is easy to check that \(T'\) is downward closed subtree of \(T\), and that it is pruned (except it may have more than one root; in that keep only the part of \(T'\) above a particular root). [To see it is pruned, take \(x \in T'\). Let \(\alpha > |x|_T\). Let \(S \subseteq T\) be \(\kappa\) many extensions of \(x\), all of which have height \(> \alpha\). \(\kappa\) many elements of \(S\) must extend a single \(y \in T\) of height \(\alpha\). Then \(y \in T'\).]

Before investigating these trees, we first make the connection with Suslin’s hypothesis.

3. **Suslin Trees and Suslin’s Hypothesis**

The following lemma connects Suslin’s hypothesis with Suslin trees.

**Lemma 3.1.** (ZFC) There is a Suslin line iff there is a Suslin tree.

**Proof.** Suppose first that \((X, <)\) is a Suslin line. We construct the tree \(T\) out of the intervals \(I = (x, y)\) in \(X\). Rather than take all intervals (which does not give a tree), we pick the intervals \(I_\alpha = (x_\alpha, y_\alpha)\), for \(\alpha < \omega_1\), inductively so that they do form a tree. If \(I_\beta\) for \(\beta < \alpha\) have been defined, let \(C = \bigcup_{\beta < \alpha} \{x_\beta, y_\beta\}\) be the set of endpoints so far constructed. This set is countable, so it is not dense in \(X\) (recall \(X\) is not separable). Let \(I_\alpha\) be a non-empty interval with \(I_\alpha \cap C = \emptyset\). Continue to define \(I_\alpha\) for all \(\alpha < \omega_1\). If \(\alpha \neq \beta\), then \(I_\alpha\) and \(I_\beta\) are either disjoint, or one is contained in the other. Let \(T\) be the set whose elements are the intervals \(I_\alpha\) constructed, and define \(I <_T J\) iff \(J \subseteq I\). \((T, <_T)\) is easily a tree (note that if \(I_\alpha\) and \(I_\beta\) both contain \(J\), and \(\alpha < \beta\) then \(I_\alpha \supseteq I_\beta\). Thus the \(<_T\) predecessors of \(J\) are ordered by their indices.). We show that \(T\) is a Suslin tree. An uncountable antichain of \(T\) would be an uncountable family of intervals of \(X\) which are pairwise disjoint, a contradiction since \(X\) is c.c.c. (if \(I, J\) in \(T\) are not disjoint, then one contains the other so they are comparable in \(T\)). Suppose there were an uncountable branch, say \(J_0 <_T J_1 < \cdots < J_\alpha < \cdots\). If \(x_\alpha\) denote the left endpoint of \(J_\alpha\), then the \(x_\alpha\) form an \(\omega_1\) increasing sequence from \(X\). Then the intervals \((x_\alpha, x_{\alpha+1})\) are non-empty, pairwise disjoint, a contradiction. Thus, \(T\) is a Suslin tree.

Suppose next that \(T\) is a Suslin tree, and we construct a Suslin line. Let \(X\) be the set of maximal branches of \(T\). We fix an order on \(T\) (say by identifying it with \(\omega_1\)) and order the branches of \(T\) lexicographically. This defines the linear ordering \((X, <)\). To see it is c.c.c., suppose \((x_\alpha, y_\alpha)\), \(\alpha < \omega_1\) was an \(\omega_1\) sequence of pairwise disjoint non-empty intervals. Let \(z_\alpha \in (x_\alpha, y_\alpha)\). Let \(\beta_0\) be the least ordinal \(< |z_\alpha|\) such that \(x_\alpha(\beta_0) \neq z_\alpha(\beta_0)\), and let \(\beta_1\) be the least ordinal \(< |z_\alpha|\)
such that $y_\alpha(\beta) \neq z_\alpha(\beta)$. Let $\beta = \max\{\beta_0, \beta_1\}$. Let $a_\alpha = z(\beta) \in T$. Then $\{a_\alpha\}$ is an uncountable antichain in $T$, a contradiction.

To see $(X, \prec)$ is not separable, let $A = \{b_\gamma\}$ be a countable subset of $X$. Choose $\alpha < \omega_1$ of height greater than the supremum of the heights of the branches $b_\gamma$. There is some $z \in T$ of height $\alpha$ which has three distinct extensions (in fact, $\omega_1$ many) in $T$ (as $T$ is an $\omega_1$ tree). This defines a non-empty interval in $X$ which contains only branches of length $> \alpha$. Hence, this gives a non-empty interval missing $A$, so $A$ is not dense.

