1. Determinacy

We introduce the axiom of determinacy, which provides a powerful strengthening of ZF. Although the full axiom is inconsistent with AC, various restrictions of it are, and provide a way to extend the ZF theory of the reals presented earlier. We begin with the basic terminology. We emphasize that our background theory throughout this discussion is ZF set theory.

Let $X$ be a set. By a game on $X$ we mean a setting where two players, called I and II alternate playing elements from the set $X$ as follows:

\[
\begin{array}{c}
\text{I} & x_0 & x_2 & x_4 & \ldots \\
\text{II} & x_1 & x_3 & x_5 & \ldots
\end{array}
\]

The sequence $\vec{x} = (x_0, x_1, \ldots)$ that they jointly produce is called a run of the game. The winning condition for the game is given by a payoff set $A \subseteq X^\omega$. We say I wins the run of the game if $\vec{x} \in A$, and II wins the run. We frequently denote the game with payoff set $A$ by $G_A$.

A strategy for player I (respectively player II) is a function $\sigma$ from the sequences in $X^{<\omega}$ of even length (respectively odd length) into $X$. If $\sigma$ is a strategy for I, we say a run $\vec{x}$ is according to $\sigma$ (or I has followed $\sigma$) if for all even $n \in \omega$, $x_n = \sigma(\vec{x} \upharpoonright n)$. The definition for a strategy for II is similar, using "odd" instead of "even." If $\sigma$ is a strategy for I, and $\vec{x} \in X^\omega$, we let $\sigma(\vec{x}) \in X^\omega$ be the run according to $\sigma$ where II has made moves $\vec{z} = (z_0, z_1, \ldots)$. Thus,

$$\sigma(\vec{z}) = (\sigma(\emptyset), z_0, \sigma(\emptyset), z_0, z_1, \ldots).$$

We likewise define $\tau(\vec{z})$ when $\tau$ is a strategy for II. We let $\sigma_0(\vec{z})$ denote the sequence of moves I makes against II play of $\vec{z}$, and likewise $\tau_1(\vec{z})$ denotes II's moves against I's play of $\vec{z}$.

Exercise 1. Let $d$ be the standard metric on $\omega^\omega$, that is, $d(x, y) = \frac{1}{2^n}$ where $n \in \omega$ is least such that $x(n) \neq y(n)$ (and of course $d(x, y) = 0$ if $x = y$). Let $\sigma$ be a strategy for I or II in an integer game. Show that the map $x \mapsto \sigma(x)$ from $\omega^\omega$ to $\omega^\omega$ is one-to-one and Lipschitz continuous, that is, $d(\sigma(x), \sigma(y)) \leq d(x, y)$.

Exercise 2. Show that for any strategy $\sigma$ for I or II in an integer game that $\sigma[\omega^\omega] = \{\sigma(x) : x \in \omega^\omega\}$ is perfect. Show that $\sigma_0[\omega^\omega]$ is $\Sigma^1_1$. Give an example to show $\sigma_0[\omega^\omega]$ is not necessarily Borel [hint: use the fact that every $\Sigma^1_1$ set $A$ in $\omega^\omega$ is the projection of a closed set in $\omega^\omega \times \omega^\omega$. It may help to assume that $A$ is dense in $\omega^\omega$.]

We say a strategy for I (respectively II) is a winning strategy if for all runs $\vec{x}$ according to $\sigma$, $\vec{x} \in A$ (respectively, $\vec{x} \notin A$). We say the game $G_A$ is determined if one of the players has a winning strategy.

Exercise 3. Show that for any game $G_A$ on any set $X$, that it cannot be the case that both players have a winning strategy.

If $X$ cannot be wellordered in ZF (e.g., $X = \mathbb{R}$), then the notion of a winning strategy is too strong. One uses instead the notion of a quasi-strategy. A quasi-strategy $\sigma$ for I is a function with domain the set of $s \in X^{<\omega}$ of even length and such that $\sigma(s)$ is a non-empty subset of $X$. The definition for II is similar. Thus, a
quasi-strategy is like a strategy except that instead of giving a single move for the player, it produces a non-empty set of possible moves. We say \( \vec{x} = (x(0), x(1), \ldots) \) is a run according to \( \sigma \) if for all even \( n \), \( x(n) \in \sigma(x \upharpoonright n) \). Likewise for II. The notion of a winning quasi-strategy is then defined in the obvious manner. We say \( A \subseteq X^\omega \) is quasi-determined if one of the players has a winning quasi-strategy. We sometimes still use the term “determined” in this case. To illustrate the difference, suppose there is a relation \( R \subseteq \omega^\omega \times \omega^\omega \) which has no uniformization (this will be the case, for example, assuming \( AD + V = L(\mathbb{R}) \)). Consider the game on \( X = \mathbb{R} \) where I plays \( x \) in the first move, and II plays \( y \) in the next move. All moves after this are irrelevant. II wins the run iff \( R(x, y) \). Then II has a winning quasi-strategy, namely \( R \) itself, but has no winning strategy as a winning strategy for II produces a uniformization for \( R \).

**Exercise 4.** We say a game on a set \( X \) is finite if there is an \( n \in \omega \) such that the payoff only depends on \( \vec{x} \upharpoonright n \). Show that any finite game on any set \( X \) is determined. [hint: Say the game is of length \( n \) which is even. Then I has a winning (quasi) strategy iff \( \exists x_0 \forall x_1 \exists x_2 \cdots \forall x_{n-1} (x_0, \ldots, x_{n-1}) \in A \). If this fails, consider the negation and pass the negation through the quantifiers.]

A case of particular interest in when \( X = \omega \), that is, both players play integers, in which case the run \( \vec{x} \) they produce is an element of the Baire space \( \omega^\omega \). In this case, the game can be identified with a subset \( A \subseteq \omega^\omega \) of the Baire space.

**Definition 1.1.** The axiom of determinacy, \( AD \), is the assertion that for every \( A \subseteq \omega^\omega \) the game \( G_A \) is determined.

So, \( AD \) asserts that every two-player integer game is determined. We will see below that \( AD \) contradicts \( AC \). However, restricted form are consistent with \( AC \).

**Definition 1.2.** Let \( \Gamma \subseteq \mathcal{P}(\omega^\omega) \) be any collection of subsets of \( \omega^\omega \). By \( \text{det}(\Gamma) \) we mean the assertion that \( G_A \) is determined for all \( A \in \Gamma \).

Usually \( \Gamma \) will be a poinclass, in which case we interpret \( \text{det}(\Gamma) \) as meaning that \( G_A \) is determined for all \( A \in \Gamma \) which are subsets of \( \omega^\omega \).

The fact that \( AD \) contradicts \( AC \) does not mean that the full axiom \( AD \) loses interest. It simply means we must restrict our attention to an inner model of \( V \) in which choice fails. A natural such model, which we discuss later, is the inner model \( L(\mathbb{R}) \), the smallest inner-model of set theory containing the reals. On the one hand, \( AD \) suffices to give a reasonably complete theory for this model, while on the other hand this model contains all the sets of real which are reasonably definable. In particular \( L(\mathbb{R}) \) contains all the projective sets, and much more.

We next discuss some of the basic facts about games and determinacy. First we show that \( AD \) is inconsistent with \( ZFC \). Note that a strategy for an integer game is a subset of \( \omega^{<\omega} \times \omega \), and can thus be identified with a real. We frequently implicitly make this identification.

**Lemma 1.3.** \( AD \) is inconsistent with \( ZFC \).

**Proof.** By \( AC \), let \( \{x_\alpha\}_{\alpha < 2^{\omega}} \) enumerate the reals \( \omega^\omega \) (and hence all the strategies for either I or II in integer games). We define inductively sets \( A_\alpha, B_\alpha \subseteq \omega^\omega \) which are monotonically increasing (i.e., \( \alpha < \beta \rightarrow A_\alpha \subseteq A_\beta \)) and disjoint at each step (i.e., \( A_\alpha \cap B_\alpha = \emptyset \)). We also assume inductively that \( |A_\alpha|, |B_\alpha| < 2^\omega \). We think of the reals in \( A_\alpha \) as being reals we wish to add to the set we are building, and the
real numbers in \( B_\alpha \) as those we wish to forbid from being in our set. At stage \( \alpha \), first let \( A_{<\alpha} = \bigcup_{\beta<\alpha} A_\beta \) and \( B_{<\alpha} = \bigcup_{\beta<\alpha} B_\beta \). Note that \( A_{<\alpha} \cap B_{<\alpha} = \emptyset \), assuming the above inductive hypotheses. Consider then \( x_\alpha \). If \( x_\alpha \) is not a strategy for I or II, let \( A_\alpha = A_{<\alpha}, B_\alpha = B_{<\alpha} \). Since \( x_\alpha \) is a strategy for I, let \( A_\alpha = A_{<\alpha}, B_\alpha = B_{<\alpha} \cup \{ x_\alpha(z) \} \). If \( x_\alpha \) is a strategy for II, let \( A_\alpha = A_{<\alpha} \cup \{ x_\alpha(z) \}, B_\alpha = B_{<\alpha} \). Since \( z \notin C \), this maintains our disjointness hypothesis. Clearly \( x_\alpha \) cannot be a winning strategy for I for any set disjoint from \( B_\alpha \), and cannot be a winning strategy for II for any set containing \( A_\alpha \). Let \( A = \bigcup_{\alpha<\omega} A_\alpha \). So, \( A \cap B = \emptyset \), where \( B = \bigcup_{\alpha<\omega} B_\alpha \). So, no \( x \in \omega^\omega \) can be a winning strategy for either I or II in \( G_A \).

In view of lemma 1.3, in working with the full axiom of determinacy \( AD \) we work in the background theory \( ZF + AD \). In fact, a weak form of choice, \( DC \), is frequently also added to the background theory; we discuss this in more detail below. Actually, \( AD \) implies a weak form of choice, namely countable choice for reals. We recall the definitions.

**Definition 1.4.** For any cardinal \( \kappa \), \( AC_\kappa \) is the statement that for any \( \kappa \)-sequence \( \{ A_\alpha \}_{\alpha<\kappa} \) of non-empty sets, there is a function \( f \) with domain \( \kappa \) such that \( \forall \alpha < \kappa ( f(\alpha) \in A_\alpha ) \). *Countable choice* is the statement \( AC_{\omega} \). Countable choice for reals is the statement \( AC_{\omega} \) restricted to sequences \( \{ A_n \}_{n<\omega} \) for which \( A_n \subseteq \omega^\omega \) for all \( n \).

**Lemma 1.5.** \( AD \) implies countable choice for reals.

**Proof.** Let \( \{ A_n \}_{n<\omega} \) be given, where each \( A_n \subseteq \omega^\omega \) is non-empty. Consider the game

\[
\begin{array}{ccc}
\text{I} & n \\
\text{II} & x(0) & x(1) & x(2) & \ldots
\end{array}
\]

where I plays an integer \( n \) and II plays out a real \( x \). II wins the run if \( x \in A_n \). Clearly I cannot have a winning strategy \( \sigma \) since as soon as I plays \( n \), can play any \( x \in A_n \) to defeat \( \sigma \). A winning strategy \( \tau \) for II gives a choice function. \( \square \)

A small variation of this argument shows the following.

**Lemma 1.6.** Assume \( AD \). Then countable choice holds in \( L(\mathbb{R}) \).

**Proof.** There is a definable map \( \pi \) in \( L(\mathbb{R}) \) from \( \omega^\omega \times \text{On} \) onto \( L(\mathbb{R}) \). Given the sequence \( \{ A_n \}_{n<\omega} \) of non-empty sets, let \( A'_n = \{ (x, \alpha) \in \omega^\omega \times \text{On} : \pi(x, \alpha) \in A_n \} \). It clearly suffices to get a choice function for the \( A'_n \). For each \( n \), let \( \alpha_n \in \text{On} \) be least such that \( \exists x ( x, \alpha_n ) \in A'_n \). Let \( A''_n = \{ x \in \omega^\omega : (x, \alpha_n) \in A'_n \} \). From lemma 1.5, let \( \langle x_n \rangle_{n<\omega} \) be a choice function for the \( A''_n \). Then \( \langle \pi(x_n, \alpha_n) \rangle_{n<\omega} \) is a choice function for the \( A_n \). \( \square \)

The axiom of dependent choice is a strengthening of countable choice.

**Definition 1.7.** \( DC \) is the following statement. Let \( X \) be a set and \( R \subseteq X^{<\omega} \) such that for all \( s \in R \) there is an \( x \in X \) with \( s \cup x \in R \). Then there is a \( \bar{x} \in X^\omega \) such that \( \forall n \bar{x} \upharpoonright n \in R \). Let \( DC_X \) be the statement of \( DC \) for sets \( R \subseteq X^{<\omega} \).

**Exercise 5.** Show that \( DC \) implies countable choice.
In fact, AD implies that DC holds in L(\mathbb{R}) (Kechris). This shows that if AD is consistent, then so is AD+DC. The proof of this theorem requires more techniques.

DC is equivalent to the assertion that every illfounded tree on a set X has a branch. In fact the following exercise shows a bit more.

**Exercise 6.** Show that DC is equivalent to the assertion that every relation R on a set X which is illfounded has an infinite decreasing sequence. Recall a relation is wellfounded if every non-empty subset of X has an R-minimal element. [hint: If R is illfounded, let S ⊆ X^\omega be the set of sequences (x_0, x_1, \ldots, x_n) such that x_nRx_{n-1}R\ldots Rx_0 (i.e., the chain is R-decreasing) and \{x: xRx_n\}, R is illfounded. Apply DC to S.]

If X cannot be wellordered in ZF, then we need DC to even produce a run according to a given quasi-strategy. More precisely, we need DC_X to produce such a run.

We consider next the situation for other X. We let AD_X denote the assertion that every game on X (i.e., A ⊆ X^\omega) is determined. So, AD = AD_\omega. First note the following simple fact.

**Exercise 7.** Show that if X ⊆ X' then AD_X' ⇒ AD_X.

In particular, AD_\omega ⇒ AD_2. The converse is also true by the following lemma.

**Lemma 1.8.** AD_2 ⇒ AD_\omega.

**Proof.** Assume AD_2, and let A ⊆ \omega^\omega. We show that G_A is determined. We define an A' ⊆ 2^\omega which simulates the game G_A, and such that whichever player has a winning strategy in G_{A'} has one in G_A. Call a finite sequence s ∈ 2^{<\omega} good if it is an initial segment of a sequence of the form

\[0^{\alpha_0}\langle 1, 0\rangle \langle 0, 1\rangle \langle 0, 1\rangle \langle 0, 1\rangle \ldots.\]

In other words, s is good if it is a partial play of a game on \{0, 1\} of the form:

<table>
<thead>
<tr>
<th></th>
<th>0, 0, \ldots, 0</th>
<th>1</th>
<th>0, 0, \ldots, 0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>\underbrace{0, 0, \ldots, 0}_a</td>
<td>1</td>
<td>\underbrace{0, 0, \ldots, 0}_a</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>0, 0, \ldots, 0</td>
<td>0</td>
<td>0, 0, \ldots, 0</td>
<td>0</td>
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</tbody>
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We now define G_{A'}. The first player to play so that the sequence is not good loses. Suppose they produce x ∈ 2^\omega, and all initial segments of x are good. If x is eventually equal to 0, then the last player to play a 1 wins (if they both play all 0’s then II wins). Otherwise, consider the sequence y = (a_0, a_1, \ldots) ∈ \omega^\omega. Then I wins the run of G_{A'} iff y ∈ A. Suppose \sigma' is a winning strategy for I in G_{A'} (the case for II is similar). We define a strategy \sigma for I in G_A as follows. To define \sigma(\emptyset), have I follow \sigma' in G_{A'} where II plays 0’s. I must eventually play a 1 as otherwise I loses by definition. If there have been a_0 rounds of 0’s played before the round where I plays a 1, then we let \sigma(\emptyset) = a_0.

To define \sigma(a_0, a_1), have II play a_1 more 0’s followed by a 1. Following \sigma', I must must respond with a 0 to all of these moves, as otherwise I loses, contradicting \sigma' being a winning strategy. Have II then continue to play 0’s until I plays a 1, which I
must do as otherwise I loses by definition. This defines \( a_2 = \sigma(a_0, a_1) \) as the above diagram illustrates. Continuing in this manner defines the strategy \( \sigma \).

For any run \( x = (a_0, a_1, \ldots) \) of \( \sigma \), there is a corresponding run \( x' \) of \( \sigma' \) where \( x' \) is good and both players have played infinitely many 1's. Furthermore, \( x' \) and \( x \) correspond as in the diagram above. So, by definition, \( x' \in a' \) iff \( x \in A \). Since \( \sigma' \) is a winning strategy for I in \( G_{A'} \), \( x' \in A' \). Thus, \( x \in A \). □

**Corollary 1.9.** \( AD \) is equivalent to \( AD_X \) for any countable set \( X \) with at least 2 elements.

Determinacy is not symmetrical between the two players in the sense that I having a winning strategy for a game \( G_A \) is not the same as II having a winning strategy for the complement \( G_{A'} \). However, the asymmetry is minor according to the following lemma.

**Lemma 1.10.** If \( \Gamma \) is any pointclass then \( \det(\Gamma) \iff \det(\overline{\Gamma}) \). In fact for any \( A \subseteq \omega^\omega \) there is a \( B \) recursively reducible to \( \omega^\omega - A \) such that I (resp. II) has a winning strategy in \( G_B \) iff II (resp. I) has a winning strategy in \( G_A \).

**Proof.** Given \( A \subseteq \omega^\omega \), let \( B = \{ x : x' \in \omega^\omega - A \} \), where \( x'(n) = x(n + 1) \). If \( \sigma \) is a winning strategy for I in \( G_A \), then \( \sigma' \) is winning for II in \( G_B \), where \( \sigma'(a_0, a_1, a_2, \ldots, a_{2n}) = \sigma(a_1, \ldots, a_{2n}) \), that is, II ignores I’s first move and then follows \( \sigma \). Also, if \( \sigma' \) is winning for II in \( G_{A'} \), then \( \sigma \) is winning for I in \( G_A \).

Similarly, if \( \tau \) is winning for II in \( G_A \) iff \( \tau' \) is winning for I in \( G_{A'} \) where \( \tau' \) makes an arbitrary first move, and then follows \( \tau \).

Thus, \( G_A \) is determined iff \( B_B \) is determined. The first statement of the lemma follows. □

**Exercise 8.** Show assuming \( ZFC \) that there are two determined games \( A \) and \( B \) such that \( A \cap B \) is not determined, and likewise for unions. [hint: Let \( C \subseteq \omega^\omega \) be a non-determined game. Let \( A = \{ x : x(0) = 1 \} \). Let \( B = \{ x : x(0) = 0 \} \). Then I wins both \( G_A \) and \( G_B \), but \( G_{A \cap B} \) is not determined.

