Coordinatewise decomposition of Borel functions

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1. Descriptive set theory

**Definition.** A topological space $X$ is *Polish* if it is separable and completely metrizable.

**Definition.** The *Borel subsets of* $X$ are those obtained from the open sets via complements and countable unions (and countable intersections).

**Definition.** A function $f : X \to Y$ is *Borel* if

$$\forall B \subseteq Y \text{ Borel } (f^{-1}(B) \text{ is Borel}).$$

A central part of descriptive set theory is the study of Borel sets and functions on Polish spaces.
§1. Descriptive set theory

Definition. An equivalence relation $E$ on $X$ is $Borel$ if it is Borel as a subset of $X \times X$.

The study of Borel equivalence relations on Polish spaces is a relatively recent, but rapidly developing, piece of modern descriptive set theory.

One motivation comes from foundational questions regarding effective cardinality and classification.

Another comes from ergodic theory, where the importance of equivalence relations induced by group actions has been apparent for some time.

Here we will focus on some descriptive set-theoretic questions which are intimately linked to their ergodic-theoretic counterparts.
Suppose that $Z$ is a Polish space and $E$ is a Borel equivalence relation on $Z$.

**Definition.** $E$ is *smooth* if there are Borel sets $B_0, B_1, \ldots \subseteq Z$ such that, for all $z_1, z_2 \in Z$,

$$z_1 E z_2 \iff \forall n \in \mathbb{N} \ (z_1 \in B_n \iff z_2 \in B_n).$$

**Definition.** A *transversal* of $E$ is a set $B \subseteq Z$ such that $\forall z \in Z \ \exists! y \in B \ (zEy)$.

**Definition.** $E$ is *countable* if all of its equivalence classes are countable.

**Fact.** If $E$ is countable, then $E$ is smooth if and only if $E$ admits a Borel transversal.
§2. Smooth equivalence relations

Example. The Vitali equivalence relation is the equivalence relation on \( \mathbb{R} \) given by

\[ x E_V y \iff \exists q \in \mathbb{Q} \ (x + q = y). \]

Fact. If \( m \) is Lebesgue measure on \( \mathbb{R} \), then

\( \forall B \subseteq \mathbb{R} \) Borel (\( m(B) > 0 \Rightarrow E_V|B \) is non-smooth).

Theorem (Harrington-Kechris-Louveau). Exactly one of the following holds:

1. \( E \) is smooth;

2. There is a Borel set \( B \subseteq \mathbb{Z} \) with \( E|B \cong_B E_V \).

Corollary. If \( E \) is non-smooth, then there is a probability measure \( \mu \) on \( \mathbb{Z} \) such that

\( \forall B \subseteq \mathbb{Z} \) Borel (\( \mu(B) > 0 \Rightarrow E|B \) is non-smooth).
§3. Marriage problems

Definition. A marriage problem is a set $S \subseteq X \times Y$, where $X \cap Y = \emptyset$.

Definition. A solution to a marriage problem is a bijection $f : X \to Y$ such that $\text{graph}(f) \subseteq S$.

Example. Let $X = 2\mathbb{Z}$, $Y = 2\mathbb{Z} + 1$, and

$$S = \{(m, n) \in X \times Y : |m - n| = 1\}.$$

The function $f : 2\mathbb{Z} \to 2\mathbb{Z} + 1$, given by

$$f(m) = m + 1,$$

is a solution to the marriage problem.
Example. Define \( s : 2\mathbb{Z} \to 2\mathbb{Z} \) by
\[
[s(x)](n) = x(n + 1).
\]
Define \( i : 2\mathbb{Z} \to 2\mathbb{Z} \) by
\[
[i(x)](n) = x(-n).
\]
Clearly \( i^2 = \text{id} \), and since \( i \circ s = s^{-1} \circ i \), we have
\[
(s \circ i)^2 = s \circ i \circ s \circ i
= s \circ s^{-1} \circ i \circ i
= \text{id}.
\]
Let \( X = \{ x \in 2\mathbb{Z} : x \text{ is not periodic} \} \) and
\[
Y = \{ \{x_1, x_2\} : i(x_1) = x_2 \text{ or } s \circ i(x_1) = x_2 \}.
\]
Define \( S \subseteq X \times Y \) by
\[
S = \{ (x, \{x_1, x_2\}) \in X \times Y : x \in \{x_1, x_2\} \}.
\]
§3. Marriage problems

Claim. There is a solution to $S$.