This shows $(X, \prec)$ is c.c.c. but not separable. From lemma 1.3 this gives a Suslin line. Alternatively, we can modify the tree $T$ directly so that $(X, \prec)$ as just constructed is dense in itself. To do this, first prune $T$ so that, without loss of generality, every $x \in T$ has extensions to arbitrarily high levels (the pruned subtree is clearly still a Suslin tree). Then consider levels $T_0, T_1, \ldots$ of $T$ such that all points of $T$ of level $\alpha_n$ have $\omega$ many extensions at level $\alpha_{n+1}$. This defines a subtree $T'$ of $T$ (the union of the points at some level $\alpha_n$) which is still an $\omega_1$ tree (and thus still a Suslin tree). The tree $T'$ is $\omega$-splitting (i.e., every element has infinitely many immediate extensions). If we order the extensions of any point of $T'$ in order type $\mathbb{Q}$, then $X$ will clearly be dense in itself. We can then take the completion of $(X, \prec)$ to get a Suslin line.

4. ARONSZAJN TREES

Recall an Aronszajn tree is an $\omega_1$ with no $\omega_1$ branch. The next lemma shows we can construct them in ZFC.

**Lemma 4.1.** (ZFC) There is an $\omega_1$ Aronszajn tree.

We’ll give two construction of an Aronszajn tree.

**first proof.** We construct the tree as a subtree of $\mathbb{Q}^{\omega_1}$. The $\alpha$th level of the tree will consist of increasing sequences $t \in \mathbb{Q}^\alpha$ with sup$(t)$ finite (i.e., ran$(t)$ is a bounded set of rationals). We will have $t \prec u$ iff $u$ extends $t$. We construct the levels of the tree, $T\alpha$ inductively and will also satisfy $(\ast)$: for any $t \in T\alpha$ and any $\beta > \alpha$ and $q >$ sup$(t)$, there is a $u \in T\beta$ with $t \prec_T u$ and sup$(u) < q$.

If $T\alpha$ is defined, we let $T\alpha+1$ consist of all $t^\omega\prec q$ where $q \in \mathbb{Q}$ and $q >$ sup$(t)$. This clearly maintains $(\ast)$.

Suppose now $\alpha$ is a limit and $T\beta$ has been defined for all $\beta < \alpha$. For each $t \in T_{<\alpha} = \bigcup_{\beta < \alpha} T\beta$ and each $q >$ sup$(t)$, choose a sequence of ordinals $|t| \prec_\alpha \alpha_0 < \alpha_1 < \ldots$ with sup$_n \alpha_n = \alpha$ and choose rationals sup$(t) < r_0 < r_1 < \ldots$ with sup$_n r_n < q$. By $(\ast)$, choose then $t <_T t_0 <_T t_1 <_T t_2 <_T \ldots$ where $t_i \in T\alpha_i$ and sup$(t_i) < r_i$. Put then $u = \bigcup t_n$ in $T\alpha$. Clearly we have maintained $(\ast)$, and $T\alpha$ is countable. Thus, $T$ is an $\omega_1$ tree. It clearly has no $\omega_1$ branch, since that would give an $\omega_1$ sequence of distinct rationals.

**second proof.** We now give a second construction due to Kunen. We will construct a sequence $\{s_\alpha\}_{\alpha < \omega_1}$ satisfying:

1. $S_\alpha$ is a one-to-one function from $\alpha$ to $\omega$.
2. If $\alpha < \beta$ then $s_\beta \upharpoonright \alpha$ agrees with $s_\alpha$ except on a finite set.
3. $\omega - \text{ran}(s_\alpha)$ is infinite.

Granting this, we let $T$ be the set of all $s_\alpha \upharpoonright \beta$ where $\beta \leq \alpha$, that is, all initial segments of all of the $s_\alpha$. We order $T$ again by extension. Clearly $T$ is a tree. $T$ is
an $\omega_1$ tree from property (2), since there are countably many $s \in \alpha^\omega$ which agree with $s_\alpha$ up to a finite set. From (1) there are clearly no $\omega_1$ branches through $T$.

It remains to construct the $s_\alpha$, which we do by induction. For successor steps, let $s_{\alpha+1} = s_\alpha \cup u$ where $u \not\in \text{ran}(s_\alpha)$. Suppose $\alpha$ is a limit, and let $\{\alpha_n\}$ be an increasing sequence with supremum $\alpha$. Begin with $s_{\alpha_0}$. Get $t_{\alpha_1} = s_{\alpha_0} \cup u$ where $u$ is the result of changing $s_{\alpha_1}$ on a finite subset of $\alpha_1 - \alpha_0$ so that $t_{\alpha_1}$ is one-to-one. In general, let $t_{\alpha_{n+1}} = t_{\alpha_n} \cup u$ where $u$ is the result of changing $s_{\alpha_{n+1}}$ on a finite subset of $\alpha_{n+1} - \alpha_n$ so that $t_{\alpha_{n+1}}$ is one-to-one. We can do this from properties (2) and (3). Let $t = \bigcup_{n} t_{\alpha_n}$, then $t$ satisfies properties (1) and (2). We can then modify $t$ to get $s_\alpha$ by, for example, changing the values at the $\alpha_n$, say by $s_\alpha(\alpha_n) = t(\alpha_{2n})$. $s_\alpha$ now satisfies (1)-(3). \qed

The second proof modifies to get $\kappa$-Aronszajn trees for $\kappa$ a successor of a regular, assuming GCH.