Two natural generalizations of \( AD = AD_\omega \) suggest themselves. One is allowing \( X \) to be a larger ordinal (we must go to at least \( \omega_1 \) to have something potentially different). Another is to allow the players to play reals (since integers can be viewed as simple reals).

Regarding the ordinal case, we have the following lemma. In the proof, we will borrow from an upcoming fact, namely that \( AD \) implies the perfect set property for all sets of reals.

**Lemma 1.11 (ZF).** For any uncountable well-ordered set \( X \), the axiom \( AD_X \) is inconsistent.

**Proof.** Assuming \( ZF \), it suffices to show that \( AD_{\omega_1} \) is inconsistent (from exercise 7). Suppose \( AD_{\omega_1} \). From exercise 7 we also have \( AD \). Consider the following game on \( \omega_1 \):

\[
\begin{align*}
\text{I} & \quad \alpha \\
\text{II} & \quad x(0) \quad x(1) \quad x(2) \quad \cdots
\end{align*}
\]
where I plays an ordinal \( \alpha < \omega_1 \), and plays integers \( x(0), x(1), \ldots \), thereby playing out a real \( x \in \omega^\omega \). II wins iff \( x \in \text{WO} \) and \( |x| = \alpha \). First note that I cannot have a winning strategy \( \sigma \), since if \( \sigma \) calls for I to play \( \alpha \), II can defeat \( \sigma \) by playing any \( x \in \text{WO} \) with \( |x| = \alpha \). So, since we are assuming this game is determined, II must have a winning strategy \( \tau \). Let \( A = \{ \tau(\alpha): \alpha < \omega_1 \} \). Then \( A \) is a wellordered subset of \( \text{WO} \) of size \( \omega_1 \) (for each \( \alpha < \omega_1 \) there is a unique \( x \in A \) with \( |x| = \alpha \)). We will show in theorem 2.2 below that under \( \text{AD} \), every uncountable subset of \( \omega^\omega \) contains a perfect set. So, let \( P \subseteq A \) be perfect. In particular \( A \) is closed in \( \omega^\omega \). But every \( \Sigma^1_1 \) subset of \( \text{WO} \) codes only boundedly many ordinals since its \( \Pi^1_1 \)-norm on \( \text{WO} \).

\[ \text{Proof.} \]

Let \( \emptyset \in X \). For if \( \emptyset \in X \), then \( 0 \in X \). For if \( 0 \in X \), let \( y \in X \) be such that \( y \in x \). From the fact mentioned at the end of the previous paragraph, we have that any open \( (\text{or closed}) \) game on an ordinal is determined.

As an immediate corollary we have that any open or closed game on an ordinal is determined.

Although the full axiom \( \text{AD}_\kappa \) for uncountable \( \kappa \) is inconsistent, nevertheless ordinal games are very useful in determinacy theory. We will see later that important classes of ordinal games are determined.

When \( X = \mathbb{R} \), the axiom \( \text{AD}_X \) becomes a powerful strengthening of \( \text{AD} \). \( \text{AD}_\mathbb{R} \), as we will see later, is strictly stronger than \( \text{AD} \). Its extra strength, however, does not generally become apparent until one goes beyond the model \( L(\mathbb{R}) \), and so will not directly concern us for a while.

The next result is one of the most basic results in determinacy theory and is used in many arguments. For \( X \) a set, we topologixe \( X^\omega \) by giving \( X \) the discrete topology and \( X^\omega \) the corresponding product topology. So, a basic open set is of the form \( N_x = \{ \bar{x} \in X^\omega: (\bar{x} \upharpoonright \text{lh}(s)) = \bar{s} \} \) where \( s \in X^{<\omega} \).

**Theorem 1.12** (Gale-Stewart). Assume \( \text{ZF} \). Then for any set \( X \), and any open (or closed) \( A \subseteq X^\omega \), the game \( G_A \) is (quasi) determined.

**Proof.** Let \( E \) be the set of all \( s = (s(0), \ldots, s(2n-1)) \) of even length. We define a subset \( W \subseteq E \) of winning positions for I. Let \( W_0 = \{ s \in e: N_S \subseteq A \} \). In general, for \( \alpha \in \text{On} \), first let \( W_{\alpha} = \bigcup_{\beta < \alpha} W_\beta \). Then set

\[
W_\alpha = W_{\alpha} \cup \{ s \in E: \exists x \in X \forall y \in X \ s \upharpoonright x \downharpoonright y \in W_{\alpha} \}.
\]

Let \( \theta \) be the least ordinal such that \( W_\theta = W_{\theta} \). So, \( W_\alpha = W_\theta \) for all \( \alpha \geq \theta \). Let \( W = W_\theta \). For \( s \in W \), let \( |s| \) be the least ordinal \( \alpha < \theta \) such that \( s \in W_{\alpha} \). Note that if \( s \in W \) and \( |s| > 0 \), then \( \exists x \forall y (s \upharpoonright x \downharpoonright y < |s|) \).

Suppose first that \( \emptyset \in W \). Then I has a winning strategy \( \sigma \) in \( G_A \). Namely, let \( (s(0), s(1), \ldots, s(2n-1)) \in \Sigma \) if for all \( i < n \) either \( s \upharpoonright (2i) \in W_0 \) or \( |s \upharpoonright 2i| > |s \upharpoonright (2i + 2)| \). From the fact mentioned at the end of the previous paragraph, \( \sigma \) is a quasi-strategy for I. From the wellfoundedness of \( \text{On} \) it follows that I is a winning quasi-strategy. that is, for any \( \bar{s} \) a run according to \( \Sigma \), we have the ranks \( |s \upharpoonright 2i| \) decrease until we reach an \( i \) such that \( s \upharpoonright 2i \notin W_0 \), and thus \( \bar{s} \in A \).

Suppose next that \( \emptyset \notin W \). We claim that II has a winning quasi-strategy \( \Sigma \) in \( G_A \). Namely, let \( \Sigma = E - W \). Note that \( \emptyset \notin \Sigma \). Also, if \( s \in \Sigma \) then \( \forall x \exists y (s \upharpoonright x \downharpoonright y \in \sigma) \). For if \( s \in \sigma \) and \( \exists x \forall y (s \upharpoonright x \downharpoonright y \notin \sigma) \), then \( \exists x \forall y (s \upharpoonright x \downharpoonright y \in W) \).

So, \( s \in W_{\theta + 1} = W_\theta = W \), a contradiction. So, \( \Sigma \) is a quasi-strategy for II. If \( \bar{s} \in X^\omega \) is a run according to \( \Sigma \), then \( \bar{s} \notin A \). For if \( \bar{s} \in A \), then for some \( i \) we would have \( N_{\bar{s} \upharpoonright i} \subseteq A \) as \( A \) is open. But then \( \bar{s} \upharpoonright 2i \notin W_0 \subseteq W \), a contradiction (as \( \bar{s} \upharpoonright 2i \in E = W \)).
The proof of theorem 1.12 produces for any open or closed game $G_A$ a canonical winning quasi-strategy for $G_A$. This does not use any form of choice.

An important and deep theorem of Martin says that every Borel game on any set is determined. We will give this proof later, but for now we just extend theorem 1.12 one step further.

**Theorem 1.13** (Wolff). Assume ZF. Every $\Sigma^0_2$ game on a set $X$ is (quasi) determined.

*Proof.* Let $A \subseteq X^\omega$ be $\Sigma^0_2$, say $A = \bigcup_n F_n$ where each $F_n$ is closed in $X^\omega$. Let $T_n$ be the canonical tree on $X$ such that $F_n = [T_n]$. We again define a set $W$ of winning positions for $I$. Again let $E$ be the set of all $s \in X^{<\omega}$ of even length. Let $W_0$ be the set of all $s \in E$ such that there is an $n \in \omega$ such that $I$ has a winning quasi-strategy starting at $s$ for the game $F_n$. For general $\alpha$, let $W_{<\alpha} = \bigcup_{\beta<\alpha} W_\beta$ and then define $W_\alpha$ to be $W_{<\alpha}$ together with the set of $s \in E$ such that for some $i$, $I$ has a winning quasi-strategy starting at $s$ for $A_{<\alpha}^i$, where $A_{<\alpha}^i$ is the set of $x \in X^\omega$ such that either $\forall j x \upharpoonright 2j \in T_i$ or for the least $j$ such that $x \upharpoonright 2j \notin T_i$ we have $x \upharpoonright 2j \in W_{<\alpha}$. Note that $A_{<\alpha}^i$ is a closed set, so is quasi-determined.

Define $\theta$ and $W = W_\theta = W_{<\theta}$ as in theorem 1.12. Also as before define, for $s \in W$, $|s|$ to be the least $\alpha < \theta$ such that $s \in W_\alpha$.

First suppose $\emptyset \in W$, and we define a winning quasi-strategy $\Sigma$ for $I$ in $G_A$. We give an informal description of $\Sigma$, the underlying formal definition will be apparent. $s = (s(0), s(1), \ldots, s(2n-1))$ is according to $\Sigma$ provided the following holds. Let $n_0$ be least such that $I$ has a winning quasi-strategy starting at $\emptyset$ in $A^\omega_{<\theta}$. I follows the canonical winning quasi-strategy for this closed game until an $i_0$ is reached (if it is) such that $s \upharpoonright 2i_0 \notin T_{n_0}$. If this happens, then $s \upharpoonright 2i_0 \in W_{<\theta}$. Let $\theta_0 < \theta$ be least such that $s \upharpoonright 2i_0 \in W_{\theta_0}$. Let $n_1$ be least such that $I$ has a winning quasi-strategy in $A^\omega_{<\theta_0}$ starting from $s \upharpoonright 2i_0$. I then follows the canonical winning quasi-strategy for this game until an $i_1 > i_0$ is reached (if it is) such that $s \upharpoonright 2i_1 \notin T_{n_1}$. Thus, $s \upharpoonright 2i_1 \in W_{<\theta_0}$. Let $\theta_1 < \theta_0$ be least such that $s \upharpoonright 2i_1 \in W_{\theta_1}$. $\Sigma$ continues in this manner. If $\vec{s}$ is a run according to $\Sigma$, then for some $k$ we have that $\vec{s}$ is a run starting from $\vec{s} \upharpoonright 2k$ which stays in the tree $T_{n_k}$ (as otherwise we get an infinite decreasing sequence of $\theta_i$). Thus, $\vec{s} \in F_{n_k}$, so $\vec{s}$ is a win for $I$.

Suppose next that $\emptyset \notin W$. We describe a winning quasi-strategy for $II$ in $G_A$. Start with $n = 0$. Since $\emptyset \notin W_\theta$, II has a canonical winning strategy for the game $A^0_{<\theta}$ starting from $\emptyset$. II follows this strategy until a least $i_0$ is reached such that $s \upharpoonright 2i_0 \notin T_0$ and $s \upharpoonright 2i_0 \notin W_{<\theta} = W_\theta$. This must happen as II is winning for $A^0_{<\theta}$. Consider then $n = 1$. Since $s \upharpoonright 2i_0 \notin W_\theta$, II has a winning quasi-strategy for the game $A^1_{<\theta}$ starting from $s \upharpoonright 2i_0$. II follows this canonical strategy until an $i_1$ is reached such that $s \upharpoonright 2i_1 \notin T_1$ and $s \upharpoonright 2i_1 \notin W_{<\theta} = W_\theta$. This describes $\Sigma$. If $\vec{s}$ is a run according to $\Sigma$, then clearly $\vec{s} \notin [T_n] = F_n$ for all $n$, and thus $II$ wins the run.

\[\square\]

2. Regularity Results

In this section we show that AD implies regularity properties for sets of reals, that is, AD eliminates the pathological sets produced from AC.

First we consider the perfect set property. The basic game argument is best illustrated on the Cantor space.
Lemma 2.1 (ZF + AD). Every $A \subseteq 2^\omega$ is either countable or else contains a perfect subset.

Proof. Given $A \subseteq 2^\omega$ we consider the following game $G^*_A$:

<table>
<thead>
<tr>
<th></th>
<th>$s_0$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$i_0$</td>
<td>$i_1$</td>
<td>$i_2$</td>
<td>...</td>
</tr>
</tbody>
</table>

where I plays sequences $s_k \in 2^{<\omega}$ (possibly empty) and II plays $i_k \in \{0, 1\}$. I wins the run iff $x = s_0 \wedge i_0 \wedge s_1 \wedge i_1 \wedge \cdots$ is in $A$. If I has a winning strategy $\sigma$, then clearly $\sigma[2^\omega] = \{ x = s_0 \wedge i_0 \wedge s_1 \wedge i_1 \wedge \cdots : (s_0, i_0, \ldots) \text{ is according to } \sigma \}$ is a perfect set contained in $A$. Suppose then that II has a winning strategy $\tau$. Consider $x \in A$. We say a partial run $(s_0, i_0, \ldots, s_n, i_n)$ is good for $x$ if it is according to $\tau$ and $(s_0 \wedge i_0 \wedge \cdots \wedge s_n \wedge i_n)$ is an initial segment of $x$. There must be a maximal good sequence, since if every good sequence had a good extension then there would be a run $(s_0, i_0, \ldots)$ according to $\tau$ with $x = (s_0 \wedge i_0 \wedge \cdots)$. This contradicts $\tau$ being winning for II. Let $s = (s_0, i_0, \ldots, s_n, i_n)$ be maximal good for $x$. Say $s_0 \wedge i_0 \wedge \cdots \wedge s_n \wedge i_n = x \upharpoonright j$. So, for every $t \in 2^{<\omega}$, $(x \upharpoonright j)^{\tau}(s \wedge t)$ is incompatible with $x$. This, however, allows us to compute $x$ from $s$ and $\tau$. Namely, $x(j) = 1 - \tau(s \wedge \varnothing)$ (by $s \wedge \varnothing$ we mean the partial run consisting of $s$ and the one extra move by I consisting of $s_{n+1} = \varnothing$). Then $x(j+1) = 1 - \tau(s \wedge (x(j)))$, etc. Since the set of possible $s$ is countable (and $\tau$ is a single fixed real), this shows that $A$ is countable. 

As a consequence we have the following.

Theorem 2.2 (ZF + AD). Let $X$ be a Polish space. Then every $A \subseteq X$ is either countable or else contains a perfect subset.

Proof. We may clearly assume $X$ is uncountable, and thus is Borel isomorphic to $2^\omega$. Let $\pi : 2^\omega \to X$ be a Borel isomorphism. If $A \subseteq X$ is uncountable, then so is $B = \pi^{-1}(A)$. Let $P \subseteq B$ be perfect. Then $\pi(P) \subseteq A$ is uncountable and Borel, and so contains a perfect subset. 

We can also prove theorem 2.2 directly on an arbitrary Polish space $X$ by a game argument. We sketch this alternate proof. Let $\mathcal{U}$ be a countable base for $X$. Consider the game $G$ played as follows.

<table>
<thead>
<tr>
<th></th>
<th>$U_0^0$, $U_1^0$</th>
<th>$U_0^1$, $U_1^1$</th>
<th>$U_0^2$, $U_1^2$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$i_0$</td>
<td>$i_1$</td>
<td>$i_2$</td>
<td>...</td>
</tr>
</tbody>
</table>

For the first move, I plays a pair $U_0^0, U_1^0 \in \mathcal{U}$ where $U_0^0 \cap U_1^0 = \varnothing$, and $\text{diam}(U_0^0), \text{diam}(U_1^0) < \frac{1}{2}$. II plays an $i_0 \in \{0, 1\}$ signifying a choice of one of these sets. At the next round, I again plays a pair $U_0^1, U_1^1$ with $U_0^1 \cap U_1^1 = \varnothing$, $\text{diam}(U_0^1), \text{diam}(U_1^1) < \frac{1}{4}$, and $U_0^1 \subseteq U_0^0, U_1^1 \subseteq U_1^0$. II plays $i_1 \in \{0, 1\}$ which picks one of these sets. The play continues in this manner, and I wins the run iff $x \in A$ where $x = \bigcap_{n} U_{i_n}^n$. If I has a winning strategy $\sigma$, it is clear that following $\sigma$ against all possible plays of II
produces a perfect subset of $A$. Suppose II has a winning strategy $\tau$. Say a partial run of the game

$$(U^0_0, U^0_1), i_0, \ldots , (U^n_0, U^n_1), i_n$$

is $x$-good if it is a run according to $\tau$ where I has followed the rules above and $x \in U^n_1$. Since $\tau$ is winning for II, there is a maximal $x$-good sequence, say $s = (U^0_0, U^0_1), i_0, \ldots , (U^n_0, U^n_1), i_n$. So, for any pair $(U^0_0, U^0_1)$ satisfying the requirements for I, if $i_{n+1} = \tau(s^- (U^n_0, U^n_1))$, then $x \notin U^{n+1}_1$. Let $W$ be the union of all open sets $U^{n+1}_n$ which are attained in this manner. So, $W \subseteq U^n_1$. Also, $x \notin W$. We claim that $U^n_1 - W$ is a singleton. To see this, suppose $x \neq y$ are both in $U^n_1 - W$. Let $U^{n+1}_0, U^{n+1}_1$ be disjoint basic open sets of diameters $< \frac{1}{2^n}$ whose closures are contained in $U^n_1$, and with $x \in U^{n+1}_0$, $y \in U^{n+1}_1$. Let $i_{n+1}$ be II’s response. If $i_{n+1} = 0$, then $x \in W$, and if $i_{n+1} = 1$, then $y \in W$, a contradiction.

The determinacy used in theorem 2.2 was local, that is, to get the perfect set property for a pointclass $\Gamma$ requires only $\text{det}(\Gamma)$. In fact, we can improve this a little using an “unfolding argument” for an existential quantifier.

**Theorem 2.3.** Let $\Gamma$ be a pointclass and assume $\text{det}(\Gamma)$. Then for any Polish space $X$ and every $A \subseteq X$ in $\exists^* \Gamma$, $A$ is either countable or else contains a perfect set.