Proof. Associated with $S$ is the graph $G_S$ on the disjoint union $Z_S = X \sqcup Y$, given by

$$G_S = \{(z_1, z_2) : (z_1, z_2) \in S \text{ or } (z_2, z_1) \in S\}.$$  

Let $E_S$ be the equivalence relation whose classes are the connected components of $G_S$.

By AC, there is a transversal $B \subseteq X$ of $E_S$. 

Next, we must consider the structure of $\mathcal{G}_S$:

\[
\begin{align*}
&\ s^2(i(x)) \quad s(i(x)) \quad i(x) \quad s^{-1}(i(x)) \quad s^{-2}(i(x)) \\
&\ s^{-2}(x) \quad s^{-1}(x) \quad x \quad s(x) \quad s^2(x)
\end{align*}
\]

**Definition.** The $s$-saturation of $B$ is given by

\[
[B]_s = \bigcup_{n \in \mathbb{Z}} s^n(B).
\]

Define $f : X \rightarrow Y$ by

\[
f(x) = \begin{cases} 
\{x, i(x)\} & \text{if } x \in [B]_s, \\
\{x, s \circ i(x)\} & \text{otherwise}.
\end{cases}
\]

The function $f$ is a solution to $S$. \square
Claim. There is no Borel solution to $S$.

Proof. Suppose, towards a contradiction, that $f : X \to Y$ is a solution to $S$.

Then the set $A = \{x \in X : f(x) = \{x, i(x)\}\}$ is Borel and $s$-invariant, and $X = A \cup i(A)$.

Let $\mu$ be the product measure on $2^\mathbb{Z}$.

Then $\mu(X) = 1$ and $\mu(A) = \mu(i(A))$.

The ergodicity of $s$ ensures that $\mu(A) \in \{0, 1\}$.

These conditions are mutually exclusive. \qed
As with the non-smoothness of $E_V$, we have shown the inexistence of a Borel solution by verifying a measure-theoretic strengthening.

**Definition.** A set $S \subseteq X \times Y$ is 2-regular if $|S_x| = |S^y| = 2$, for all $x \in X$ and $y \in Y$.

**Theorem.** Suppose that $S \subseteq X \times Y$ is a 2-regular Borel marriage problem with no Borel solution. Then there is a probability measure $\mu$ on $X$ such that, for every Borel set $B \subseteq X$ of positive measure, there is no Borel injection $f : B \to Y$ whose graph is contained in $S$.

**Remark.** In contrast, Graf-Mauldin have shown that $c$-regular Borel marriage problems always admit measure-theoretic solutions, even though Borel (or even just Baire measurable) solutions need not exist.
§4. Coordinatewise decomposition

Suppose $X \cap Y = \emptyset$, $S \subseteq X \times Y$, $G$ is a non-trivial standard Borel group, and $f : S \to G$.

**Definition.** A *coordinatewise decomposition* of $f$ is a pair $(u, v)$, where $u : X \to G$, $v : Y \to G$, and

$$\forall (x, y) \in S \ (f(x, y) = u(x)v(y)).$$

**Definition.** A set $S \subseteq X \times Y$ admits decompositions if every function $f : S \to G$ admits a coordinatewise decomposition.
Claim. $S$ admits decompositions $\iff G_S$ is acyclic.