**Lemma 4.2.** Let $\kappa = \lambda^+$ where $\lambda$ is regular and $2^{<\lambda} = \lambda$. Then there is a $\kappa$-Aronszajn tree.

**Proof.** We construct $s_\alpha$ for $\alpha < \kappa$ satisfying:

1. $S_\alpha$ is a one-to-one function from $\alpha$ to $\lambda$.
2. If $\alpha < \beta$ then $S_\beta|\alpha$ agrees with $S_\alpha$ except on a set of size $< \lambda$.
3. $\text{ran}(s_\alpha)$ is non-stationary in $\lambda$.

Granting this, we again let $T$ be the tree os initial segments of the $s_\alpha$. This is a $\kappa$ tree since for each $\alpha < \kappa$ there are at most $\lambda^{<\lambda} = 2^{<\lambda} = \lambda$ many $s \in \lambda^\alpha$ which agree with $s_\alpha$ except on a set of size $< \lambda$.

We construct the $s_\alpha$ by induction as before. Successor steps are trivial. Suppose $\alpha$ is a limit ordinal. We assume $\text{cof}(\alpha) = \lambda$, as the other case is easier. Fix $\{\alpha_i\}_{i < \lambda}$ increasing, continuous, and cofinal in $\alpha$. We construct sequence $t_{\alpha_i} \in \lambda^{\alpha_i}$ by induction on $i$ as before. For $i < \lambda$ a limit we take unions. Properties (1) and (2) are immediate, and (3) follows since a $< \lambda$ intersection of sets c.u.b. in $\lambda$ is c.u.b. in $\lambda$. We let $t_{\alpha_{i+1}} = t_{\alpha_i} \cup u$, where $u = s_{\alpha_{i+1}}|\{\alpha_{i+1} - \alpha_i\}$, except we change the values on $< \lambda$ many points of $\{\alpha_{i+1} - \alpha_i\}$ to get $t_{\alpha_{i+1}}$ one-to-one. Using (2) and (3) and the trivial fact that every c.u.b. subset of $\lambda$ has size $\lambda$, there is no problem defining $t_{\alpha_{i+1}}$ (we redefine $s_{\alpha_{i+1}}|\{\alpha_{i+1} - \alpha_i\}$ on a set of size $< \lambda$ to have values in a c.u.b. set which ran($t_\alpha$) misses). We still clearly have that each ran($t_\alpha$) misses a c.u.b. subset of $\lambda$, say $C_i$.

Let $t = \bigcup_{i < \lambda} t_{\alpha_{i+1}}$. $t$ then satisfies (1) and (2), and we must adjust it to get $s_\alpha$ also satisfying (3). Let $\{\beta_i\}_{i < \lambda}$ be an increasing, continuous sequence with $\beta_i \in \bigcap_{j < i} C_j$. Define $s_\alpha$ to be $t$, except $t$ takes value $\beta_i$ we define $s_\alpha$ to take value $\beta_{i+1}$. Clearly $s_\alpha$ then satisfies (3) (since $\{\beta_i\}_{i < \lambda}$ is c.u.b.). $s_\alpha$ still satisfies (2) since the modification of $t$ to $s_\alpha$ changes $< \lambda$ many values of $t|\alpha_i$ for any $i < \lambda$ (since ran($t|\alpha_i$) doesn’t contain any $\beta_j$ for $j > i$). \qed

If we don’t assume CH, then there may or may not be $\omega_2$ Aronszajn trees (Mitchell). For $\kappa$ strongly inaccessible, there is a $\kappa$ Aronszajn tree iff $\kappa$ is weakly compact, a mild large cardinal axiom (the existence of weakly compact cardinals is consistent with $V = L$). On the other hand, Jensen showed that in $L$, there is a $\kappa$ Suslin, hence a $\kappa$ Aronszajn, tree for all regular $\kappa$ which are not weakly compact.
5. Suslin Trees

Unlike Aronszajn trees, we cannot construct a Suslin tree in ZFC. We first show that ♦ implies that there is a Suslin tree.