**Proof.** We give the proof in the case $X = 2^\omega$, leaving the general case to an exercise. Let $A \subseteq 2^\omega$ be in $\exists^* \Gamma$. Let $B \subseteq 2^\omega \times \omega^\omega$ be such that $A(x) \leftrightarrow \exists y B(x, y)$. Consider the following “unfolded perfect set game”:

<table>
<thead>
<tr>
<th>I</th>
<th>y(0)</th>
<th>y(1)</th>
<th>y(2)</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s_0</td>
<td>s_1</td>
<td>s_2</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>i_0</td>
<td>i_1</td>
<td>i_2</td>
<td>...</td>
</tr>
</tbody>
</table>

where I plays sequence $s_n \in 2^{<\omega}$ and integers $y(n) \in \omega$, and II plays $i_n \in \{0, 1\}$. Let $x = s_0 \upharpoonright i_0 \upharpoonright s_1 \upharpoonright i_1 \upharpoonright \cdots$, and $y = (y(0), y(1), \ldots)$. Then I wins the run iff $(x, y) \in B$. The payoff set is in $\Gamma$, and so the game is determined. If I has a winning strategy, then we clearly get a perfect set contained in $A$ ( $\{(\sigma(z))_0 : z \in 2^\omega \}$ is a perfect set contained in $A$). Suppose that II has a winning strategy $\tau$. Fix for the moment $x \in A$, and fix $y \in \omega^\omega$ such that $(x, y) \in B$. We say a partial run $\vec{s}$ according to $\tau$ is $x$-good if it is of the form $\vec{s} = ((y(0), s_0), i_0, \ldots , (y(n), s_n), i_n)$ where $s_0 \upharpoonright i_0 \upharpoonright \ldots \upharpoonright s_n \upharpoonright i_n = x \upharpoonright j$ is an initial segment of $x$. Since $\tau$ is winning for II, there is a maximal $x$-good sequence. Say $\vec{s} = ((y(0), s_0), i_0, \ldots , (y(n), s_n), i_n)$ is maximal good. Thus, if I plays $(y(n+1), s_{n+1})$ as the next move, where $s_{n+1} = (x(j), \ldots, x(t-1))$, and $\tau$ responds with $i_{n+1}$, then $x(t) = 1 - i_{n+1}$. This again gives an algorithm for computing $x$ from $\vec{s}$ and the fixed strategy $\tau$. Since there are only countable many possible $\vec{s}$, this shows $A$ is countable. □

**Exercise 9.** Give a direct game proof of theorem 2.3 on a general Polish space. [hint: modify the previously given perfect set game on a general $X$ to include the witness $y$.]

If $A$ is $\Sigma^1_1$, then there is a closed $B \subseteq X \times \omega^\omega$ such that $A(x) \leftrightarrow \exists y B(x, y)$. In this case, the unfolded perfect set game is closed, and so it is determined in $\text{ZF}$. 

This gives another proof of the perfect set property for $\Sigma^1_1$. On the other hand, $\det(\Pi^1_1)$ suffices to show the perfect set property for $\Sigma^1_2$. We will investigate the strength of $\det(\Pi^1_1)$ later. Note that projective determinacy give the perfect set property for all projective sets.

We next consider the Baire property, which is the topological notion of regularity. Recall from definition ?? that a set $A$ in a topological space is meager if it is a countable union of nowhere dense sets (equivalently, contained in a countable union of closed nowhere dense sets). Also, $A$ has the Baire property if there is an open set $U$ such that $A \triangle U$ is meager. An easy argument (c.f. lemma ??) shows every Borel set has the Baire property, and in theorem ?? we showed that every $\Sigma^1_1$ (or $\Pi^1_1$) set has the Baire property. This last fact will also follow by a game argument below.

We again first consider the case $X = \omega^\omega$ where the basic ideas are more clear. Let $A \subseteq \omega^\omega$. We consider the game $G^*_A$ played as follows:

$$
\begin{array}{cccc}
I & s_0 & s_2 & s_4 & \ldots \\
II & s_1 & s_3 & s_5 & \ldots \\
\end{array}
$$

where each player plays $s_i \in \omega^{<\omega}$. II wins the run iff $x \in A$ where $x = s_0 \upharpoonright s_1 \upharpoonright \ldots$.

**Lemma 2.4.** II has a winning strategy in $G^*_A$ if and only if $A$ is comeager in $\omega^\omega$. I has a winning strategy in $G^*_A$ if and only if there is a neighborhood on which $\omega^\omega - A$ is comeager.

**Proof.** First suppose $A$ is comeager in $\omega^\omega$. Say $A \supseteq \bigcap_n D_n$ where $D_n$ is dense open. Define the strategy $\tau$ for II as follows. Let $\tau(s_0, s_1, \ldots, s_{2n})$ be the least sequence $s_{2n+1}$ such that $N_s \subseteq D_n$, where $s = s_0 \upharpoonright s_1 \upharpoonright \ldots \upharpoonright s_{2n}$. This exists since $D_n$ is dense open. Clearly $\tau$ is a winning strategy for II in $G^*_A$.

Suppose next that II has a winning strategy $\tau$ in $G^*_A$. We define a sequence $D_n$ of dense open sets. Say a sequence $s \in \omega^{<\omega}$ is 1-good if it is of the form $s = s_0 \upharpoonright \tau(s_0)$ for some $s_0$. In general, say $s$ is $n$-good if it is of the form $s = s_0 \upharpoonright s_1 \upharpoonright \ldots \upharpoonright s_{2n+1}$ where $(s_0, s_1, \ldots, s_{2n+1})$ is a partial run according to $\tau$.

Let $M_1$ be a maximal set of pairwise incompatible 1-good elements. Let $D_1 = \bigcup \{N_s : s \in M_1 \}$. To see that $D_1$ is dense, consider a basic open set $N_t$. Then $u = t \upharpoonright \tau(t)$ is 1-good, so there is an $s \in M_1$ compatible with $u$. So, $N_s \cap N_t \neq \emptyset$, and so $N_s \cap N_t \neq \emptyset$. For each 1-good sequence $s$, pick a canonical partial run $(s_0, \tau(s_0))$ with $s_0 \upharpoonright \tau(s_0) = s$.

In general, suppose an antichain $M_n$ of $n$-good sequences has been defined and $D_n = \bigcup \{N_s : s \in M_n \}$ is dense. Assume also that for each $s \in M_n$ we have defined a canonical partial run $(s_0, s_1, \ldots, s_{2n-1})$ according to $\tau$ with $s = s_0 \upharpoonright \ldots \upharpoonright s_{2n-1}$, and that for all $m < n$, $(s_0, s_1, \ldots, s_{2m-1})$ is the canonical run associated to $s_0 \upharpoonright \ldots \upharpoonright s_{2m-1}$. Let $M_{n+1}$ be maximal subject to being an antichain, every $s \in M_{n+1}$ is $n+1$-good, and every $s \in M_{n+1}$ is of the form $s' \upharpoonright t \upharpoonright \tau(s_0, s_1, \ldots, s_{2n-1}, t)$ for some $s' \in M_n$ with associated partial run $(s_0, \ldots, s_{2n-1}, t)$. Let $D_{n+1} = \bigcup \{N_s : s \in M_{n+1} \}$. To see that $D_{n+1}$ is dense, consider a basic open set $N_t$. Let $s' \in M_n$ be compatible with $t$ and let $(s_0, s_1, \ldots, s_{2n-1})$ be the partial run associated to $s'$ (so $s' = s_0 \upharpoonright \ldots \upharpoonright s_{2n-1}$). Note that $u = s' \upharpoonright (t - s') \upharpoonright \tau(s_0, s_1, \ldots, s_{2n-1}, t - s')$ is $n+1$-good and extends $t$ (where $t - s'$ is the sequence such that $s' \upharpoonright (t - s') = t$).
if \( \text{lh}(s') \leq \text{lh}(t) \), and otherwise \( t - s' = \emptyset \)). By maximality, there is an \( s \in M_{n+1} \) such that \( s \parallel u \). Then \( N_s \cap N_\emptyset \neq \emptyset \), and so \( N_s \cap N_t \neq \emptyset \).

If \( x \in \bigcap_n D_n \), then for each \( n \) there is an initial segment \( x \upharpoonright i_n \) of \( x \) in \( M_n \), \( x \upharpoonright i_n \) is of the form \( s_0 \upharpoonright s_1 \upharpoonright \ldots \upharpoonright s_{2n-1} \) where \( (s_0, s_1, \ldots, s_{2n-1}) \) is the canonical partial run associated to \( x \upharpoonright i_n \). Note that \( (s_0, \ldots, s_{2n-3}) \) is the canonical partial run associated to \( x \upharpoonright i_{n-1} \) since \( M_{n-1} \) is an atichain. Thus, the \( s_n \) give a run according to \( \tau \) which produces the real \( x \). Hence, \( x \in A \).

If there is a neighborhood on which \( \omega^\omega - A \) is comeager, let \( s_0 \) be such that \( \omega^\omega - A \) is comeager on \( N_{s_0} \). Have I play \( s_0 \) for the first move, and then follow a strategy for II to get into \( \omega^\omega - A \) as in the first part of the proof. Conversely, if I has a winning strategy \( \sigma \), let \( s_0 = \sigma(\emptyset) \). \( \sigma \) then gives a strategy for II in the \( \ast \ast \) game starting from \( s_0 \) to get into \( \omega^\omega - A \). The first part of the proof shows that \( \omega^\omega - A \) is comeager on \( N_{s_0} \).

Lemma 2.4 says that every \( A \) is either comeager or else there is a basic open set \( U \) on which it is meager. Applying this to \( \omega^\omega - A \) gives that every \( A \) is either meager or else there is a basic open set \( U \) on which it is comeager (i.e., \( U - A \) is meager).

As a corollary we have the following.

**Theorem 2.5 (ZF + AD).** Every \( A \subseteq \omega^\omega \) has the Baire property.

**Proof.** Let \( A \subseteq \omega^\omega \). Let \( \{U_i\}_{i \in \omega} \) be a maximal, pairwise disjoint collection of basic open sets such that \( A \) is comeager of \( U_i \) (i.e., \( U_i - A \) is meager). By countable additivity, \( A \) is comeager on \( U = \bigcup_i U_i \). It suffices to show that \( A - U \) is meager. If \( A - U \) is not meager, then by lemma 2.4, there is a basic open set \( V \) on which \( A - U \) is comeager. By maximality, \( V \cap U_i \neq \emptyset \) for some \( i \). Let \( W \subseteq V \cap U_i \) be basic open. Then \( \omega^\omega - W \) is comeager on \( W \), a contradiction since \( U_i \subseteq W \) is nonmeager.

We can also define an analog of the \( \ast \ast \) game on a general Polish space \( X \). In fact, it makes sense to define this game on a general topological space. We assume for this discussion that \( X \) has a wellorderable base \( U \) (which, of course, holds if \( X \) is second countable). Given \( A \subseteq X \), \( X \) a topological space, the general game \( G^\ast_A \) is played as follows.

\[
\begin{array}{cccccc}
I & U_0 & U_1 & U_2 & U_3 & \ldots \\
II & U_1 & U_2 & U_3 & U_4 & \ldots \\
\end{array}
\]

where the \( U_i \) are basic open sets and \( U_i \subseteq U_{i-1} \) (the first player to violate this rule loses). II wins the run iff \( \bigcap_n U_n \subseteq A \). If \( A \) is comeager, then easily II has a winning strategy as before. Assume now II has a winning strategy \( \tau \) in \( G^\ast_A \). Let \( M_1 \) be a maximal pairwise disjoint collection of basic open sets \( U_1 \) such that for some \( U_0 \), \( (U_0, U_1) \) is a partial run of \( \tau \). As before, \( \bigcup M_1 \) is dense in \( X \). For each \( U_1 \in M_1 \), let \( s_1(U_1) = (U_0, U_1) \) be a partial run of \( \tau \) ending with \( U_1 \). We can do this since \( U \) is wellordered. In general, assume \( M_{2n-1} \) has been defined and is a pairwise disjoint collection of open sets \( U_{2n-1} \), and each \( U_{2n-1} \) is the last set played in a partial run following \( \tau \). In fact, assume for each \( U_{2n-1} \in M_{2n-1} \), a canonical run \( s_{2n-1}(U_{2n-1}) = (U_0, U_1, \ldots, U_{2n-3}, U_{2n-2}, U_{2n-1}) \) according to
\( \tau \) is given, where each \( U_{2j-1} \in M_{2j-1} \) and \( s_{2n-1}(U_{2n-1}) \) extends \( s_{2n-3}(U_{2n-3}) \). Assume also that \( \cup M_{2n-1} \) is dense in \( X \). Let then \( M_{2n+1} \subseteq U \) be maximal subject to being an antichain and for every \( U_{2n+1} \in M_{2n+1} \), there is a partial run \( (U_0, U_1, \ldots, U_{2n-1}, U_{2n}, U_{2n+1}) \) according to \( \tau \) such that \( U_{2n-1} \in M_{2n-1} \) and \( s_{2n-1}(U_{2n-1}) = (U_0, U_1, \ldots, U_{2n-1}) \). As before, \( \cup M_{2n+1} \) is dense. Using the wellordering of \( U \), we can easily define \( s_{2n+1} \) (the sequence of functions \( (s_n)_{n \in \omega} \) is actually being constructed from the wellordering of \( U \)).

Let \( D_{2n+1} = \cup M_{2n+1} \). Suppose \( x \in \bigcap_n D_{2n+1} \). For each \( n \), let \( U_{2n+1} \in M_{2n+1} \) be such that \( x \in U_{2n+1} \). Let \( s_{2n+1} = s_{2n+1}(U_{2n+1}) \). The antichain property of the \( M_{2n+1} \) gives that \( s_1 \subseteq s_3 \subseteq \cdots \subseteq s_{2n+1} \) (that is, these sequences extend each other). Thus, there is a run of \( \sigma \) where the odd moves are the \( U_{2n+1} \). Since \( \tau \) is winning for \( \lll \), \( \bigcap_n U_{2n+1} \subseteq A \), and so \( x \in A \). Thus, \( A \) is comeager.

The same argument essentially works when \( I \) has a winning strategy, provided \( X \) satisfies a technical assumption (*): there are \( U_n \subseteq U \), each \( U_n \) a base, such that if \( U_n \in U_n \) for all \( n \), then \( \bigcap_n U_n \) contains at least one point. For example, any metric space satisfies (*). Let \( U_0 = \sigma(\emptyset) \). \( I \)'s strategy \( \sigma \) gives a strategy \( \tau \) for \( \lll \) in the game stating from \( U_0 \) to produce a sequence \( U_0, U_1, \ldots \) such that \( \bigcap_n U_n \not\in A \). From \( \tau \) we can easily get a strategy \( \tau' \) for \( \lll \) which is also winning for \( \lll \) in this game starting from \( U_0 \), and with the additional property that for every \( U_{2n} \) which \( \tau \) plays at stage \( n \), there is a \( U \in U_{2n} \) such that \( U_n \subseteq U \) [after \( I \) plays \( U_{2n-1} \), two picks the least basic open set \( V \subseteq U_{2n-1} \) with \( V \in U_n \) and follows \( \tau \) against \( I \)'s last move of \( V \).] Define the sets \( D_{2n} \) as before using \( \tau' \), so the \( D_{2n} \) are open and dense in \( U_0 \). If \( x \in \bigcap_n D_{2n} \), then as before there is a run \( (U_0, U_1, \ldots) \) following \( \tau' \) such that \( \forall n (x \in U_{2n}) \) and \( \bigcap_n U_{2n} \not\in A \). From (*) we have \( \bigcap_n U_{2n} = \{x\} \). Thus, \( x \in A \).

So, for \( X \) satisfying (*), and \( A \subseteq X \), \( \lll \) has a winning strategy if \( A \) is comeager, and \( I \) has a winning strategy if \( A \) is meager on some non-empty open \( U_0 \). If \( X \) is second countable, then the game \( G_{A^*} \) is essentially an integer game, and thus determined. Thus, for every \( A \subseteq X \), either \( A \) is comeager or else meager in some open set \( U \). Equivalently, every \( A \subseteq X \) is either meager or else comeager in some open \( U \). This gives the following.

**Theorem 2.6 (ZF + AD).** Every subset \( A \) of a second countable space \( X \) satisfying (*) has the Baire property.

The proof is exactly as before (Theorem 2.5), using the following exercise.

**Exercise 10.** Assume ZF + AD, and let \( X \) be second countable. Show that a countable union of meager sets is meager. [Hint: Let \( A_n \) be meager. A sequence of closed, nowhere sets whose union is \( A_n \) can be coded by a real. This reduces the choice needed to countable choice for reals, which follows from AD.]

**Exercise 11.** Assume ZF + AC. Let \( \{U_\alpha\}_{\alpha < \kappa} \) be a pairwise disjoint collection of non-empty open sets in a topological space \( X \). Suppose \( \{A_\alpha\}_{\alpha < \kappa} \) are such that each \( A_\alpha \subseteq U_\alpha \) is meager. Show that \( \bigcup_{\alpha < \kappa} A_\alpha \) is meager. [Hint: use AC to write each \( A_\alpha \) as a union \( A_\alpha = \bigcup_n F^n_\alpha \), where \( F^n_\alpha \) is closed nowhere dense in \( X \). Show that \( \bigcup_n F^n_\alpha \) is nowhere dense for each \( n \).]

Recall a function \( f : X \rightarrow Y \) is said to be Baire measurable if for every open \( U \subseteq Y \), \( f^{-1}(U) \) has the Baire property in \( X \).
Lemma 2.7. Let \( X, Y \) be Polish and \( F : X \to Y \) have the Baire property. Then there is a comeager \( A \subseteq X \) such that \( f \upharpoonright A \) is continuous.

Proof. Let \( \{V_i\}_{i \in \omega} \) be a base for \( Y \). For each \( i \), let \( U_i \subseteq X \) be open such that \( M_i = U_i \triangle f^{-1}(V_i) \) is meager. Let \( M = \bigcup_i M_i \), so \( M \) is meager in \( X \), and let \( A = X - M \). Then \( (f \upharpoonright A)^{-1}(V_i) = U_i \cap A \) and so \( f \upharpoonright A \) is continuous. \( \Box \)

As a corollary we have the following theorem.

Theorem 2.8 (ZF + AD). Let \( f : X \to Y \), where \( X, Y \) are Polish. Then there is a comeager \( A \subseteq X \) such that \( f \upharpoonright A \) is continuous.

Another important consequence of AD is comeager uniformization.

Theorem 2.9 (ZF + AD). Let \( X, Y \) be Polish and \( R \subseteq X \times Y \) be such that \( \forall x \exists y R(x,y) \). Then there is a comeager \( A \subseteq X \) and a function \( f : A \to Y \) such that \( \forall x \in A R(x,f(x)) \).

Proof. We give the proof for the case \( X = \omega^\omega \), \( Y = 2^\omega \) leaving the general case to an exercise below. So, suppose \( R \subseteq \omega^\omega \times 2^\omega \) and \( \text{dom}(R) = \omega^\omega \). Consider the following "unfolded" variation of the \( \ast \ast \) game.

\[
\begin{array}{cccccc}
I & s_0 & s_2 & s_4 & \ldots \\
II & s_1 & s_3 & s_5 & \ldots \\
& y(0) & y(1) & y(2) & \ldots \\
\end{array}
\]

where \( s_i \in \omega^{<\omega} \) and \( y(i) \in \{0, 1\} \). Let \( x = s_0 \upharpoonright s_1 \upharpoonright \ldots \), and let \( y = (y(0), y(1), \ldots) \). Then II wins the run iff \( R(x,y) \). Suppose first that II has a winning strategy \( \tau \). We define a sequence of dense open sets \( D_i \) as in the proof of lemma 2.4. Let \( M_1 \subseteq \omega^{<\omega} \) be a maximal antichain of 1-good sequences, where \( s \) is 1-good if it is of the form \( s = t \upharpoonright \tau_0(t) \) for some \( t \) (here \( \tau_0 \) denotes the sequence part of \( \tau \)'s response). Let \( D_1 = \bigcup \{N_s : s \in M_1 \} \), so \( D_1 \) is dense open as in lemma 2.4. Suppose the maximal antichain \( M_1 \) has been defined and consists of \( i \)-good sequences, where \( s \) is \( i \)-good if there is a partial run \( (s_0, (s_1, y(0)), \ldots, s_{2i-1}, (s_{2i-1}, y(i-1))) \) according to \( \tau \) with \( s = s_0 \upharpoonright s_1 \upharpoonright \ldots \upharpoonright s_{2i-1} \). Assume we have associated to each \( s \in M_1 \) a canonical such partial run of \( \tau \).