**Theorem 5.1.** ♦ implies that there is a Suslin tree.

*Proof.* Fix a ♦ sequence \( \{A_\alpha\}_{\alpha<\omega_1} \). We construct the levels of the tree \( T_\alpha = \{ x \in t : |x|_T \leq \alpha \} \) by induction. We will have that \( T \subseteq \omega_1 \). We will also maintain that every \( x \in T \) has extensions to all higher levels. Let \( T_0 \) consist of just the ordinal 0. Given \( T_\alpha \), let \( T_{\alpha+1} \) be defined by extending every \( x \in T_\alpha \) to \( \omega \) many immediate extensions in \( T_{\alpha+1} \). Suppose now \( \alpha \) is limit. Let \( T_{<\alpha} = \bigcup_{\beta<\alpha} T_\beta \) be the part of the tree so far constructed. If \( A_\alpha \) is not a maximal antichain in \( T_{<\alpha} \), then define \( T_\alpha \) by picking for every \( x \in T_{<\alpha} \) a branch \( b_x \) of \( T_{<\alpha} \) of length \( \alpha \) containing \( x \), and extending this branch to a point in \( T_\alpha \). Suppose now that \( A_\alpha \) is a maximal antichain of \( T_{<\alpha} \). We define \( T_\alpha \) to “seal-off” this antichain, that is, prevent it from growing further. For each \( x \in T_{<\alpha} \), let \( b_x \) be a branch of \( T_{<\alpha} \) of length \( \alpha \) which contains \( x \) and some element of \( A_\alpha \). We can do this since every \( x \) is comparable with an element of \( A_\alpha \). \( T_\alpha \) is defined by extending each such \( b_x \) to a point of \( T_\alpha \).

Let \( T = \bigcup_{\alpha<\omega_1} T_\alpha \). Clearly \( T \) is a pruned \( \omega_1 \) tree. Suppose \( A \subseteq \omega_1 \) is a maximal antichain of \( T \). Let \( C \subseteq \omega_1 \) be c.u.b. such that for \( \alpha \in C \), \( T_{<\alpha} \subseteq A \cap \alpha \) is a maximal antichain of \( T_{<\alpha} \). From ♦, let \( \alpha \in C \) be such that \( A \cap \alpha = A_\alpha \). Then at stage \( \alpha \) in the construction we defined \( T_\alpha \) so that all elements of height \( \alpha \) extend an element of \( A_\alpha \). This shows that \( A \cap \alpha \) is a maximal antichain of \( T \), so \( A = A \cap \alpha \), and thus \( A \) is countable.

**Corollary 5.2.** It is consistent with ZFC that there is a Suslin line.

Constructing \( \kappa \)-Suslin trees for higher regular (non weakly compact) cardinals requires more that ♦. However, it is easier to force directly the existence of these trees. In fact, this was the original argument of Jech and Tennenbaum. To get a Suslin tree we can force with either countable trees (Jech) or finite trees (Tennenbaum). We sketch both proofs. The first proof works for all regular cardinals as well.

For the first proof (\( \kappa \) now a regular cardinal of \( \mathcal{M} \)), let the partial order \( \mathbb{P} \) consist of pruned trees \( T, |T|, \kappa \), of height \( \alpha + 1 \) for some \( \alpha < \kappa \) (i.e., for any \( x \in T \) has an extension to the highest level \( \alpha \) of \( T \)). For convenience we also require \( T \) to be splitting, and we assume also \( T \leq \kappa \), that is, the elements of \( T \) are ordinals less than \( \kappa \). We define \( T_1 \leq T_2 \) iff \( |T_1| \geq |T_2| \) and \( T_1 \upharpoonright |T_2| = T_2 \) (here \( T \upharpoonright \beta \) denotes the elements of \( T \) of height \( < \beta \) in \( T \)). Thus, \( T_1 \) must extend \( T_2 \) “vertically.”

\( \mathbb{P} \) is countably closed, but not in general \( < \kappa \) closed (the problem is with the pruned condition; an increasing union of conditions of length \( \omega_1 \) may fail to be extendible to a condition as there may be no branches cofinal in the union). However, \( \mathbb{P} \) is \( < \kappa \) distributive, which is enough to get that \( \mathbb{P} \) preserves all cofinalities and cardinalities \( \leq \kappa \). To see this, let \( \{D_\rho\}_{\rho<\kappa}, \rho < \kappa, \) be a \( < \kappa \) collection of dense sets in \( \mathbb{P} \). We define conditions \( p_\eta \) of height \( \alpha_\eta + 1 \) inductively. We will have \( p_0 \geq p_1 \geq \ldots \).

As we define \( p_\eta \) we will also define for each \( x \in p_\eta \) a function \( f_x \) which gives a branch of \( p_\eta \) containing \( x \) of height \( \alpha_\eta + 1 \). If \( x \in p_\eta \geq p_{\eta_2} \), then we will have that the \( f_x \) functions are compatible. At successor steps, if \( p_\eta \) is defined we let \( p_{\eta+1} \in D_{\eta+1} \) be any extension of \( p_\eta \) of some successor height \( \alpha_{\eta+1} + 1 > \alpha_\eta + 1 \), and extend all of the \( f_x \) function for \( x \in p_\eta \) as well as define the \( f_x \) functions for \( x \in p_{\eta+1} - p_\eta \).
There is no problem doing this as \( p_{i+1} \) extends \( p_i \) and is pruned. For \( \eta \) limit, let 
\[
\beta = \sup \{ \alpha_i : \alpha_i < \eta \}.
\]
Let \( T = \bigcup_{i < \eta} p_i \). Our branch functions define for each \( x \in T \) a branch \( f_x \) of \( T \) of height \( \beta \). Let \( T' \leq T \) have height \( \beta + 1 \) and obtained by extending each branch \( f_x \) to level \( \beta + 1 \) of \( T' \). \( T' \) is now a condition in \( P \). Let \( p_\eta \in D_\eta \) extend \( T' \), and extend the branch functions of \( T' \) appropriately. Continuing, we define a condition \( p_\eta \) which extends conditions in all of the \( D_\eta \). Thus, \( \bigcap_{\eta < \rho} D_\eta \) is dense, so \( P \) is \( < \kappa \) distributive.

Let \( G \) be \( M \) generic for \( P \), where \( M \) is a transitive model of ZFC. We may identify \( G \) with a pruned \( \kappa \) tree \((G \) has height \( \kappa \) since any condition \( T \) can be extended to a condition \( T' \) of height \( \alpha + 1 \) for any \( |T| \leq \alpha + 1 < \kappa \).

We claim that \( G \) is a \( \kappa \)-Suslin tree. Suppose \( \tau \in M^P \) and \( T_0 \models \tau \) (\( \tau \) is a maximal antichain of \( G \)). Let \( D \subseteq P \) be those conditions \( T \) such that for some \( A \subseteq T \) we have

\begin{enumerate}
  \item \( A \) is a maximal antichain of \( T \).
  \item \( \sup \{|a| : a \in A\} < |T| \) (i.e., there is a level of \( T \) such that all elements of \( A \) are below that level).
  \item \( T \models (A \subseteq \tau) \).
\end{enumerate}

We claim that \( D \) is dense below \( T_0 \). For let \( T \leq T_0 \). As \( P \) is \( < \kappa \) distributive, we may get \( T_1 \leq T \) such that for all \( x \in T_0 \), there is a \( a \in T_1 \) such that \( x \) is compatible with \( a \) and \( T_1 \models (a \in \tau) \). In general, define \( T_{\alpha+1} \leq T_{\alpha} \) so that for all \( x \in T_{\alpha} \), there is a \( a \in T_{\alpha+1} \) such that \( x \) is compatible with \( a \) and \( T_{\alpha+1} \models (a \in \tau) \). Let \( T = \bigcup_{\alpha \in \kappa} T_{\alpha} \). Then there is a maximal antichain \( A \) of such that \( T \models (A \subseteq \tau) \). Extend \( T \) to \( T' \) of height \( |T'| + 1 \) as follows. For each \( x \in T \), let \( b_x \) be a branch of \( T \) containing \( x \) and an element of \( A \), with \( b_x \) of height \( |T'| \). For each \( x \in T \), put a point in \( T' \) which extends all the elements of \( b_x \) (i.e., extend the branch \( b_x \)). This defines \( T' \), and we have \( T' \in D \). Thus, \( D \) is dense in \( P \). Let \( T \in G \cap D \). Let \( A \subseteq T \) witness \( T \in D \). We must have \( A = \tau_G \), since if \( T' \leq T \) and \( x \in T' \setminus T \), then \( x \) is above some element of \( A \). Thus, \( \tau_G = A \) has size \( < \kappa \) in \( M[G] \). This shows \( G \) is a Suslin tree.

**Corollary 5.3.** Let \( M \) be a transitive model of ZFC and \( \kappa \) a regular cardinal of \( M \). Then there is a \( \kappa \)-distributive forcing (hence preserves all cofinalities and cardinalites \( \leq \kappa \)) such that in \( M[G] \) there is a \( \kappa \)-Suslin tree.

The second proof uses finite trees. \( P \) now consists of finite trees \( T \subseteq \omega_1 \) satisfying:

if \( \alpha \prec \beta \) then \( \alpha < \beta \). We define \( T_1 \leq T_2 \) iff \( T_1 \cap (T_2 \times T_2) \) and is below that level. We again identify a generic \( G \) with a tree \( G \) on \( \omega_1 \) (to see it is a tree, note that if \( \alpha \in G \) then the \( \leq_G \) predecessors of \( \alpha \) are \( \leq_G \) ordered in their usual order as ordinals, hence the \( \leq_G \) predecessors are well-ordered).