Let \( M_{i+1} \) be maximal subject to being an antichain and every for \( s \in M_{i+1} \) there is a (unique) \( t \in M_i \) and associated partial run \( (s_0, (s_1, y(0)), \ldots, s_{2i-1}, (s_{2i-1}, y(i-1))) \) (so \( t = s_0 \upharpoonright s_1 \upharpoonright \ldots \upharpoonright s_{2i-1} \)) such that \( s = t \upharpoonright s_{2i} \upharpoonright s_{2i+1} \) where

\[
(s_0, (s_1, y(0)), \ldots, s_{2i-1}, (s_{2i-1}, y(i-1)), s_{2i}, (s_{2i+1}, y(i)))
\]

is according to \( \tau \) (for some \( y(0), \ldots, y(i) \)). As in lemma 2.4, each \( M_i \) is a maximal antichain, and so \( D_i = \bigcup \{N_s : s \in M_i \} \) is dense. If \( x \in \bigcap_i D_i \), then there is a unique run according to \( \tau \) in which \( x \) is the concatenation of the sequences played. Let \( y \) be the corresponding real resulting from the integer moves that \( \tau \) makes in this run. since \( \tau \) is winning for II, \( R(x,y) \). Note, in fact, that if \( A = \bigcap_i D_i \), then the map \( x \in A \leftrightarrow y \) is continuous.

To finish, we show that I cannot have a winning strategy. Suppose \( \sigma \) were a winning strategy for I. We build a particular \( x \in \omega^\omega \) such that for any \( y \in \omega^\omega \), there is a run according to \( \sigma \) which produces \((x,y)\). This contradicts \( \sigma \) being
winning for I, as we can consider the run that produces \((x, y)\) where \(y\) is such that \(R(x, y)\). We define sequences \(u_0, u_1, \cdots\) with \(u_{i+1}\) extending \(u_i\) and let \(x\) be the union of the \(u_i\).

To begin, let \(u_0 = \sigma(\emptyset)\). To define \(u_1\), we define two sequences \(t_0 \subseteq t_1\) extending \(u_0\). We let \(t_0 = u_0 \prec \sigma(u_0, (\emptyset, 0))\) and \(t_1 = u_0 \prec \sigma(u_0, (t_0 - u_0, 1))\), where \(t_0 - u_0\) denotes the sequence \(v\) such that \(u_0 \prec v = t_0\). Let \(u_1 = t_1\). Note, there are two partial runs according to \(\sigma\), in one of which II plays 0 as the first integer move and the other in which II plays 1, both of which produce sequence moves whose concatenation is an initial segment of \(u_1\).

To define \(u_2\) we define sequences \(t_{00} \subseteq t_{01} \subseteq t_{10} \subseteq t_{11}\). There is a partial run \(s_0, (s_1, 0), s_2\) of \(\sigma\) such that \(s = s_0 \prec s_1 \prec s_2\) is an initial segment of \(u_1\). Let \(t_{00} = u_1 \prec \sigma(s_0, (s_1, 0), s_2, ((u_1 - s), 0))\), where \(u_1 - s\) means the sequence \(v\) such that \(s \prec v = u_1\). Let \(t_{01} = u_1 \prec \sigma(s_0, (s_1, 0), s_2, ((t_{00} - s), 1))\). Note that \(t_{00} \subseteq t_{01}\). There is also a partial run \(s_0', (s_1', 1), s_2'\) according to \(\sigma\) in which \(s' = s_0' \prec s_1' \prec s_2'\) is an initial segment of \(u_1\). Let \(t_{10} = u_1 \prec \sigma(s_0', (s_1', 1), s_2', ((t_{00} - s'), 0))\). Note that \(t_{01} \subseteq t_{10}\). Continuing, we define \(t_{11}\). Let \(u_2 = t_{11}\). For any \(y(0), y(1) \in \{0, 1\}\) there is a partial run according to \(\sigma\) of the form \(s_0, (s_1, y(0)), s_2, (s_3, y(1))\), \(s_4\) such that \(s = s_0 \prec \cdots \prec s_4\) is an initial segment of \(u_2\).

In general, we define \(u_n\) by defining increasing sequences \(t_{y|n}\) extending \(u_{n-1}\) (the \(y \upharpoonright n\) are ordered lexicographically), and letting \(u_n = t_{1|n-1}\). Let \(x\) be the limit of the \(u_n\). By construction, \(x\) has the property stated above, which gives a contradiction. \(\square\)

**Corollary 2.10 (ZF+AD).** Let \(R \subseteq X \times Y\) with \(X, Y\) Polish. Suppose \(\forall x \exists y R(x, y)\). Then there is a comeager \(A \subseteq X\) and a continuous function \(f: A \to Y\) such that \(\forall x \in A \ R(x, f(x))\).

**Exercise 12.** Prove theorem 2.9 for arbitrary Polish spaces. \([\text{hint: Play the game where I and II plau basic open sets } U_i \text{ with } U_0 \supseteq U_1 \supseteq \cdots, \text{diam}(U_i) < \frac{1}{2}\text{, and } U_i \subseteq U_{i-1}\]. II also plays basic open sets \(V_i\) satisfying these same requirements. Let \(\{x\} = \bigcap_i U_i\), and \(\{y\} = \bigcap_i V_i\). II wins the run iff \(R(x, y)\). If II has a winning strategy, get the dense open sets \(D_i\) as in the discussion after theorem 2.5. Argue that I cannot have a winning strategy as in the proof of theorem 2.9.

From these results on category we now have the following basic fact.

**Theorem 2.11 (ZF + AD).** There does not exist an uncountable wellordered set of reals.

**Proof.** Suppose \(\{x_\alpha\}_{\alpha < \omega_1} \subseteq \omega^\omega\) was a sequence of distinct reals. Let \(P \subseteq \{x_\alpha\}_{\alpha < \omega_1}\) be perfect. Since \(P\) is in bijection (in fact, homeomorphic to) to \(2^\omega\), we may assume that we have a wellordering \(\prec\) of \(2^\omega\) of length \(\omega_1\). Consider \(\prec\) as a subset of \(2^\omega \times 2^\omega\). For every \(y\), \(\{x: x \prec y\}\) is countable, and therefore is meager. So, \(\forall^* y \forall^* x \neg(x \prec y)\). Since \(\prec\) has the Baire property, by Kuratowski-Ulam we have \((2^\omega \times 2^\omega) - \prec\) is comeager, and thus \(\forall^* x \forall^* y \neg(x \prec y)\). This is a contradiction since for any \(x\), \(\{y: x \prec y\}\) is co-countable and thus comeager. \(\square\)

**Remark 2.12.** The proof of theorem 2.11 shows in ZF that if there is a wellordering of the reals, then there is a set without the Baire property. See also remark 2.19.

**Exercise 13.** Show from AD that there does not exists an \(\omega_1\) sequence of distinct closed or open sets in a Polish space. \([\text{remark: the same is actually true for any level of the Borel hierarchy; we’ll show this later.}]\)
Another application of this argument gives the following important fact.

**Theorem 2.13 (ZF + AD).** A wellordered union of meager sets is meager.

**Proof.** Suppose \( \{A_\alpha\}_{\alpha < \theta} \) is given with each \( A_\alpha \) meager, but \( \bigcup_{\alpha < \theta} A_\alpha \) non-meager. Let \( \rho \leq \theta \) be the least ordinal such that \( \bigcup_{\alpha < \rho} A_\alpha \) is non-meager. Since \( A \) has the Baire property, there is a basic open set \( U \) in \( X \times X \) such that \( A \) is comeager on \( U \times U \). Let

\[
\prec = \{(x,u) \in U \times U : \exists \alpha < \beta (x \in A_\alpha - \bigcup_{\alpha' < \alpha} A_{\alpha'} \land y \in A_\beta - \bigcup_{\beta' < \beta} A_{\beta'})\}.
\]

For all \( x \in U \), \( \{y \in U : x \prec y\} \) is comeager in \( U \). So, for comeager many \( y \in U \), \( \{x \in U : x \prec y\} \) is comeager in \( U \). However, the last set is meager by the minimality of \( \rho \). \( \square \)

We now discuss the situation concerning measure.

**Exercise 14.** Let \( \mu \) be a Borel probability measure on \( 2^\omega \). Let \( A \) be a \( \mu \) measurable set. Show that for any \( \epsilon > 0 \), there are basic open sets \( N_j \) (in the usual base for \( 2^\omega \)) such that \( A \subseteq \bigcup_j N_j \) and \( |\mu(A) - \sum_j \mu(N_j)| < \epsilon \). [Hint: Since \( \mu \) is regular, it is enough to show this for open \( A \). Write \( A \) as a countable disjoint union of basic open sets.]

**Theorem 2.14.** Let \( \mu \) be a Borel probability measure on a Polish space \( X \). Then every \( A \subseteq X \) is \( \mu \)-measurable.

**Proof.** We first argue that it is enough to prove the theorem in the case \( X = 2^\omega \). Let \( \mu \) be a Borel measure on the Polish space \( X \), and let \( \pi : X \rightarrow 2^\omega \) be a Borel bijection (we assume \( X \) is uncountable, or the result is trivial). Let \( \nu = \pi(\mu) \). That is, \( \nu \) is the measure on \( 2^\omega \) given by \( \nu(B) = \mu(\pi^{-1}(B)) \). Suppose \( A \subseteq X \) is given.

Let \( B = \pi[A] \). By the \( 2^\omega \) case of the theorem, there is a Borel set \( C \subseteq 2^\omega \) such that \( Z = B \triangle C \) has \( \nu \) meaquase 0. So, there is a Borel set \( D \) with \( Z \subseteq D \) and \( \nu(D) = 0 \). Then \( \pi^{-1}(C) \) and \( \pi^{-1}(D) \) are Borel sets in \( X \) and \( \mu(\pi^{-1}(D)) = 0 \). Since \( \pi \) is a bijection, \( A \triangle \pi^{-1}(C) = \pi^{-1}(D) \). Thus, \( A \) is \( \mu \)-measurable.

Suppose now \( A \subseteq 2^\omega \), and \( \mu \) is a Borel probability measure on \( 2^\omega \). Let \( \mathcal{M} \) be a maximal collection of Borel sets of positive measure which are pairwise almost disjoint, and almost contained in \( A \) (“almost” refers to the measure \( \mu \)). \( \mathcal{M} \) must be countable, say \( \mathcal{M} = \{B_i\}_{i<\omega} \). Let \( B = \bigcup_i B_i \). So, \( \mu(B - A) = 0 \) and every Borel subset of \( A - B \) has \( \mu \) measure 0. It suffices to show that \( A - B \) has measure 0, as then \( A = (A - B) \cup (B - (B - A)) \), which is a union of a measure 0 set and a Borel set minus a measure 0 set.

Changing notation, let us assume that \( A \subseteq 2^\omega \) and every Borel subset of \( A \) has \( \mu \) measure 0. We must show that \( A \) has \( \mu \) measure 0. Note that in fact every \( \mu \)-measurable subset of \( A \) has \( \mu \) measure 0.

For every fixed \( \epsilon > 0 \) we consider the following “covering” game \( G_\epsilon(A) \) (due to Harrington).

<table>
<thead>
<tr>
<th>I</th>
<th>x(0)</th>
<th>x(1)</th>
<th>x(2)</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>a_0</td>
<td>a_1</td>
<td>a_2</td>
<td>...</td>
</tr>
</tbody>
</table>
where I plays $x(i) \in \{0,1\}$ building a real $x \in 2^\omega$, and II plays integers $a_i \in \omega$ which we think of as coding a finite sequence $N^i_0, \ldots, N^i_k$ of basic open sets in $2^\omega$. II must must play so that $\sum_{j \leq k} \mu(N^i_j) \leq \frac{\epsilon}{2^i}$ (otherwise II loses). If II follows the rules, then II wins the run iff $(x \in A \rightarrow x \in \bigcup_{i \leq k} \bigcup_{j \leq k} N^i_j)$.

Suppose I had a winning strategy $\sigma$. Note that $B = \sigma[\omega^\omega] \subseteq A$, and $B \in \Sigma^1_1$. In particular, $B$ is $\mu$ measurable, and so $\mu(B) = 0$. Let $N_1, N_2, \ldots$ be a sequence of basic open sets in $2^\omega$ with $A \subseteq \bigcup_j N_j$ and such that $\sum_j \mu(N_j) < \epsilon$. Let $a_0$ code the first $k_0$ of the $N_j$, where $k_0$ is large enough so that $\sum_{j > k_0} \mu(N_j) < \frac{\epsilon}{2}$. Let $a_1$ code the sets $N_{k_0+1}, \ldots, N - k_1$ where $k_1$ is large enough so that $\sum_{j > k_1} \mu(N_j) < \frac{\epsilon}{2^2}$. Continue in this manner to define $a_i$, so that the sum of the $\mu(N_j)$ for $j$ coded by $a_i$ is less than $\frac{\epsilon}{2^i}$. If II plays the $a_i$, then II wins the run, a contradiction.

Suppose now that $\tau$ is a winning strategy for II. For each $i$, let $U_i$ be the union of all the $N^i_j = N^i_j(s)$ coded by some $a_i$ of the form $\tau(s)$ for $s \in 2^i$. The measure of $U_i$ is at most $2^i \cdot \frac{\epsilon}{2^i} = \frac{\epsilon}{2}$. Then $U = \bigcup U_i$ contains $A$ and $\mu(U) \leq \sum_i \frac{\epsilon}{2^i} = 2\epsilon$.

So, for every $\epsilon > 0$, $A$ can be covered by an open set of $\mu$ measure less than $\epsilon$. This shows that $A$ has $\mu$ measure 0. 

Many of the sets constructed with $\text{AC}$ give rise to sets which are non-measurable and without the Baire property. For example, consider an ultrafilter $U$ on $\omega$. We view $U$ as a subset of $2^\omega$ by identifying an $A \subseteq \omega$ with its characteristic function $\chi_A \in 2^\omega$. We then have the following.

**Theorem 2.15 (ZF).** Suppose $U$ is a non-principal ultrafilter on $\omega$. Then $U$ is non-measurable and does not have the Baire property (measurable here refers to the standard Bernoulli measure on $2^\omega$).

**Proof.** Suppose first that $U$ has the Baire property. Note that for any $x, y \in 2^\omega$, if $\exists n \forall m \geq n \ (x(m) = y(m))$, then $x \in U$ iff $y \in U$ (by $x \in U$ we mean $\{n : x(n) = 1\} \in U$). This is because every co-finite set is in the ultrafilter. If $s \in 2^{<\omega}$ is such that $U$ is comeager on $N_s$, then it follows that $U$ is comeager on every $N_t$ with $\text{lth}(t) = \text{lth}(s)$. This is because the natural bijection between $N_s$ and $N_t$ is a homeomorphism, and so preserves category. Thus, either $U$ is meager or else comeager. Consider now the map $\pi : 2^\omega \to 2^\omega$ defined by $\pi(x)(n) = 1 - x(n)$. Clearly $\pi$ is a homeomorphism, and so $U$ is meager (or comeager) iff $\pi(U)$ is meager (or comeager). However, $x \in U$ if $\pi(x) \notin U$, and so $U$ is meager (or comeager) iff $2^\omega - U$ is meager (or comeager), a contradiction.

Suppose next that $U$ is measurable. If $U$ has positive measure, then for any $\epsilon > 0$ there is a basic open set $N_s$ such that $\mu(U \cap N_s) > (1 - \epsilon)\mu(N_s)$ (Lebesgue density theorem). Again, the natural bijection between $N_s$ and $N_t$, where $\text{lth}(s) = \text{lth}(t)$ is measure preserving, and so $\mu(U \cap N_t) > (1 - \epsilon)\mu(N_t)$. It follows that $\mu(U) \geq 1 - \epsilon$.

Since this holds for all $\epsilon > 0$, we have $\mu(U) = 1$. So, either $\mu(U) = 0$ or $\mu(U) = 1$.

Consider again the map $\pi$, and note that $\pi$ is also measure preserving. So, $\mu(U) = 0$ (or $= 1$) iff $\mu(\pi(U)) = 0$ (or $= 1$). Since $\pi$ flips membership in $U$, we have that $\mu(U) = 0$ (or $= 1$) iff $\mu(2^\omega - U)) = 0$ (or $= 1$), a contradiction. 

As a consequence of this, we have the following.

**Theorem 2.16 (AD).** Every ultrafilter on a set $X$ is countably additive.

**Proof.** Let $U$ be an ultrafilter on $X$. We may assume $U$ is non-principal since principal ultrafilters are arbitrarily additive. Suppose $U$ is not countable additive,
say $A_n \subseteq X$ are such that $A_n \notin U$ but $A = \bigcup_n A_n \in U$. We may assume the $A_n$ are increasing by finite additivity of $\mathcal{U}$. Define $f: \omega \to \omega$ by $f(x) =$ the least $n$ such that $x \in A_n$. Let $\mu = f(U)$, so $\mu$ is an ultrafilter on $\omega$. $\mu$ is non-principal since each $A_n \notin U$. From theorem 2.15, $\mu$ does not have the Baire property as a subset of $2^\omega$. This contradicts theorem 2.5. \qed

**Definition 2.17.** A measure on a set $X$ is a countably additive ultrafilter on $X$.

So, assuming AD, every ultrafilter on a set $X$ is a measure. As another example we have the following.

**Lemma 2.18.** If there is a wellordering of the reals, then there is a set without the Baire property, and there is a set which is not measurable.

**Proof.** Suppose $\{x_\alpha\}_{\alpha<\omega}$ is a wellordering of $2^\omega$. Suppose that every subset of $2^\omega$ has the Baire property. Let $\rho \leq c$ be least such that $A_\rho = \{x_\alpha\}_{\alpha<\rho}$ is non-meager. Let $\prec = \{(x_\alpha, x_\beta): \alpha < \beta < \rho\}$. We have that $\forall y \in 2^\omega \forall^* x \in 2^\omega ((x,y) \not\in \prec)$. Since $\prec$ has the Baire property, by Kuratowski-Ulam we have $\forall^* x \forall^* y ((x,y) \not\in \prec)$. Since $A_\rho$ is non-meager, there is an $x \in A_\rho$ such that $\forall^* y ((x,y) \not\in \prec)$. Say $x = x_\alpha$, where $\alpha < \rho$. Since $\{x_\beta: \alpha < \beta < \rho\}$ must be non-meager, there is an $x_\beta$ in this set such that $(x_\alpha, x_\beta) \notin \prec$, contradicting the definition of $\prec$.