First we show that \( T \) is c.c.c. If \( \{T_\alpha\}_{\alpha < \omega_1} \) were an antichain, then by the \( \Delta \) system lemma we may assume that each \( T_\alpha \) consists of a root \( R \) and a set \( A_\alpha = \{a_\alpha^n : n \leq \alpha \} \) (of some fixed size \( n \)), where the \( A_\alpha \) are pairwise disjoint. We may further assume that the \( T_\alpha \) orderings on the root \( R \) are all the same. Further, we may assume that for \( r \in R \), \( r \prec T_\alpha \) \( a_\alpha^k \) if \( r \prec T_\beta \) \( a_\beta^k \) for all \( \alpha, \beta \). We may also assume that \( a_\alpha^k \prec T_\alpha a_\alpha^l \) if \( a_\alpha^k \prec T_\beta a_\beta^l \) for all \( k, l \leq n \) and all \( \alpha, \beta \). Thus, \( T_\alpha \) and \( T_\beta \) look the same except for the values of the ordinals in \( A_\alpha, A_\beta \). However it is now easy to get a common extension of any two of the \( T_\alpha \) (e.g., the union of \( T_\alpha \) and \( T_\beta \) is now a condition).

We show that \( G \) has no uncountable antichain (from which it also follows that \( G \) is an \( \omega_1 \) tree (for another argument, see the exercise below). Suppose \( T \models (\tau \) is an uncountable antichain of \( G \)). Get an \( \omega_1 \) sequence \( T_\alpha \) of conditions extending...
$T$ and ordinals $\eta_\alpha \in T_\alpha$ with $T_\alpha \models (\eta_\alpha \in \tau)$, and the $\eta_\alpha$ are distinct. Thin the $T_\alpha$ to a $\Delta$ system as above, $T_\alpha = R \cup A_\alpha$. It is easy to see that for any $\alpha \neq \beta$ we can get a common extension of $T_\alpha$ and $T_\beta$ in which $\eta_\alpha$ is comparable with $\eta_\beta$, a contradiction.

**Corollary 5.4.** The existence of a Suslin tree is consistent with $\text{ZFC + } \neg \text{CH}$. In particular, the existence of a Suslin tree does not imply $\Diamond$.

6. **Kurepa Trees**

Recall a Kurepa tree is an $\omega_1$ tree with at least $\omega_2$ branches of length $\omega_1$. We can give an easy reformulation of this which does not mention trees.

**Lemma 6.1.** There is a Kurepa tree iff there is a family $F$ of subsets of $\omega_1$ satisfying:

1. $|F| \geq \omega_2$.
2. $\forall \alpha < \omega_1 \ (|A \cap \alpha : A \in F|) \leq \omega$.

We call a family $F$ as in lemma 6.1 a Kurepa family.

**Proof.** Assume first that $T$ is a Kurepa tree. Without loss of generality we may assume $T \subseteq \omega_1$ and if $\alpha < T \beta$ then $\alpha < \beta$. Then $F = \{A \cap \alpha : A \in F\}$ is a family of characteristic functions of $A \in F$. Easily $T$ is a Kurepa tree (note that if $b$ is a branch through $T$, then $b \cap \alpha$ is determined by the $\alpha^{th}$ level of $b$).

Conversely, assume $F$ is a Kurepa family. Let $T$ be the subtree of $2^{<\omega_1}$ consisting of all initial segments of characteristic functions of $A \in F$. Thus $T$ is a Kurepa tree (note that the element of $T$ of height $\alpha$ correspond to the elements $A \cap \alpha$ for $A \in F$).

Just as $\Diamond$ implies the existence of a Suslin tree, there is a combinatorial principle $\Diamond^*$ which implies the existence of a Kurepa tree. $\Diamond^*$ implies $\Diamond$, however the existence of a Kurepa tree does not imply the existence of a Suslin tree. Although we show here the consistency of the existence of Kurepa tree directly by forcing, we state the principle $\Diamond^*$. We note also that $\Diamond^*$, like $\Diamond$ holds in $L$.

**Definition 6.2.** $\Diamond^*$ is the statement that there is a sequence $\{A_\alpha\}_{\alpha < \omega_1}$, where each $A_\alpha$ is a countable family of subsets of $\alpha$, such that for all $A \subseteq \omega_1$, there is a c.u.b. $C \subseteq \omega_1$ such that for all $\alpha \in C$ we have $A \cap \alpha \in A_\alpha$ and $C \cap \alpha \in A_\alpha$.

In view of theorem 0.7, it is clear that $\Diamond^*$ implies $\Diamond$.

**Theorem 6.3.** Let $M$ be a transitive model of $\text{ZFC + CH}$. Then there is a countably closed $P \in M$ such that if $G$ is $M$-generic for $P$ then $M[G]$ satisfies that there is Kurepa tree.