The argument for measure is similar, using Fubibi’s theorem instead of Kuratowski-Ulam. \qed

**Remark 2.19.** Shelah has shown in ZF + DC that if there is an uncountable wellordered set of reals, then there is a non-measurable set. The proof of this also shows that if every $\Sigma^1_3$ is measurable, then $\forall x (\omega^L[x] < \omega_1)$. Thus, the statement that every $\Sigma^1_3$ set is measurable has the consistency strength of an inaccessible cardinal. On the other hand, Shelah has shown that if ZF is consistent then so is ZF + “every set of reals has the Baire property”. So, even the Baire property for all sets of reals doesn’t have consistency strength beyond ZF. This is an interesting asymmetry between measure and category.

### 3. Turing and Wadge Degrees

If $x, y$ are in $\omega^\omega$ or $2^\omega$, then we say $x$ is Turing reducible to $y$, $x \leq_T y$ iff $x$ can be effectively computed from $y$. This can be made precise in several different ways. For example, one use the notion of computation from an oracle. In this approach, the ordinary notion of a Turing machine is modified to include an auxiliary tape (the “oracle”) which is read-only, and which the machine is allowed to scan at any time (in the current position). The action the machine takes at any step is allowed to depend on this scanned value as well as the value on the read-write tape (and the state of the machine as usual). Thus, when we say “$x$ is computed from $y$,” we mean that there is a Turing machine which when started with input $n$ on the read-write tape and $y$ on the read-only tape will terminate with the correct value of $x(n)$.

We identify subsets of $\omega$ with elements of $2^\omega$ via their characteristic functions, and we frequently pass between the two points of view. We say an $x$ in $\omega^\omega$ or $2^\omega$ (ar a subset of $\omega$) is recursive if $x$ is computable from the constant 0 sequence (i.e., $x$ is just outright computable). We say $x \equiv_T y$ if $x \leq_T y$ and $y \leq_T x$. It is clear that $\equiv_T$ is an equivalence relation on the reals and $\leq_T$ is a partial order on the equivalence classes. By a Turing degree we mean an equivalence class $[x]_T = \{y: y \equiv_T x\}$. 

Clearly each Turing degree is a countable set of reals. We sometimes write 0 for the Turing equivalence class of the constant 0 real, that is, the class of all recursive reals. We write $D$ for the set of Turing degrees. We frequently use $d$ to denote a Turing degree (i.e., $d = [x]_T$ for some $x \in \omega^\omega$).

By the cone above $x$, we mean the set $\{y : x \leq_T y\}$. This is clearly a set of Turing degrees. We have the following fundamental theorem of Martin.

**Theorem 3.1** (Martin). Every set of Turing degrees either contains or omits a cone of degrees.

*Proof.* Let $A \subseteq D$ be a set of degrees. Consider the usual game $G_A$:

<table>
<thead>
<tr>
<th>I</th>
<th>$x_0$</th>
<th>$x_2$</th>
<th>$x_4$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$x_1$</td>
<td>$x_3$</td>
<td>$x_5$</td>
<td>...</td>
</tr>
</tbody>
</table>

So, I wins the run iff $x \in A$. Suppose that I has a winning strategy $\sigma$. Let $d \in D$ and $\sigma \leq_T d$ (i.e., $d$ is in the cone above $\sigma$). Consider the run of the game where I follows $\sigma$ and II plays any $x$ with $d = [x]_T$. The resulting run $\sigma(x)$ is computable from $\sigma$ and $x$, and thus computable from $x$ (since $\sigma \leq_T x$). Clearly $x$ is also computable from $\sigma(x)$ ($x = (\sigma(x))_1$). So, $d = [x]_T = [\sigma(x)]_T$. Since $\sigma$ is winning for I, $\sigma(x) \in A$, and since $A$ is a set of degrees, $d \in A$. So, $A$ contains the cone above $\sigma$. A similar argument shows that if II has a winning strategy $\tau$, then $D - A$ contains the cone above $\tau$. □

In view of theorem 3.1 we make the following definition.

**Definition 3.2.** The Martin measure on the set of turing degrees $D$ is the measure defined by cones, that is, $A \subseteq D$ has measure one iff $A$ contains a cone of degrees.

Theorem 3.1 says that the Martin measure is in fact a measure (i.e., a countably additive ultrafilter) on $D$ (recall that from AD every ultrafilter is countably additive, although here it is obvious directly that the Martin measure is countably additive).

We can give an improvement to theorem 3.1. To state this, we make the following definition.

**Definition 3.3.** A tree $T$ on $\{0, 1\}$ (or on $\omega$) is said to be pointed if for any $x \in [T]$, $T \leq_T x$.

Thus, a tree $T$ is pointed if $T$ is computable from any of its branches. We now state our improvement to theorem 3.1.

**Theorem 3.4** (ZF + AD). Let $A \subseteq 2^\omega$ (or $\omega^\omega$). Then there is a perfect pointed tree $T$ on $\{0, 1\}$ (or on $\omega$) such that $[T] \subseteq A$ or $[T] \subseteq 2^\omega - A$.

*Proof.* Suppose $A \subseteq 2^\omega$, and consider again the basic game $G_A$. Suppose I has a winning strategy $\sigma$ (the case for II is similar). For any $x = (x(0), x(1), \ldots) \in 2^\omega$, let $\sigma(x) = (y(0), x(0), y(1), x(1), \ldots)$ be the real produced when II plays $x$ and I follows $\sigma$. So, for any $x \in 2^\omega$, $\sigma(x) \in A$. Consider the set $C \subseteq 2^\omega$ of those $x$ such that for all $n$, $x(2n) = 1$ iff $\sigma(n) = 1$ (viewing $\sigma$ as an element of $2^\omega$). Clearly $\sigma[C] = [T]$ for some perfect tree $T$. So, $[T] \subseteq A$. If $z \in [T]$, then $((z)_1)_0 = \sigma$, and so $\sigma \leq_T z$. However, $T$ is computable from $\sigma$, and so $T \leq_T z$. Thus, $T$ is pointed. □
We note that theorem 3.4 is indeed a strengthening of theorem 3.1. For suppose $A \subseteq D$, and $T$ is perfect pointed with $[T] \subseteq A$. Suppose $x \geq_T T$. Define a branch $y$ of $T$ as follows. Let $y = s_0 \bowtie x(0) \bowtie s_1 \bowtie x(1) \ldots$, where $s_0$ is the least splitting node of $T$ (i.e., such that $s_0 \bowtie 0$, $s_0 \bowtie 1$ are both in $T$), $s_1$ is the least splitting node extending $s_0 \bowtie x(0)$, etc. Clearly $y$ is computable from $T$ an $x$. Since $x \geq_T T$, we have $y \leq_T x$. On the other hand, $x$ is computable from $y$ and $T$. Since $T$ is pointed, $T \leq_T y$, and so $x \leq_T y$. Thus, $x \equiv_T y$. Since $y \in [T]$, $y \in A$ and thus $x \in A$ as $A \subseteq D$. So, $A$ contains the cone above $T$.

To give an application of the Martin measure we use the following fact, whose proof uses the coding lemma, which we give later (in fact, the coding lemma provides a much stronger version of the fact).

**Fact 1 (ZF + AD).** If there is a map from $\omega^\omega$ onto an ordinal $\lambda$, then there is a map from $\omega^\omega$ onto $P(\omega^\omega)$.

We make the following important definition.

**Definition 3.5.** $\Theta$ is the supremum of lengths of the prewellorderings of $\omega^\omega$.

Of course, it makes no difference in the definition of $\Theta$ whether we use $\omega^\omega$, $2^{\omega^\omega}$, or any other uncountable polish space (as they are all isomorphic). Easily $\Theta$ is a limit ordinal. Equivalently, we can say that $\Theta$ is the supremum of the ordinals $\lambda$ such that there is a surjection from $\omega^\omega$ onto $\lambda$. Assuming AC, where $\omega^\omega$ has a welldefined cardinality, $\Theta = (2^\omega)^+$ (so $\Theta = \omega_2$ assuming CH). However, the definition is meant primarily in the determinacy context, and provides an AD version of the cardinality of the reals.

Assuming countable choice gives that $\operatorname{cof}(\Theta) > \omega$ according to the following exercise.

**Exercise 15.** Assume ZF+ countable choice. Show that $\operatorname{cof}(\Theta) > \omega$. [hint: suppose $\{\alpha_n\}_{n<\omega}$ were a cofinal $\omega$ sequence in $\Theta$. By countable choice, let $\preceq_n$ by a prewellordering of $\omega^\omega$ of length $\alpha_n$. Glue the $\preceq_n$ together to get a prewellordering of length $\Theta$, a contradiction.]

**Theorem 3.6 (ZF + AD).** Every countably additive filter $\mathcal{F}$ on an ordinal $\lambda < \Theta$ can be extended to a measure on $\lambda$.

**Proof.** Since $\lambda < \Theta$, from fact 1 there is a map $\pi: \omega^\omega \rightarrow P(\lambda)$. For every degree $d$, define $A_d = \bigcap \{\pi(x): x \in d \land \pi(x) \in \mathcal{F}\}$. Let $f(d)$ be the least element of $A_d$, which makes sense since $\mathcal{F}$ is countably additive. So, $f: \mathcal{D} \rightarrow \lambda$. Let $\mu = f(\nu)$, where $\nu$ is the Martin measure on $\mathcal{D}$. So, $\mu$ is a measure on $\lambda$. If $F \in \mathcal{F}$, then consider $x$ such that $\pi(x) = F$. If $d \in \mathcal{D}$ with $x \leq_T d$, then $A_d \subseteq F$, and so $f(d) \in F$. So, $\forall^*_d (f(d) \in F)$. Thus, $\mu(F) = 1$. So, $\mu$ extends the filter $\mathcal{F}$. \hfill \Box

**Corollary 3.7.** If $\lambda < \Theta$ and $\operatorname{cof}(\lambda) > \omega$, then there is a measure $\mu$ on $\lambda$ such that $\mu([0, \alpha]) = 0$ for all $\alpha < \lambda$.

**Proof.** Let $\mathcal{F}$ be the co-bounded filter on $\lambda$. Since $\operatorname{cof}(\lambda) > \omega$, $\mathcal{F}$ is countably additive. From theorem 3.6, there is a measure $\mu$ on $\lambda$ extending $\mathcal{F}$. Since $\mu$ extends $\mathcal{F}$, every co-bounded set has $\mu$ measure 1, and so every bounded in $\lambda$ set has $\mu$ measure 0. \hfill \Box
As another application of the Martin measure, we give the following theorem of Kunen.

**Theorem 3.8 (ZF + AD).** Let \( \lambda < \Theta \). Then the set of measures on \( \lambda \) is wellorderable.

**Proof.** Again let \( \pi : \mathcal{P}(\lambda) \rightarrow \). Suppose \( \mu_1, \mu_2 \) are measures on \( \lambda \). For \( \mu \) a measure on \( \lambda \) and \( d \in D \), again let

\[
A_d^\mu = \{ \pi(x) : x \in d \land \mu(\pi(x)) = 1 \}.
\]

Let also \( f^\mu(d) \) be the least element of \( A^\mu(d) \). Define \( \mu_1 \prec \mu_2 \) iff \( \forall d \ (f^{\mu_1}(d) < f^{\mu_2}(d)) \). For any two measure \( \mu_1, \mu_2 \) we have either \( \mu_1 \prec \mu_2 \) or \( \mu_2 \prec \mu_1 \). To see this, let \( A \subseteq \lambda \) such that (without loss of generality) \( \mu_1(A) = 0, \mu_2(A) = 1 \). Let \( \pi(x_0) = A, \pi(x_1) = \lambda - A \). If \( x_0, x_1 \leq_T d \in D \), then \( f^{\mu_1}(d) \in \lambda - A \) and \( f^{\mu_2}(d) \in A \), and thus \( f^{\mu_1}(d) \neq f^{\mu_2}(d) \). So, \( \forall d \ (f^{\mu_1}(d) \neq f^{\mu_2}(d)) \), and it follows that that either \( \mu_1 \prec \mu_2 \) or \( \mu_2 \prec \mu_1 \). Easily \( \prec \) is transitive and irreflexive. So, \( \prec \) is a linear ordering of the measures on \( \lambda \). To see it is wellfounded, suppose \( \mu_n \prec \mu_n \) for all \( n \). Let \( A_n \subseteq D \) be such that for all \( d \in A_n \), \( f^{\mu_n+1}(d) < f^{\mu_n}(d) \). By countable additivity of \( \nu \), \( \bigcap_n A_n = \emptyset \). Let \( d \in \bigcap_n A_n \). Then \( f^{\mu_1}(d) > f^{\mu_2}(d) > \cdots \), a contradiction. \( \Box \)

In view of theorem 3.8, we make the following definition.

**Definition 3.9 (ZF + AD).** For \( \lambda < \Theta \), let \( \beta(\lambda) \) denote the cardinality of the set of measures on \( \lambda \).

We will give estimates for \( \beta(\lambda) \) later. We now turn to a discussion of the basic facts concerning Wadge degrees. The objects of study now are not reals but sets of reals. It is most convenient for this discussion to work in the space \( \omega^\omega \), which we do for the rest of this section. We will frequently write \( A^c \) for the complement \( \omega^\omega - A \).

Recall that a function \( f \) from a metric space \( (X, d) \) to a metric space \( (Y, \rho) \) is said to be Lipschitz continuous, with Lipschitz constant \( C \), if \( \rho(f(x), f(y)) \leq Cd(x, y) \) for all \( x, y \in X \). In the case \( X = Y = \omega^\omega \), and \( d = \rho = \) the usual metric: \( d(x, y) = \frac{1}{n} \) where \( n \) is least such that \( x(n) \neq y(n) \) (and \( d(x, y) = 0 \) if \( x = y \)), to say that \( f \) is Lipschitz continuous (with constant 1) simply means that if \( x \upharpoonright n = y \upharpoonright n \), then \( f(x) \upharpoonright n = f(y) \upharpoonright n \). We simply call such a function Lipschitz continuous. Note that Lipschitz continuous functions are essentially strategies for \( I \) in integer games (a strategy for \( I \) is also Lipschitz continuous, and in fact slightly better).

**Definition 3.10.** Let \( A, B \subseteq \omega^\omega \). We say \( A \) is Wadge reducible to \( B \), \( A \leq_w B \), if there is a continuous function \( f: \omega^\omega \rightarrow \omega^\omega \) such that \( A = f^{-1}(B) \). We say \( A \) is Lipschitz reducible to \( B \), \( A \leq_\ell B \), if there is a Lipschitz continuous function \( f: \omega^\omega \rightarrow \omega^\omega \) such that \( A = f^{-1}(B) \).

Note that to say \( A = f^{-1}(B) \) means that for all \( x \) we have \( x \in A \) iff \( f(x) \in B \). Thus \( f \) reduces the question of membership in \( A \) to that of membership in \( B \). Trivially, if \( A \leq_\ell B \) then \( A \leq_w B \). Both \( \leq_\ell \) and \( \leq_w \) are clearly reflexive and transitive (the composition of two Lipschitz functions is Lipschitz), that is, both are partial orders. Note also that \( \Gamma \subseteq \mathcal{P}(\omega^\omega) \) is a pointclass if \( \Gamma \) is closed under Wadge reduction.

Note that \( A \leq_\ell B \) iff \( A^c \leq_\ell B^c \), and likewise for \( \leq_w \). The following fundamental lemma, due to Wadge, is known as Wadge’s lemma.
Lemma 3.11 (ZF + AD). Let $A, B \subseteq \omega^\omega$. Then either $A \leq_\ell B$ or $B \leq_\ell A^c$.

Proof. Consider the game $G_{A,B}$ where I plays out $x \in \omega^\omega$ and II plays out $y \in \omega^\omega$, and where II wins the run iff $(x \in A \iff y \in B)$. If II has a winning strategy for $G_{A,B}$, then a winning strategy $\tau$ gives a Lipschitz continuous function, which we also call $Gx\omega$ from strategy $G$. A terminal sequence of passes, II loses. Otherwise, the payoff is exactly as in the game, the players make integer moves but II is also allowed to pass. If II makes $\leq B$ set their complements, then they are. We make this precise in the next definition.

We can also define a game $G_{A,B}^w$ corresponding to Wadge reduction. In this game, the players make integer moves but II is also allowed to pass. If II makes a terminal sequence of passes, II loses. Otherwise, the payoff is exactly as in the Lipschitz game $G_{A,B}$. It is easy to see that $A \leq_w B$ iff II has a winning strategy in the game $G_{A,B}$.

Of course, it follows immediately that for any $A, B$ that either $A \leq_w B$ or $B \leq_w A$.

So, $\leq_\ell$ and $\leq_w$ are not linear orders on $\mathcal{P}(\omega^\omega)$, but if we amalgamate sets with their complements, then they are. We make this precise in the next definition.

Definition 3.12. By a Lipschitz degree we mean the equivalence class $\{A, A^c\}$ of a set $A \subseteq \omega^\omega$ together with its complement $A^c$ under the relation $\{A, A^c\} \equiv_\ell \{B, B^c\}$ iff $A$ is Lipschitz reducible to either $B$ or $B^c$, and $B$ is Lipschitz reducible to either $A$ or $A^c$. Likewise, we define a Wadge degree to be the equivalence class of a pair $\{A, A^c\}$ using $\leq_w$ instead of $\leq_\ell$.

Note that $\leq_\ell$ is welldefined on the Lipschitz degrees, and likewise $\leq_w$ is welldefined on the Wadge degrees. That is, $\{A, A^c\} \leq_\ell \{B, B^c\}$ iff $A \leq_\ell B$ or $A \leq_\ell B^c$. We write $[A]_\ell$ and $[A]_w$ for the Lipschitz and Wadge degrees of a set $A$ (that is, $[A]_\ell$ is the equivalence class of $\{A, A^c\}$). From Wadge’s lemma it follows that $\leq_\ell$, and thus also $\leq_w$, is a linear ordering of the Lipschitz (resp. Wadge) degrees. As usual, we write $[A]_\ell \prec [B]_\ell$ to mean $[A]_\ell \leq [B]_\ell$ and $[B]_\ell \nprec [A]_\ell$.

We next present the following important result of Martin and Monk which states that the Lipschitz and Wadge degrees are actually wellordered by these orders.

Theorem 3.13 (ZF + DC + AD). The Lipschitz degrees are wellordered under $\leq_\ell$. Likewise, the Wadge degrees are wellordered by $\leq_w$.

Proof. Suppose $[A_0]_\ell > [A_1]_\ell > [A_2]_\ell > \cdots$. For each $n$, II does not have a strategy in the Lipschitz game $G_{A_n, A_{n+1}}$, and so I has a winning strategy $\sigma_n$ for this game. So, $\sigma_n(x) \in A_n$ iff $x \notin A_{n+1}$. Likewise, II does not win $G_{A_n, A_{n+1}}$, so let $\tau_n$ be a winning strategy for I in this game. Thus, $\tau_n(x) \in A_n$ iff $x \in A_{n+1}$.

So, $\sigma_n$ “flips” membership between $A_{n+1}$ and $A_n$, and $\tau_n$ preserves membership. It is important for the following argument that all of the $\sigma_n$, $\tau_n$ are strategies for I. For $z \in 2^\omega$ we fill in the following diagram in such a way that for all $n$, $x_n = \sigma_n(x_{n+1})$ if $z(n) = 1$ and $x_n = \tau_n(x_{n+1})$ if $z(n) = 0$. 


<table>
<thead>
<tr>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0(0)$</td>
<td>$x_1(0)$</td>
<td>$x_2(0)$</td>
<td>$x_3(0)$</td>
<td>$x_4(0)$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$x_0(1)$</td>
<td>$x_1(1)$</td>
<td>$x_2(1)$</td>
<td>$x_3(1)$</td>
<td>$x_4(1)$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$x_0(2)$</td>
<td>$x_1(2)$</td>
<td>$x_2(2)$</td>
<td>$x_3(2)$</td>
<td>$x_4(2)$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$x_0$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_4$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

For $n$ such that $z(n) = 1$, let $x_n(0) = \sigma_n(\emptyset)$, and for $n$ such that $z(n) = 0$ let $x_n(0) = \tau_n(\emptyset)$. In a similar manner we fill in all of the $x_n(i)$. Let $x_0(z)$ be the value of $x_0$ produced as described above using $z \in 2^\omega$. Let $B_0 = \{z : x_0(z) \in A_0\}$, and $B_1 = \{z : x_0(z) \notin A_0\}$. At least one of $B_0$, $B_1$ must be non-meager. Say $N_\omega$ be a basic open set in $2^\omega$ on which one of these sets is comeager. Say, $B_0$ is comeager on $N_\omega$ (the other case is similar). So, $B_0$ is comeager on $N_{\omega^0}$ and on $N_{\omega^1}$. Consider the map $\pi$ from $N_{\omega^0}$ to $N_{\omega^1}$ obtained by flipping the $lh(s)$ digit. So, $\pi$ is a homeomorphism between $N_{\omega^0}$ and $N_{\omega^1}$. Since $B_0$ is comeager in $N_{\omega^0}$, $\pi(B_0)$ is comeager in $N_{\omega^1}$. However, for any $z$, $x_0(z) \in A_0$ iff $x_0(\pi(z)) \notin A_0$ (as $\tau_n$ does). Thus, $\pi(B_0 \cap N_{\omega^0}) = B_1 \cap N_{\omega^1}$, and for each $n$ that $B_0 \cap N_{\omega^0} \supsetneq B_1 \cap N_{\omega^1}$. Thus, $B_1$ is also comeager in $N_{\omega^1}$, a contradiction (as $B_0$, $B_1$ are disjoint).

The proof also works for Wadge degrees, since if $[A_0]_w >_w [A_1]_w >_w \cdots$, then it is also the case that for each $n$ that $[A_0]_\ell >_\ell [A_{n+1}]_\ell$. □

Variations of the Martin-Monk method are frequently used in the abstract theory of pointclasses.

As an immediate consequence of Wadge’s lemma we have the following.

**Theorem 3.14 (ZF + AD).** If $\Gamma$ is a non-selfdual pointclass, then $\Gamma \upharpoonright \omega^\omega$ has a universal set.

**Proof.** Let $A \in \Gamma - \hat{\Gamma}$. If $B \in \Gamma$, then we must have $B \leq_\ell A$, as otherwise we would have $A \leq_\ell B^c$ which would give $A \in \hat{\Gamma}$. Define then $U(x, y) \iff x(y) \in A$, where we view every real $x$ as coding a Lipschitz continuous map from $\omega^\omega$ to $\omega^\omega$ (for example $x(s) = m$, for $s \in \omega^{<\omega}$, iff $x)((s)) = m$). The map $(x, y) \mapsto x(y)$ is continuous, and so $U \in \Gamma$. Since every $B \in \Gamma$ is Lipschitz reducible to $A$, it follows that $U$ is universal for $\Gamma \upharpoonright \omega^\omega$. □

**Definition 3.15.** If $A \subseteq \omega^\omega$, we define the Wadge rank of $A$, $o(A)$ to be the rank of $\{A, A^c\}$ in $\leq_w$. Likewise we define the Lipschitz rank $o_\ell(A)$. If $\Gamma$ is a pointclass, we define $o(\Gamma) = \sup\{o(A) : A \in \Gamma\}$.

So, if $\Gamma$ is non-selfdual, then $o(\Gamma) = o(A)$ for any $A \in \Gamma - \hat{\Gamma}$.

We develop some of the facts concerning the Lipschitz and Wadge hierarchies. Note that every Wadge degree is a union of Lipschitz degrees. The following lemma is another connection.

**Definition 3.16.** We say $\{A, A^c\}$ is a selfdual Lipschitz (or Wadge) degree if $A \equiv_\ell A^c$, otherwise we say the degree is non-selfdual.

Note that if $A \leq_\ell A^c$ then $A^c \leq_\ell A$, and so $A \equiv_\ell A^c$. The following important lemma is due to Steel.
Lemma 3.17 (ZF + AD). $A \equiv_w A^c$ iff $A \equiv_{\ell} A^c$.

Proof. Suppose $A \subseteq_w A^c$. Let $f : \omega^\omega \to \omega^\omega$ be continuous such that $x \in A$ iff $f(x) \in A^c$. Suppose $A \notin f A$. Then I wins the game $G_A, A$, say by $\sigma$. For each $z \in 2^\omega$, we consider the diagram as in the proof of theorem 3.13, where for $n$ such that $\sigma(n) = 1$ we use $\sigma$ (that is $x_n = \sigma(x_{n+1})$, and if $\sigma(n) = 0$ we use $f$ (i.e., $x_n = f(x_{n+1})$. Since $f$ is just assumed to be continuous, it is no longer the case that the diagram can always be filled in. However, we claim that for comeager many $z \in 2^\omega$, the diagram can be filled in. This is because the set of $z$ such that $x_0 \nmid n, \ldots, x_n \nmid n$ are defined is dense open in $2^\omega$. To see this, note that for any $s \in 2^{\omega^\omega}$, there is a $t$ extending $s$ of the form $t = s^1 n$ such that a sufficiently large initial segment of $x_n$ is defined so that $x_0 \nmid n, \ldots, x_n \nmid n$ are all defined. Let $C \subseteq 2^\omega$ be the comeager set of $z$ such that the diagram can be completely filled in. Let $x_i(z)$ denote the corresponding reals produced. Note that $\sigma$ preserves membership in $A$ and $f$ flips membership.

The argument is now similar to that of theorem 3.13. Let $C_0 = \{ z \in C : x_0(z) \in A \}$, and $C_1 = \{ z \in C : x_0(z) \notin A \}$. At least one of the $C_i$ is nonmeager, and so comeager on some neighborhood $N_s$ of $2^\omega$. Let $C_i$ be comeager in $N_s \cap N_s^{-1}$. Let $\pi$ be the natural homeomorphism between $N_s \cap N_s^{1-1}$, and $\pi(D) \subseteq C_{1-i}$, a contradiction since $C_i$ is also comeager in $N_s \cap N_s^{1-1}$ and $C_i, C_{1-i}$ are disjoint. □

In view of lemma 3.17, we may speak unambiguously of a degree being selfdual or non-selfdual.

Definition 3.18. If $A, B \subseteq \omega^\omega$, their join is defined by

$$A \oplus B = \{ x : (x(0) \text{ is even and } x(0) \notin A) \lor (x(0) \text{ is odd and } x(0) \in A) \},$$

where $x'(n) = x(n + 1)$. Similarly, if $A_n \subseteq \omega^\omega$ are given for each $n$, their join is

$$\oplus_n A_n = \{ x : x' \in A_n(0) \}. $$

Lemma 3.19 (ZF + AD). For any $A, B \subseteq \omega^\omega$, $A, B \subseteq_\ell A \oplus B$. Furthermore, if $\{ C, C^c \} \subset A \oplus B$ then $\{ C, C^c \} \subset A$ or $\{ C, C^c \} \subset B$.

Proof. Clearly $A, B \subseteq_\ell A \oplus B$. For example, to see that $A \subseteq_\ell A \oplus B$, have II play first a 0 in the game $G_{A, A \oplus B}$, and then copy I’s moves. Suppose now that $C \subseteq_\ell A \oplus B$, but $A \oplus B \notin_\ell C$. Thus, I wins the game $G_{A \oplus B, C}$ by $\sigma$. Suppose $\sigma$’s first move is even (the other case is similar). Then $\sigma$ gives a winning strategy for II in the game $G_{C, A^c}$. Thus, $C \subseteq_\ell A^c$. So, $\{ C, C^c \} \subseteq_\ell \{ A, A^c \}$. So, $o(C) \leq \max\{ o(A), o(B) \}$. □

Note that lemma 3.19 does not say that $o(A \oplus B) = \max\{ o(A), o(B) \}$ (it does say that $o(A \oplus B) \leq \max\{ o(A), o(B) \} + 1$). We do get the following.

Lemma 3.20. If $A$ is non-selfdual, then $A \oplus A^c$ is the least $\ell$-degree strictly above the degree of $A$. Also, $A \oplus A^c \notin_\ell A$, so $A \oplus A^c$ is also the least Wadge degree strictly above $A$. Furthermore, $A \oplus A^c$ is selfdual.

Proof. First note that $A \oplus A^c \notin_w A$. For if $A \oplus A^c \subseteq_w A$, then $A^c \subseteq_w A$. This contradicts $A$ being non-selfdual. To see that II wins $G_{A \oplus A^c, (A \oplus A^c)^c}$, if II plays $i$ for the first move, have II play $i + 1$ (to change the parity), and then copy I’s moves. Thus, $A \oplus A^c$ is selfdual. So, $A \oplus A^c$ is selfdual and $o(A \oplus A^c) > o(A)$. Finally,
suppose \( \{B, B^c\} <_\ell A \oplus A^c \). So, I has a winning strategy \( \sigma \) in \( G_{A \oplus A^c,B} \). If I's first move is even (the other case is similar), then \( \sigma \) gives a strategy for II in the game \( G_{B,A^c} \), and so \( o(B) \leq o(A) \).

\[ \square \]

Corollary 3.20 identifies the next Lipschitz (in fact Wadge) degree after \( \{A,A^c\} \) when \( A \) is non-selfdual. We now identify the next Lipschitz degree when \( A \) is selfdual.

**Lemma 3.21 (ZF + AD).** Suppose \( A \) is selfdual. Then \( A' = \{0^c x : x \in A\} \) is the next Lipschitz degree above \( A \). Also, \( A' \) is selfdual.

**Proof.** Clearly \( A \leq \ell A' \). Suppose \( A' \leq \ell \{A,A^c\} \). Since \( A \) is selfdual, we have \( A' \leq \ell A^c \). So, II has a winning strategy \( \tau \) in the game \( G_{A',A^c} \). From the definition of \( A' \), it follows that \( \tau \) give a strategy for I in the game \( G_{A,A} \) (make a 0 for the first move, and then follow \( \tau \) as a strategy for I). II can defeat this, however, by copying. To see that II wins \( G_{A'(A^c)<_\ell} \), if I plays 0, then have II play 0 and then follow a strategy for II in \( G_{A,A^c} \). If I makes a first move other than 0, have II play a 0, and then any sequence in \( A \). Thus, \( A' \) is selfdual. Finally, suppose \( \{B, B^c\} <_\ell A \). So, I has a winning strategy \( \sigma \) in \( G_{A',B} \). \( \sigma \) must make a first move of 0 as otherwise II could defeat \( \sigma \) by playing a real not in \( B \). Ignoring I’s first move, \( \sigma \) then gives a winning strategy for II in the game \( G_{B,A^c} \), and so \( o(B) \leq o(A) \).

**Exercise 16.** Show that if \( A \) is selfdual, then \( A' \equiv_w A \) (where \( A' \) is as in lemma 3.21).

**Exercise 17.** Show that if \( A \) is non-selfdual, then \( A' \not\equiv_\ell A \) [hint: Since \( A \) is non-selfdual, I wins the game \( G_{A,A^c} \). This gives a winning strategy for II in the game \( G_{A,A} \)]

Lemmas 3.20, 3.21 identify the Lipschitz degrees at successor stages in the Lipschitz hierarchy. We next consider limit stages. First we consider limit stages of cofinality \( \omega \).

**Lemma 3.22.** Suppose \( \{A_n\}_{n \in \omega} \) is given with \( o_\ell(A_n) < o_\ell(A_{n+1}) \) for all \( n \). Then \( o_\ell(\oplus_n A_n) = \sup_n o_\ell(A_n) \). Also, \( \oplus_n A_n \) is selfdual. Furthermore, if \( A_n \equiv_w A_0 \) for all \( n \), then \( \oplus_n A_n \equiv_w A_0 \).

**Proof.** Let \( A = \oplus_n A_n \). Clearly \( A_n \leq \ell A \) for all \( n \). To see that \( A \) is selfdual, we show that II wins \( G_{A,A^c} \). If I makes first move \( n \), II plays any \( m \) such that \( o_\ell(A_n) < o_\ell(A_m) \). II then follows any winning strategy for II in \( G_{A_n,A_m} \). Suppose \( \{B, B^c\} <_\ell A \). Since \( \not\leq_\ell B \), I has a winning strategy \( \sigma \) in \( G_{A,B} \). From the definition of join, this gives a winning strategy for II in the game \( G_{B,A^c} \), where \( n = \sigma(\emptyset) \). Thus, \( \{B, B^c\} \leq_\ell \{A_n, A_n^c\} \). Thus, \( o_\ell(A) = \sup_n o_\ell(A_n) \).

Finally, suppose \( A_n \equiv_w A_0 \) for all \( n \). II has a winning strategy in \( G_{A,A}^{\omega} \), defined as follows. If I makes first move \( n \), II passes, and then follows a strategy for II in the game \( G_{A_n,A}^{\omega} \).

Thus, at limit stages of cofinality \( \omega \) in the Lipschitz hierarchy, there is a selfdual degree. From lemma 3.22 it follows that after a non-selfdual degree \( \{A,A^c\} \), the next \( \omega_1 \) Lipschitz degrees are all selfdual and all of the same Wadge degree as \( \{A,A^c\} \).

Now we consider stages in the Lipschitz hierarchy of uncountable cofinality.

**Lemma 3.23.** Suppose \( \text{cof}(\alpha) > \omega \). Then the \( A \) such that \( o_\ell(A) = \alpha \) is non-selfdual.
Proof. Suppose $o(A) = \alpha$, and $A$ is selfdual. For each $n \in \omega$, let $A_n = \{x : n \cap x \in A\}$. Thus, $A = \bigoplus_n A_n$. Let $\alpha_n = o(A_n)$. If $\alpha_n < \alpha$ for all $\alpha$, then $o(A) = \sup_n \alpha_n < \alpha$, a contradiction. So, fix $n$ such that $o(A_n) = o(A)$, in particular, $A_n$ is selfdual. Thus, II has a winning strategy in $G_{A,A_n}$. If we fix I’s first move as $n$, and then copy II’s moves, we defeat II’s winning strategy.

From the previous lemmas, we now have a complete picture of the Lipschitz and Wadge hierarchies.

**Theorem 3.24 (ZF + AD).** The non-selfdual and the selfdual Wadge degrees alternate. At limit stages of countable cofinality there is a selfdual Wadge degree which is the degree of the join of Wadge degrees from a cofinal $\omega$ sequence. At limit stages of uncountable cofinality there is a non-selfdual Wadge degree. Every non-selfdual Lipschitz degree is non-selfdual as a Wadge degree. Every selfdual Wadge degree consists of an $\omega_1$ block of selfdual Lipschitz degrees.

Along the way, we have also obtained some additional information about the degrees. For $A \subseteq \omega^\omega$ and $n \in \omega$, recall $A_n = \{x : n \cap x \in A\}$. Also, for $s \in \omega^{<\omega}$, let $A_s = \{x : s \cap x \in A\}$.

**Lemma 3.25.** If $A$ is a selfdual degree, then for every $n \in \omega$, $A_n \leq_\ell A$. Also, for every $x \in \omega^\omega$ there is an $n$ such that $A_{x|n} <_w A$. If $A$ is non-selfdual, then there is an $x \in \omega^\omega$ such that for all $n$, $A_{x|n} \equiv_w A$ (thus, this property characterizes the non-selfdual degrees).

Proof. Clearly $A_s \leq_\ell A$ for any $s$ and any $A$. If $A$ is selfdual, then we showed in the proof of lemma 3.23 that for every $n$ that $A_n <_\ell A$. So, for any $x \in \omega^\omega$, the sequence $A_{x|0}, A_{x|1}, \ldots$ is strictly decreasing as Lipschitz degrees until we hit a non-selfdual degree $A_{x|n}$ (which we must eventually do by wellfoundedness). We then have that $A_{x|n} <_w A$ as otherwise $A$ is non-selfdual.

Suppose now that $A$ is non-selfdual. If for each $n$ we had $A_n <_w A$, then $A = \bigoplus_n A_n$ would be selfdual. Namely, II could win $G_w(A,A')$ as follows: I plays $n$, II passes, and then II follows a strategy Wadge reducing $A_n$ to $A'$. So, for some $n$ we have $A_n \equiv_w A$, and in particular $A_n$ is also non-selfdual. Repeating the argument gives an $x$ such that for all $n$, $A_{x|n} \equiv_w A$.

**Exercise 18.** Show the property of lemma 3.25 for Wadge degrees directly, that is without mentioning Lipschitz degrees. [hint: suppose $A$ is selfdual, and suppose $x \in \omega^\omega$ were such that for all $n$, $A_{x|n} \equiv_w A$. For each $n$, fix winning strategies $\sigma_n$, $\tau_n$ for $I$ in the Wadge games $G_w(A,A_{x|n})$ and $G_w(A,A_{x|n}^c)$. Define a sequence of integers $k_n$ inductively so that for all $n$, if we set $s_n = x \upharpoonright k_n$ and for $m < n$ let $s_m = (x \upharpoonright k_m) \upharpoonright \rho_{k_m}(s_{m+1})$, then $lh(s_0) \geq n$ (for all choices of $\rho_{k_m} \in \{\sigma_{k_m}, \tau_{k_m}\}$). For every $z \in 2^\omega$, this defines a filling-in of a diagram producing $x_0, x_1, \ldots$ with $x_n \upharpoonright k_n = x \upharpoonright k_n$, and where $x_n = \sigma_{k_n}(x_{n+1})$ if $x(n) = 0$ and $x_n = \tau_{k_n}(x_{n+1})$ if $x(n) = 1$. This gives a contradiction as in theorem 3.13.]