**Proof.** $P$ consists of pairs $(T,f)$ where $T$ is a pruned countable subtree of $2^{<\omega_1}$ of height $\alpha < \omega_1$, and $f$ is a function with domain a countable subset of $\omega_3$ such that for $\alpha \in \text{dom}(f)$, $f(\alpha)$ is a branch through $T$. We say $(T_1,f_1) \leq (T_2,f_2)$ if $T_1 \cap T_2 = T_2$ (i.e., $T_1$ extends $T_2$ vertically), $\text{dom}(f_2) \subseteq \text{dom}(f_1)$, and for all $\alpha \in \text{dom}(f_2)$, $f_1(\alpha)$ extends $f_2(\alpha)$ (i.e., $f_1$ extends all of the branches of $f_2$, and may give new ones as well).

Clearly $P$ is countably closed. Let $T$ be the tree produced by the generic, and $F$ the function produced (in the obvious manner). Thus, $F$ is a map from $\omega_2^M$ to the length $\omega_1$ branches of $T$ (note: $\omega_1^M = \omega_1^{M[G]}$). It is easy to see that for a generic
$G$, the resulting function $F$ will also be one-to-one (we may extend any $(T, f)$ with $\alpha, \beta \in \text{dom}(f)$ to a $(T', f')$ where $f'(|\alpha|) \neq f(|\beta|)$). Thus, in $M[G]$ the $\omega_1$ tree $T$ has $\geq \omega_2^M$ branches of length $\omega$. We show finally that $\mathbb{P}$ is $\omega_2$-c.c. in $M$, which shows that $\omega_2^M = \omega_2^{M[G]}$ and completes the proof.

Suppose $\{(T_\alpha, f_\alpha)\}_{\alpha < \omega_2}$ were an antichain of $\mathbb{P}$. We may assume that all of the trees have the same height $\beta < \omega_1$. By CH in $M$, there are $2^\beta = \omega_1$ many trees of height $\beta$, so we may assume that all of the $T_\alpha$ are equal to a fixed tree $T_0$. Again using CH, we may thin the antichain so the $\text{dom}(f_\alpha)$ form a $\Delta$ system on $\omega_2$, say $\text{dom}(f_\alpha) = R \cap A_\alpha$ where the $A_\alpha$ are pairwise disjoint. We may assume that that the $f_\alpha$ all agree on the root $R$, as there are only $\omega_1^\omega = 2^\omega = \omega_1$ many choices for $f \upharpoonright R$. At this point, any two members of the antichain are compatible, a contradiction. \hfill \Box

7. A Model in which there are no Kurepa Trees

Starting with a model $M$ of ZFC+ there is an inaccessible cardinal, we produce a generic extension $M[G]$ in which there are no Kurepa trees. The inaccessible cardinal is necessary since if $M$ satisfies ZFC+ there are no Kurepa trees then $\omega_2^M$ is inaccessible in $L$ [Work in $M$. If $\omega_2$ is not inaccessible in $L$, then $\omega_2 = (\omega_2)^{L[A]}$ for some $A \subseteq \omega$. However $\Diamond^\omega$ holds in any $L[A]$ for $A \subseteq \omega$. Thus there is a Kurepa tree in $L[A]$, and this remains a Kurepa tree in $L[A]$ as $\omega_2 = (\omega_2)^{L[A]}$.]

To motivate the forcing, note that if $T$ is a Kurepa tree, then $T$ will remain a Kurepa tree in any larger model unless $\omega_2$ is collapsed in the larger model. Conversely, collapsing $\omega_2$ by a countably closed forcing will kill the Kurepa trees of the ground model (by lemma 7.3), but may introduce new Kurepa trees, so it will be necessary to collapse the new $\omega_2$, etc. This suggests we collapse to $\omega_1$ all the ordinals below some large cardinal.

**Definition 7.1.** Let $\kappa$ be a regular cardinal. The Silver collapse of $\kappa$ to $\omega_2$ is the forcing $\mathbb{P}$ consisting of functions with domain a countable subset of $\{(|\alpha, \beta|): \alpha < \kappa \land \beta < \omega_1\}$ and $f(\alpha, \beta) < \alpha$ for all $(\alpha, \beta) \in \text{dom}(f)$.

It is clear that if $g$ is generic for $\mathbb{P}$, then $(|\kappa| \leq \omega_2)^{M[G]}$, as in $M[G]$ all ordinals $\alpha < \kappa$ are onto images of $\omega_1$. It is also clear that $\mathbb{P}$ is countably closed, so $\omega_1$ is preserved in forcing with $\mathbb{P}$.

**Lemma 7.2.** Let $\kappa$ be a strongly inaccessible cardinal of $M$. Then $\mathbb{P}$ is $\kappa$-c.c. Thus, $\kappa$ is a cardinal of $M[G]$, and hence $\kappa = \omega_2^{M[G]}$.