## 4. Theory Of Pointclasses

We continue with the AD theory of Wadge degrees, developing an abstract theory of pointclasses. First we consider the separation property. We have the following general result of Steel and Van-Wesep.
Theorem 4.1 (ZF+AD). For every non-selfdual pointclass $\Gamma$, exactly one of $\text{sep}(\Gamma)$ or $\text{sep}(\hat{\Gamma})$ holds.

Steel showed [?] that for every non-selfdual $\Gamma$ that either $\text{sep}(\Gamma)$ or $\text{sep}(\hat{\Gamma})$ holds. Van-wesep [?] showed that both sides of a non-selfdual class cannot have the separation property.

We first prove Steel’s theorem, which we state separately in the following.

Theorem 4.2 (Steel). Assume ZF + AD. For every non-selfdual pointclass $\Gamma$, either $\text{sep}(\Gamma)$ or $\text{sep}(\hat{\Gamma})$.

Proof. Let $\Gamma$ be non-selfdual, and let $\Delta = \Gamma \cap \hat{\Gamma}$. We say a pair of sets $A, B \subseteq \omega^\omega$ is $\Delta$-inseparable if there is no $\Delta$ set $C$ separating them, that is, $A \subseteq C$, $C \cap B = \emptyset$ (this is symmetric in $A$ and $B$).

Lemma 4.3. Let $A_0, A_1$ be sets which are $\Delta$ inseparable. Then for any pair $B_0, B_1$ of disjoint sets both of which in $\Gamma$ or both of which are in $\hat{\Gamma}$, there is a strategy $\sigma$ for $I$ (i.e., a Lipschitz $\frac{1}{2}$ function) such that for all $x \in \omega^\omega$, $(x \in B_0 \rightarrow \sigma(x) \in A_0)$ and $(x \in B_1 \rightarrow \sigma(x) \in A_1)$.

Proof. Play the game where I plays out $z \in \omega^\omega$ and II plays out $x \in \omega^\omega$ and I wins the run iff

$$(x \in B_0 \rightarrow z \in A_0) \land (x \in B_1 \rightarrow z \in A_1).$$

If I has a winning strategy $\sigma$, then we are done. If II has a winning strategy $\tau$, then note that $\forall z \ (\tau(z) \in B_0 \cup B_1)$. Also, if $z \in A_0$, then $\tau(z) \in B_1$, and if $z \in A_1$ then $\tau(z) \in A_1$. So, $\{z : \tau(z) \in B_1\} = \omega^\omega - \{z : \tau(z) \in B_0\}$, and so these are both $\Delta$ sets. This gives a separation of $A_0$ and $A_1$, a contradiction. \(\square\)

To prove theorem 4.2, suppose toward a contradiction that $\neg \text{sep}(\Gamma)$ and $\neg \text{sep}(\hat{\Gamma})$. Let $A_0, A_1$ be $\Gamma$ sets which are $\Delta$ inseparable, and let $B_0, B_1$ be $\hat{\Gamma}$ sets which are $\Delta$-inseparable. From the lemma (applied to the $\Delta$-inseparable pair $B_0, B_1$) there is a continuous function $f$ such that $(x \in A_0 \rightarrow f(x) \in B_0)$ and $(x \in A_1 \rightarrow f(x) \in B_1)$.

Let $C_0 = f^{-1}(B_0)$, $C_1 = f^{-1}(B_1)$. Thus, $C_0, C_1 \subseteq \hat{\Gamma}, A_0 \subseteq C_0, A_1 \subseteq C_1$, and $C_0 \cap C_1 = \emptyset$.

From the lemma (applied to the $\Delta$-inseparable pair $A_0, A_1$) there is a strategy $\sigma_0$ for I such that $(x \in C_0 \rightarrow \sigma_0(x) \in A_1)$, and $(x \in C_1 \rightarrow \sigma_0(x) \in A_0)$. There is also a strategy $\sigma_1$ for I such that $(x \in A_0 \rightarrow \sigma_1(x) \in A_0)$ and $(x \in C_0 \rightarrow \sigma_1(x) \in A_1)$. Finally, there is a strategy $\sigma_2$ for I such that $(x \in A_1 \rightarrow \sigma_2(x) \in A_1)$, and $(x \in C_1 \rightarrow \sigma_2(x) \in A_0)$. For any $z \in 3^{\omega}$ there is a filling-in of the diagram to produce reals $x_0, x_1, \ldots$ such that $x_n = \rho(x_{n+1})$ where $\rho = \sigma_0$ or $\sigma_1$ or $\sigma_2$ if $z(n) = 0$ or 1 or 2 respectively. Let $x_0(z)$ be the value of $x_0$ produced for this particular $z$. Note that all of the $\sigma_i$ have the property that if $x \in A_0 \cup A_1$ then $\sigma_i(x) \in A_0 \cup A_1$.

We now get the usual contradiction an in theorem 3.13. Namely, let $E_0 = \{z : x_0(z) \in C_0\}$, $E_1 = \{z : x_0(z) \in C_1\}$, and $E_2 = \{z : x_0(z) \notin C_0 \cup C_1\}$. Let $s \in \omega^{<\omega}$ be such that one of the $E_i$ is comeager on $N_s$. Suppose $E_0$ is comeager on $N_s$. So, $E_0$ is comeager on the neighborhood determined by $t = s^{<\omega}k^{<\omega}s$ for any $k$. By choosing $k$ of the appropriate parity, we have that for almost all $z$ in $N_t$ that $x_0(z) \in A_1 \subseteq C_1$, a contradiction. The argument in the case $E_1$ is comeager on $N_s$ is identical. Finally, suppose $E_2$ is comeager on $N_s$. Then $E_2$ is also comeager on $N_t$, for $t = s^{<\omega}1^{<\omega}s$. However, for almost all $z$ in $N_t$ we have $x_0(z) \in A_0 \cup A_1$ from
the definition of $\sigma_1$ (we use here the fact that if $x \notin C_0$, then $\sigma_1(x) \in A_1$, and so all further $\sigma_i$ applied to this point stay in $A_0 \cup A_1$). This is a contradiction and completes the proof of theorem 4.2.

We now give the other half of theorem 4.1.

**Theorem 4.4 (Van Wesep).** Assume $\text{ZF} + \text{AD}$. If $\Gamma$ is non-selfdual, then $\text{sep}(\Gamma)$, $\text{sep}(\tilde{\Gamma})$ cannot both hold.

**Proof.** Suppose toward a contradiction that $\text{sep}(\Gamma)$, $\text{sep}(\tilde{\Gamma})$ both hold. Let $A \in \Gamma - \tilde{\Gamma}$. We again regard every real $x \in \omega^\omega$ as coding a Lipschitz continuous function from $\omega^\omega$ to $\omega^\omega$. Define $A_0(x, y) \leftrightarrow ((x)_0(y) \in A)$ and $A_1(x, y) \leftrightarrow ((x)_1(y) \in A)$. Then $A_0, A_1 \in \Gamma$ and form a $\Gamma$-universal pair. That is, for every pair $B_0, B_1$ of $\Gamma$ sets, there is an $x$ such that $B_0 = (A_0)_x$, $B_1 = (A_1)_x$ (recall $C_x$ denotes the section $\{y : C(x, y)\}$ of the set $C$). We cannot have that $A_0 - A_1$ and $A_1 - A_0$ can be separated by a $\Delta$ set. This is the same argument that $\text{red}(\Gamma) \rightarrow \neg \text{sep}(\Gamma)$. Here briefly is the argument again. Suppose $C \in \Delta$ and $A_0 - A_1 \subseteq C$, $C \cap (A_1 - A_0) = \emptyset$. Then $C \in \Delta$ and is universal for $\Delta$ sets, a contradiction (since we then define $D(x) \leftrightarrow \neg C(x, x)$, so $D \in \Delta$, and for all $x$, $D \neq C_x$). Since we are assuming $\text{sep}(\Gamma)$, it follows that $A_0 - A_1$ and $A_1 - A_0$ cannot be separated by disjoint $\Gamma$ sets (that is, there does not exist disjoint $\Gamma$ sets $D_1, D_2$ with $A_0 - A_1 \subseteq D_0$ and $A_1 - A_0 \subseteq D_1$).

Consider the game $G_0$ where I plays out $x$, II plays out $y$, and II wins the run iff

\[
(y \in A_0 \cup A_1) \\
\land ((x \in A_0 - A_1) \rightarrow (y \in A_0 - A_1)) \\
\land ((x \in A_1 - A_0) \rightarrow (y \in A_1 - A_0)).
\]

If II had a winning strategy $\tau$, then $D_0 = \tau^{-1}(A_0^c)$ and $D_1 = \tau^{-1}(A_0^c)$ would be disjoint $\Gamma$ sets with $A_0 - A_1 \subseteq D_0$, $A_1 - A_0 \subseteq D_1$, a contradiction. Let $\sigma_0$ be a winning strategy for I in $G_0$. Then $y \in (A_0 - A_1) \rightarrow (\sigma_0(y) \in A_1 - A_0)$, $y \in (A_1 - A_0) \rightarrow (\sigma_0(y) \in A_0 - A_1)$, and $y \in (A_0 \cup A_1) \rightarrow (\sigma_0(y) \in ((A_0 - A_1) \cup (A_1 - A_0))$.

Now apply the same argument to the pair $A_0^c, A_1^c$, which is a universal pair for $\Gamma$. As before, we cannot have that $A_0^c - A_1^c = A_1 - A_0$ and $A_1^c - A_0^c = A_0 - A_1$ can be separated by disjoint $\Gamma$ sets. We consider the game $G_1$ where I plays out $x$, II plays out $y$, and II wins the run iff

\[
(y \in A_0^c \cup A_1^c) \\
\land ((x \in A_0 - A_1) \rightarrow (y \in A_1 - A_0)) \\
\land ((x \in A_1 - A_0) \rightarrow (y \in A_0 - A_1)).
\]

If II had a winning strategy $\tau$, then $D_0 = \tau^{-1}(A_0^c)$, $D_0 = \tau^{-1}(A_0^c)$ would be disjoint $\Gamma$ sets separating $A_0 - A_1$ and $A_1 - A_0$. Let $\sigma_1$ be a winning strategy for I in $G_1$. Then $y \in (A_0 - A_1) \rightarrow (\sigma_0(y) \in A_0 - A_1)$, $y \in (A_1 - A_0) \rightarrow (\sigma_0(y) \in A_1 - A_0)$, and $y \in (A_0^c \cup A_1^c) \rightarrow (\sigma_0(y) \in ((A_0 - A_1) \cup (A_1 - A_0))$.

Notice that $\sigma_0$ flips membership between $A_0 - A_1$ and $A_1 - A_0$, while $\sigma_1$ preserves membership. Also, $\sigma_0$ maps $A_0 \cup A_1$ into the union of these two sets, while $\sigma_1$ maps $(A_0 \cap A_1)^c$ into these two sets.

We get the usual Martin-Monk contradiction. For every $z \in 2^\omega$, consider the filling-in to produce $x_0, x_1, \ldots$ where $x_n = \sigma_0(x_{n+1})$ if $z(n) = 0$, and $x_n = \sigma_1(x_{n+1})$ if $z(n) = 1$. Suppose that for nonmeager many $z$ that $x_0(z) \notin (A_0 - A_1) \cup (A_1 - A_0)$. 


Say for nonmeager many $z$ that $x_0(z) \in (A_0 \cup A_1)^c$ (the case where for nonmeager many $z$ we have $x_0(z) \in A_0 \cap A_1$ is similar). Say $E = \{ z : x_0(z) \notin A_0 \cup A_1 \}$ is comeager on $N_s$. So, $E$ is also comeager on $N_s$, where $t = s^1 - s$. But $y \notin (A_0 \cup A_1)$ implies $y \in (A_0 - A_1) \cup (A_1 - A_0)$. So, for comeager in $N_1$ many $z$ we have that $x_0(z) \in (A_0 - A_1) \cup (A_1 - A_0)$, a contradiction. So, for comeager many $z$ we have $x_0(z) \in (A_0 - A_1) \cup (A_1 - A_0)$. Suppose without loss of generality that $F = \{ z : x_0(z) \in A_0 - A_1 \}$ is comeager on $N_s$. Then $F$ is comeager on $N_1$, where $t = s^1 - s$, and this is a contradiction as $\sigma_0$ flips membership between $A_0 - A_1$ and $A_1 - A_0$. provided we take $k$ of the appropriate parity.

This completes the proof of theorem 4.1.

One use of the separation property is to transfer closure properties from $\Delta$ to $\Gamma$.

**Theorem 4.5** (Steel). Assume $ZF + AD$. Let $\Gamma$ be non-selfdual and assume $sep(\Gamma)$. Then:

1. If $\Delta$ is closed under finite (countable) unions, then $\hat{\Gamma}$ is closed under finite (resp. countable) unions.
2. If $\Delta$ is closed under $\exists^\omega$, then $\hat{\Gamma}$ is closed under $\exists^\omega$.

**Proof.** To prove (1), let $A, B \in \hat{\Gamma}$ and assume $A \cup B \notin \hat{\Gamma}$. By Wadge, every $\Gamma$ set is Wadge reducible to $A \cup B$, and thus every $\Gamma$ set can be written as the union of two $\Gamma$ sets. Say $A', B'$ are $\hat{\Gamma}$ sets with $A' \cup B' \in \Gamma - \hat{\Gamma}$. By sep($\hat{\Gamma}$), let $C \in \Delta$ and separate $A'$ from $(A' \cup B')^c$, and let $D \in \Delta$ separate $B'$ from $(A' \cup B')^c$. Then $A' \cup B' = C \cup D \in \Delta$, a contradiction.

To prove (2), suppose $\exists^\omega \Gamma \notin \hat{\Gamma}$. By Wadge, $\Gamma \subseteq \exists^\omega \hat{\Gamma}$. We will show sep($\Gamma$), a contradiction. Let $A, B$ be disjoint $\Gamma$ sets. Let $A(x) \iff \exists y A'(x, y)$, and $B(x) \iff \exists y B'(x, y)$, where $A', B'$ are $\Gamma$ subsets of $\omega ^\omega \times \omega ^\omega$. Define $A''(x, y, z) \iff A'(x, y)$ and $B''(x, y, z) \iff B'(x, y)$. Clearly $A''$, $B''$ are disjoint $\Gamma$ sets. Let $A'' \subseteq D \subseteq (B'')^c$, with $D \in \Delta$, from sep($\hat{\Gamma}$). Let $E(x) \iff \exists y \forall z D(x, y, z)$. Then $E \in \Delta$ and $A \subseteq E \subseteq B''$. [Note that if $x \in B' \cap E$ then $\exists z \forall y B''(x, y, z)$ and $\exists y \forall z D(x, y, z)$. Fix $z$, $y$ witnessing these two existential statements. Then $B''(x, y, z)$ and $D(x, y, z)$, a contradiction as $B''$ and $D$ are disjoint.] \[ \square \]

**Remark 4.6.** Using the coding lemma one has that if $\exists^\omega \Delta \subseteq \Delta$ and $\alpha < cof(o(\Delta))$, then $\Delta$ is closed under $\alpha$-length unions. The proof of (1) then generalizes (using the coding lemma) to show that $\hat{\Gamma}$ is closed under $\alpha$-length unions.

An example due to Van Wesep shows that there is a non-selfdual class with neither sider having the reduction property. However, we have the following.

**Theorem 4.7** (Steel, Van Wesep). Assume $ZF + AD$. Suppose $\Gamma$ is non-selfdual, sep($\Gamma$), and the intersection of two $\Delta$ sets is in $\Gamma$. Then $\neg \text{red}(\Gamma)$.

**Proof.** The proof of (1) of theorem 4.5 shows that $\hat{\Gamma}$ is closed under finite unions (so $\Gamma$ is closed under finite intersections). Since we are assuming sep($\Gamma$), we have $\neg \text{sep}(\Gamma)$. Let $C, D$ be a disjoint pair of $\Gamma$ sets which are not separable by a $\Delta$ set. Let $A, B$ be a pair of $\Gamma$ sets. Consider the game where I plays out $x$, II plays out $y$, and II wins if $y \in A \cup B$, ($x \in C \rightarrow y \in (A - B)$), and ($x \in D \rightarrow y \in (B - A)$). If II had a winning strategy $\tau$, then $\tau^{-1}(B^c)$, $\tau^{-1}(A^c)$ would be disjoint $\Gamma$ sets with $C \subseteq \tau^{-1}(B^c)$, $D \subseteq \tau^{-1}(A^c)$. From sep($\Gamma$) we would then have a $\Delta$ set separating $C$ from $D$, a contradiction. Let $\sigma$ be a winning strategy for I. So, $y \in (A \cup B)$
Corollary 4.8. If $\Gamma$ is non-selfdual and $\Delta$ is closed under finite intersections (equivalently, finite unions), then $\text{red}(\Gamma)$ or $\text{red}(\Delta)$.

Corollary 4.9. If $\Gamma$ is non-selfdual, $\text{sep}(\Gamma)$, and $\Gamma$ is closed under finite intersections, then $\text{red}(\Gamma)$.

In particular, if $\Gamma$ is non-selfdual and $\Gamma$ is closed under finite unions and intersections, then $\text{red}(\Gamma)$ or $\text{red}(\Gamma)$ holds.

5. The Coding Lemma

The coding lemma of Moschovakis is a basic tool in determinacy theory. It provides a choice-like principle which holds assuming AD. We first prove a version of the coding lemma which applies to prewellorderings, and then prove a more general version which applies to wellfounded relations.

Theorem 5.1 (AD). Let $\Gamma$ be a nonselfdual pointclass closed under $\exists \omega^\omega$ and $\wedge$ (and assume $\Gamma \supseteq \Pi^0_1$). Let $\preceq$ be a prewellordering with both $\preceq$ and the strict part $\prec$ in $\Gamma$. Let $R \subseteq \text{dom}(\preceq) \times \omega^\omega$ be a relation with $\text{dom}(R) = \text{dom}(\preceq)$. Then there is a $\Gamma$ relation $R' \subseteq R$ such that $\text{dom}(R')$ meets every $[x]$ for $x \in \text{dom}(\preceq)$ (here $[x] = \{y : y \preceq x \land x \preceq y\}$).

Proof. Let $U \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ be universal for $\Gamma \upharpoonright \omega^\omega \times \omega^\omega$, and we assume all universal sets are good, that is, we have continuous $s$-$m$-$n$ functions. We prove the coding lemma by induction on the length $| \preceq |$ of the prewellordering $\preceq$. If $| \preceq | = \alpha + 1$ is a successor ordinal, the result follows easily by induction. Namely, given $R \subseteq \text{dom}(\preceq) \times \omega^\omega$, apply induction to $\preceq$’ where $x \preceq y \iff (x \preceq y \land y \preceq z)$, where $[z]$ is maximal in $\preceq$. Note that $\preceq'$ and the strict part $\prec'$ are in $\Gamma$. Applying induction to $\preceq'$ and $S = R \wedge (\text{dom}(\preceq') \times \omega^\omega)$ given an $S' \subseteq S \in \Gamma$. Let $R' = S' \cup \{(z, y)\}$ where $R(z, y)$. Since by induction $\text{dom}(S')$ meets every class below $[z]$, $\text{dom}(R')$ meets every class of $\preceq$. Also, $R' \in \Gamma$ since $S' \in \Gamma$, $\Gamma$ is closed under $\vee$ (since it is closed under $\exists \omega^\omega$), and $\Gamma$ contains all closed sets.

So, assume that $| \preceq | = \lambda$ is a limit. If $\alpha \leq \lambda$ we say $\epsilon \in \omega^\omega$ is $\alpha$-good if $U_\epsilon \subseteq R$ and for all for all $x \in \text{dom}(\preceq)$ with $|x| < \alpha$ we have $\text{dom}(U_\epsilon) \cap [x] \neq \emptyset$ (i.e., $U_\epsilon$ is a choice relation below $\alpha$). Note that if $\epsilon$ is $\alpha$-good for some $\alpha$, then there is a maximal $\epsilon \leq \lambda$ such that $\epsilon$ is $\alpha$-good which we denote by $\alpha(\epsilon)$. So, $\alpha(\epsilon)$ is defined iff $U_\epsilon \subseteq R$. By induction, for any $\alpha < \lambda$ there is an $\epsilon$ which is $\alpha$-good. To see this, pick $z \in \text{dom}(\preceq)$ with $|z| = \alpha$. Let $\preceq' = \preceq \cap \{(x, y) : y \prec z\}$, so $\preceq'$ and the strict part $\prec'$ are in $\Gamma$. Apply induction to $\preceq'$ and $S = R \cap \{(x, w) : x \prec z\}$ to produce $S'$. If $U_\epsilon = S'$, then $\epsilon$ is $\alpha$-good.

Consider now the following game $G$: I and II play out reals $\epsilon, \delta$ respectively in $\omega^\omega$. II wins the run provided: $(U_\epsilon \subseteq R) \rightarrow (U_\delta \subseteq R \wedge \alpha(\delta) > \alpha(\epsilon))$.

Suppose first that I has a winning strategy $\sigma$ for $G$. Let $A = \sigma[\omega^\omega]$. For every $\epsilon \in A$, $U_\epsilon \subseteq R$. Furthermore $\{\alpha(\epsilon) : \epsilon \in A\}$ is unbounded in $\lambda$ since if $\delta$ is such that $\alpha(\delta) = \beta$, then $\alpha(\sigma(\delta)) \geq \beta$, and there are $\delta$ which are $\beta$-good for arbitrarily large $\beta < \lambda$. Define then $R'(x, y) \leftrightarrow \exists \epsilon \in A (U_\epsilon(x, y))$. $R' \in \Gamma$ from our assumed closure properties, and clearly $R'$ is $\lambda$-good so we are done.
Assume next that II has a winning strategy \( \tau \) for \( G \). The relation

\[
S(\epsilon, z, x, w, u) \leftrightarrow U_\epsilon(x, w) \land (x < z)
\]

is in \( \Gamma \), and so \( S(\epsilon, z, x, w) \rightarrow U(\alpha, \epsilon, z, x, w) \rightarrow U(s(\alpha, \epsilon, z), x, w) \) for some \( a \in \omega^\omega \), and where \( s \) is our continuous \( s-m-n \) function. Let \( f(\epsilon, z) = s(\alpha, \epsilon, z) \), so \( f \) is continuous. Note that if \( z \) is in the field of \( \lesssim \), then \( U_{f(\epsilon, z)} = U_\epsilon \cap \{(x, w) : x < z\} \). If \( z \) is not in the field of \( \lesssim \), then \( U_{f(\epsilon, z)} = \emptyset \). In particular, for any \( z \) and \( \epsilon \), if \( U_\epsilon \subseteq R \), then \( U_{f(\epsilon, z)} \subseteq R \).

From the recursion theorem, let \( \epsilon \) be such that

\[
U_\epsilon(x, w) \leftrightarrow \exists z \left( z \leq x \land x \leq z \land U(\tau(f(\epsilon, z)), x, w) \right).
\]

We claim that \( R' = U_\epsilon \) works. Note that \( U_\epsilon(x, w) \rightarrow x \in \text{dom}(\lesssim) \). We show by induction on \( |x| \) that \( U_\epsilon(x, w) \rightarrow R(x, w) \). For any \( z \in [x] \), by induction we have \( U_{f(\epsilon, z)} = U_\epsilon \cap \{(x, w) : x < z\} \subseteq R \). It follows that \( U_{f(\epsilon, z)} \subseteq R \) as well, and so \( R(x, w) \).

Finally, we prove by induction on \( |x| \) that for all \( x \in \text{dom}(\lesssim) \) that \( \text{dom}(R') \cap [x] \neq \emptyset \).

Remark 5.2. The hypothesis that both \( \lesssim \) and \( \prec \) are in \( \Gamma \) from theorem 5.1 is more than necessary. The result holds just assuming the strict part \( \prec \) is in \( \Gamma \), although a different proof must be given. Theorem 5.3 below contains this stronger result, and also requires \( \prec \) to only be a wellfounded relation.

Theorem 5.3 (AD). Let \( \Gamma \) be a nonselfidual pointclass closed under \( \exists \omega^\omega \) and \( \land \) (and assume \( \Gamma \supseteq \Pi_1^\mathrm{AT} \)). Let \( \prec \) be a wellfounded relation in \( \Gamma \). Let \( R \subseteq \text{dom}(\lesssim) \times \omega^\omega \) be a relation with \( \text{dom}(R) = \text{dom}(\prec) \). Then there is a \( \Gamma \) relation \( R' \subseteq R \) such that for every \( \beta < |\prec| \), \( \text{dom}(R') \) meets \( \{x \in \text{dom}(\prec) : |x|_{\prec} = \beta\} \).

Proof. Let \( \prec_{\beta} \) denote \( \{x \in \text{dom}(\prec) : |x|_{\prec} = \beta\} \). We again proceed by induction on \( |\prec| \), and we may assume \( \prec |\prec| \) is a limit. Similar to before, we say \( \epsilon \) is \( \alpha \)-good, for \( \alpha \leq \lambda = |\prec| \), if \( U_\epsilon \subseteq R \) and for all \( \beta < \alpha \) with have \( \text{dom}(U_\epsilon) \cap \prec_{\beta} \neq \emptyset \). Also as before, if \( U_\epsilon \subseteq R \), then we let \( \alpha(\epsilon) \) be the largest \( \alpha \leq \lambda \) such that \( \epsilon \) is \( \alpha \)-good. As before we consider the game \( G \) where I and II play out reals \( \epsilon, \delta \) respectively in \( \omega^\omega \) and II wins the run provided: \( (U_\epsilon \subseteq R) \rightarrow (U_\delta \subseteq R \land \alpha(\delta) > \alpha(\epsilon)) \). Note that by induction considering \( \prec |z = \{(x, y) : x < y \land y < z\} \) for \( z \in \text{dom}(\prec) \), we have that there are \( \epsilon \) which are \( \alpha \)-good for arbitrarily large \( \alpha \) below \( \lambda \).

Suppose first that I has a winning strategy \( \sigma \) for \( G \). Let \( A = \sigma[\omega^\omega] \). So, for \( \epsilon \in A \), \( U_\epsilon \subseteq R \). Also, sup\( \\{\alpha(\epsilon) : \epsilon \in A\} = \lambda \). Define \( R' \) by \( R'(x, w) \leftrightarrow \exists \epsilon \in A \ (U_\epsilon(x, w)) \). Then \( R' \subseteq R \) and \( \text{dom}(R') \cap \prec_{\beta} \neq \emptyset \) for all \( \beta < \lambda \).

Suppose next that II has a winning strategy \( \tau \) for \( G \). Here we must argue a little differently from theorem 5.1. We attempt to define using the recursion theorem a relation \( U_\epsilon \) with \( \text{dom}(U_\epsilon) = \text{dom}(\prec) \) and such that \( U_\epsilon(x, u) \) implies \( U_u \subseteq R \) and \( \alpha(\epsilon) > |x|_{\prec} \). Consider the relation

\[
S(\epsilon, x, y, w) \leftrightarrow \exists z \exists u \ (z \prec x \land U_\epsilon(z, u) \land U_u(y, w)).
\]
From the s-m-n theorem, let \( f : \omega^\omega \times \omega^\omega \to \omega^\omega \) be continuous such that \( S(\epsilon, x, y, w) \leftrightarrow U(\epsilon, x, y, w) \). Define then, using the recursion theorem,
\[
U_\epsilon(x, u) \leftrightarrow (x \in \text{dom}(\prec)) \land u = \tau(f(\epsilon, x)).
\]

Clearly \( \text{dom}(U_\epsilon) = \text{dom}(\prec) \), and for every \( x \in \text{dom}(\prec) \) there is exactly one \( u \) such that \( U_\epsilon(x, u) \). We prove by induction on \( |x|_\prec \) that \( U_\epsilon(x, u) \) implies that \( U_\epsilon \subseteq R \) and \( \alpha(u) > |x|_\prec \). By induction, it follows that \( U_{f(\epsilon, x)} \) is a union of relations which are \( \beta \)-good for various \( \beta \) whose supremum is at least \( |x|_\prec \). Thus, \( f(\epsilon, x) \) is \( |x|_\prec \)-good. Since \( \tau \) is winning for \( \Pi \), \( u = \tau(f(\epsilon, x)) \) is \( |x|_\prec + 1 \)-good. Finally, define \( R' \) by:
\[
R'(x, w) \leftrightarrow \exists y \exists u (y \in \text{dom}(\prec) \land U_\epsilon(y, u) \land U_\epsilon(x, w)).
\]
Clearly \( R' \subseteq R \) since \( U_\epsilon(y, u) \) implies \( U_\epsilon \subseteq R \). Also, since \( \alpha(u) > |y|_\prec \), it follows that \( \text{dom}(R') \cap \prec_{\beta \neq 3} \emptyset \) for all \( \beta < \lambda \).

As a corollary it follows that if we can map the reals onto an ordinal \( \lambda \), then we can map the reals onto \( P(\lambda) \). First we note a general fact, Suppose \( A \subseteq \omega^\omega \). Define the pointclass \( \Sigma_1^\lambda(A) \) to be collection of \( B \) which can be written in the form
\[
B(x) \leftrightarrow C(x) \lor \exists y (\forall n (y)_n \in A \land D((x, y))),
\]
where \( C, D \subseteq \omega^\omega \) are \( \Sigma_1^\lambda \). We likewise define \( \Sigma_1^\lambda(A) \upharpoonright (\omega^\omega)^n \) for any \( n \), using the same formula and our recursive bijection between \( (\omega^\omega)^n \) and \( \omega^\omega \).

**Exercise 19.** Show that \( \Sigma_1^\lambda(A) \) is a pointclass which contains \( A \) and is closed under \( \exists^\omega \), \( \land \) and \( \lor \). Also, \( \Sigma_1^\lambda(A) \) has a universal set. [hint: closure under continuous preimages is immediate. Note that \( A(x) \leftrightarrow \exists y (\forall n (y)_n \in A \land D((x, y))) \) where \( D(z) \leftrightarrow \forall i, j ((z)_1)_i = ((z)_1)_j \land x = ((z)_1)_0 \). Thus, \( A \in \Sigma_1^\lambda(A) \). To see closure under \( \lor \) notice that \( \exists y (\forall n (y)_n \in A \land D((x, y))) \lor \exists z (\forall n (z)_n \in A \land D'(\langle x, y \rangle)) \leftrightarrow \exists w (\forall n (w)_n \in A \land D'(\langle x, y \rangle)) \lor D'(\langle x, w \rangle) \). To see closure under \( \land \), note that \( C(x) \land \exists y (\forall n (y)_n \in A \land D((x, y))) \leftrightarrow \exists y (\forall n (y)_n \in A \land D((x, y)) \land C(x)) \). Also, \( \exists y (\forall n (y)_n \in A \land D((x, y))) \land \exists z (\forall n (z)_n \in A \land D'(\langle x, z \rangle)) \leftrightarrow \exists w (\forall n (w)_n \in A \land \exists y (\forall n (y)_n \in A \land D'(\langle x, z, y \rangle))) \). Then \( B'(x) \leftrightarrow \exists z C(\langle x, z \rangle) \lor \exists y (\forall n (y)_n \in A \land \exists z D(\langle x, z, y \rangle)) \). Finally, if \( U \subseteq \omega^\omega \times \omega^\omega \) is universal for \( \Sigma_1^\lambda \upharpoonright \omega^\omega \), then define \( V(\epsilon, x) \leftrightarrow U(\epsilon, 0, x) \lor \exists y (\forall n (y)_n \in A \land U(\epsilon)(1, x, y))) \). Then \( V \) is universal for \( \Sigma_1^\lambda \upharpoonright \omega^\omega \).

**Corollary 5.4 (AD).** If \( \lambda < \Theta \), then there is a map \( f \) from \( \omega^\omega \) onto \( P(\lambda) \).

**Proof.** Suppose \( \lambda < \Theta \), and let \( \prec \) be a prewellordering of \( \omega^\omega \) of length \( \lambda \). Let \( \Gamma \) be a nonselfdual pointclass closed under \( \exists^\omega \), \( \land \) and with \( \leq, \in \Gamma \). Let \( U \subseteq \omega^\omega \times \omega^\omega \) be universal for \( \Gamma \upharpoonright \omega^\omega \). If \( \lambda \subseteq A \), then the coding lemma applied to the characteristic function of \( A \) show that the relation \( R(x, y) \leftrightarrow (x \in \text{dom}(\leq) \land (\langle x \rangle \in A \land y = \bar{1}) \lor (\langle x \rangle \notin A \land y = \bar{0})) \) is in \( \Gamma \). In particular, the code set of \( A, C_A \{ x \in \text{dom}(\leq) : \langle x \rangle \in A \} \) is in \( \Gamma \). Define \( f(\epsilon) = A \) if \( U_\epsilon \) is the code set for \( A \), and \( f(\epsilon) = \emptyset \) otherwise. Then, \( f \) is onto \( P(\lambda) \).

As another application of the coding lemma we prove the following.

**Theorem 5.5.** Suppose \( \Gamma \) is a nonselfdual pointclass closed under \( \forall^\omega \forall^\omega \), \( \land \), \( \lor \). Suppose also \( \text{pwo}(\Gamma) \). Let \( U \subseteq \omega^\omega \times \omega^\omega \) be a universal set for \( \Gamma \upharpoonright \omega^\omega \). Let \( \phi \) be a \( \Gamma \)-norm on \( U \). Then \( |\phi| = \delta(\Gamma) \) is the supremum of the lengths of the \( \Delta \) prewellorderings = the supremum of the lengths of the \( \Gamma \) welfounded relations.
Proof. Let $\phi$ be a $\Gamma$-norm on $U$. All the initial segments of the prewellordering are in $\Delta$ since $\pi$ is a $\Gamma$-norm. Thus, $|\phi| \leq \delta(\Gamma)$. It suffices therefore to show that any $\Gamma$ wellfounded relation $\prec$ has length less than $|\phi|$. From the recursion theorem let $\epsilon$ be such that

$$U_\epsilon(x) \iff \forall y \ (y \prec x \rightarrow (\epsilon, y) <^*_\phi (\epsilon, x)),$$

where $<^*_\phi$ is the norm relation corresponding to $\phi$ (recall for a norm $\psi$ on a set $A$ that $x <^*_\psi y$ iff $(x \in A \land (y \notin A \lor \psi(x) < \psi(y)))$). $U_\epsilon$ is attempting to define an embedding from $\prec$ into $|\phi|$ by $x \mapsto |(\epsilon, x)|$.

We show by induction on $|x|$ that $U_\epsilon(x)$ and that for all $y \prec x$ that $\phi(\epsilon, y) < \phi(\epsilon, x)$. Let $x$ be least such that the stated property fails. So, for all $y \prec x$ we have $U_\epsilon(y)$. If $\neg U_\epsilon(x)$, then by definition of $<^*_\phi$ we have that for all $y \prec x$ that $y <^*_\phi x$. From the definition of $U_\epsilon$ it now follows that $U_\epsilon(x)$, a contradiction. So, $U_\epsilon(x)$.

From the definition of $U_\epsilon$ it now follows that for all $y \prec x$ that $\phi(\epsilon, y) < \phi(\epsilon, x)$. Thus, $x \mapsto \phi(\epsilon, x)$ is order-preserving from $\prec$ to $|\phi|$, so $|\prec| \leq |\phi|$. We must in fact have $|\prec| < |\phi|$ as given any $\Gamma$ wellfounded relation $\prec$ we can easily get another $\Gamma$ wellfounded relation $\prec'$ with $|\prec'| > |\prec|$. □

As another application we have the following.

**Theorem 5.6.** Let $\Gamma$ be a nonselfdual pointclas closed under $\exists^\omega$, $\land$. Then $\delta = \sup$ the supremum of the lengths of the $\Gamma$ wellfounded relations is a regular cardinal.

Proof. Suppose $\rho = \text{cof}(\delta) < \delta$. Let $f : \rho \rightarrow \delta$ be cofinal. Let $\prec$ be a $\Gamma$ wellfounded relation of length $\rho$. Let $U \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ be a universal $\Gamma$ set. Apply the coding lemma to the relation $R(x, y) \leftrightarrow (x \in \text{dom}(\prec)) \land (U_y$ is wellfounded $\land |U_y| \geq f(|x|))$. Let $R' \subseteq R$ with $R' \in \Gamma$ be from the coding lemma, so $\text{dom}(R') \cap |\prec| \neq \emptyset$ for all $\beta < \rho$ (recall $|\prec| = \{z : |z|_\prec = \beta\}$). Let $A(y) \leftrightarrow \exists x (x \in \text{dom}(\prec) \land R'(x, y))$, so $A \in \Gamma$ and consists of codes of $\Gamma$ wellfounded relations whose lengths are unbounded in $\delta$. Define then $(y, z) \ll (y', z') \leftrightarrow (y = y' \land y \in A) \land (U_y(z, z'))$. Then $\ll$ is a wellfounded relation in $\Gamma$, and $|\ll| = \delta$. □