**Proof.** Suppose $\{p_\alpha\}_{\alpha < \kappa}$ were an antichain of size $\kappa$. Let $d_\alpha = \text{dom}(p_\alpha)$. We can view $d_\alpha$ as a countable subset of $\kappa$. We use the $\Delta$-system argument. We may assume all the $d_\alpha$ have order-type $\tau < \omega_1$. Let $\eta(\beta) = \sup\{d_\alpha(\beta): \alpha < \kappa\}$. There must be a least $\beta_0 < \tau$ such that $\eta(\beta_0) = \kappa$ [otherwise there is an $\eta < \kappa$ such that $d_\alpha \subseteq \eta$ for all $\alpha < \kappa$. Using $\kappa$ strongly inaccessible this gives $< \kappa$ many possibilities for the $p_\alpha$.] We may then thin the $\{p_\alpha\}$ sequence so the $d_\alpha$ form a $\Delta$-system with root $R \in \kappa^\kappa$. There are $< \kappa$ many possibilities for $p_\alpha \upharpoonright R$, using again the inaccessibility of $\kappa$. So we may assume $p_\alpha \upharpoonright R$ is constant, and this gives a contradiction to the $p_\alpha$ being an antichain. \hfill \Box

**Lemma 7.3.** Let $\mathbb{P}$ be a countably closed forcing in $M$. If $T \in M$ is an $\omega_1$ tree of $M$, then any branch of $T$ in $M[G]$ lies in $M$. 

Proof. Suppose $\tau \in M^P$ and $p \forces (\tau$ is a branch of $T \land \tau \notin M)$. We define conditions $p_s$ for $s \in 2^{<\omega}$ and also $x_s \in T$. Let $p_\emptyset = p$ and $x_0$ be the root of $T$. Given $p_s$ and $x_s$, let $p_{s \cdot 0}$ and $p_{s \cdot 1}$ extend $p_s$ and $x_{s \cdot 0}, x_{s \cdot 1}$ be two extensions of $x_s$ in $T$ which are incompatible in $T$, and with $p_{s \cdot i} \forces x_{s \cdot i} \in \tau$. We can do this as otherwise $p_s$ determines the branch $\tau_G$. Let $\alpha < \omega_1$ be greater than the heights of all the $x_s$ in $T$. For each $r \in 2^\omega$, let $p_r$ extend all of the $p_s$ for $s$ an initial segment of $r$, and (extending $p_r$ if necessary), let $x_r \in T$ have height $\alpha$ with $p_r \forces x_r \in T$ (we can do this as $p$ forces that $\tau$ has elements of all heights $\alpha < \omega_1$, as otherwise some $q \leq p$ would determine $\tau_G$.) We clearly have that the $\{x_r\}$ are distinct elements of $T$ of height $\alpha$, contradicting $T$ being an $\omega_1$ tree. $\square$

Theorem 7.4. Let $M$ be a transitive model of ZFC and $\kappa$ an inaccessible cardinal of $M$. Let $P \in M$ be the Silver collapse of $\kappa$ to $\omega_2$ as defined in $M$. Then if $G$ is $M$-generic for $P$, $M[G]$ satisfies that there are no Kurepa trees.

Proof. Suppose $T = \tau_G \in M[G]$ and $(T$ is a Kurepa tree$)^{M[G]}$. We may assume $T \subseteq \omega_1^{M[G]} = \omega_1^M$. Let $\sigma$ be a nice name for $T$, that is $\tau_G = \sigma_G$ and $\sigma$ is of the form $\check{\tau}_G \check{=} \sigma_G \check{=} \sigma_G$. From lemma 7.2 it follows that $|\sigma| < \kappa$. For any $\lambda < \kappa$ we may write $\check{\tau} = \check{\tau}_\lambda \check{=} \check{\tau}_\lambda$ where $\check{\tau}_\lambda$ consists of those $p \in \check{\tau}$ whose domain consists only of pairs $(\alpha, \beta)$ where $\alpha \leq \lambda$, and $\check{\tau}_\lambda$ those $p$ whose domain consists only of pairs $(\alpha, \beta)$ with $\alpha > \lambda$. Let $\lambda < \kappa$ be large enough so that $\text{tr}(\sigma) \cap \text{ON} \subseteq \lambda$. Write $G = G_{<\lambda} \times G_{>\lambda}$. Then $T = \sigma_G = \sigma_G_{<\lambda} \in M[G_{<\lambda}]$. From lemma 7.3 there are at most $(2^{\omega_1})^{M[G_{<\lambda}]}$ many branches of $T$ in $M[G]$. Since $\kappa$ is inaccessible, a simple name counting argument shows that $\rho = (2^{\omega_1})^{M[G_{<\lambda}]} < \kappa$. However, $\rho$ has cardinality $\omega_1$ in $M[G]$, as forcing with $P_{>\lambda}$ clearly collapses $\rho$ to cardinality $\omega_1$. Thus $T$ has $\leq \omega_1$ many branches in $M[G]$. $\square$