Proof of $\Rightarrow$. Suppose $x_0, y_0, x_1, y_1, \ldots, x_{n+1} = x_0$ is a cycle, fix $g_0 \in G \setminus \{1_G\}$, and define

$$f(x, y) = \begin{cases} g_0 & \text{if } (x, y) = (x_0, y_0), \\ 1 & \text{otherwise.} \end{cases}$$

If $(u, v)$ is a decomposition of $f$, then

$$g_0 = f(x_0, y_0)f(x_1, y_0)^{-1} \cdots f(x_n, y_n)f(x_{n+1}, y_n)^{-1} = (u(x_0)v(y_0))(u(x_1)v(y_0))^{-1} \cdots (u(x_n)v(y_n))(u(x_{n+1})v(y_n))^{-1} = 1_G,$$

which contradicts our choice of $g_0$. \qed
§4. Coordinatewise decomposition

Proof of \((\Leftarrow)\). Fix a transversal \(B \subseteq Z\) of \(E_S\).

For notational simplicity, assume that \(B \subseteq X\).

Let \(B_n = \{z \in Z_S : d(z, B) = n\}\), where \(d\) is the graph metric induced by \(G_S\).

For \(x \in B_0\), set \(u(x) = 1_G\).

For \(y \in B_{2n+1}\), let \(x\) be the unique neighbor of \(y\) in \(B_{2n}\), and set \(v(y) = u(x)^{-1} f(x, y)\).

For \(x \in B_{2n+2}\), let \(y\) be the unique neighbor of \(x\) in \(B_{2n+1}\), and set \(u(x) = f(x, y)v(y)^{-1}\).  

\(\square\)

If \(f\) and \(B\) are Borel, then the resulting functions \(u, v\) will be Borel as well.

So, if \(G_S\) is acyclic and \(E_S\) admits a Borel transversal, then \(u, v\) can be chosen to be Borel.
§4. Coordinatewise decomposition

Definition. A Borel set $S \subseteq X \times Y$ admits Borel decompositions if every Borel function $f : S \to G$ admits a Borel coordinatewise decomposition.

Kłopotowski-Nadkarni-Sarbadhikari-Srivastava have shown the following:

Theorem. Suppose that $G = \mathbb{Z}$ and $S \subseteq X \times Y$ is a Borel set which admits decompositions. If $S$ is 2-regular, then the following are equivalent:

1. $S$ admits Borel decompositions;
2. $E_S$ is smooth.

Proof (of $\neg(2) \Rightarrow \neg(1)$). By a Glimm-Effros style embedding argument, we can assume that:

(a) $E_S$ is non-smooth;
(b) There is a Borel solution $g : X \to Y$ to the marriage problem associated with $S$. 
§4. Coordinatewise decomposition

Define $f : S \to \mathbb{Z}$ by

$$f(x, y) = \begin{cases} 
1 & \text{if } g(x) = y, \\
-1 & \text{otherwise},
\end{cases}$$

and suppose that $(u, v)$ is a decomposition of $f$.

Then we obtain a transversal of $E_S$, by setting

$$B = \{x \in X : u(x) = 0\} \cup \{y \in Y : v(y) = 0\}.$$  

Moreover, if $u$ and $v$ are Borel, then so is $B$.  

As $E_S$ is non-smooth, it follows that there is no Borel decomposition of $f$.  

\[\square\]
This argument does not extend to all Borel sets $S \subseteq X \times Y$, even when $G = \mathbb{Z}$.

Nevertheless, we have the following:

**Theorem.** Suppose that $S \subseteq X \times Y$ is a Borel set which admits decompositions. Then the following are equivalent:

1. $S$ admits Borel decompositions;
2. $E_S$ is smooth.

**Proof.** To see that $S$ does not admit Borel decompositions when $E_S$ is non-smooth, we use the Harrington-Kechris-Louveau theorem to push a copy of $E_V$ into $E_S$.

We simultaneously push through a subequivalence relation $F_V \subseteq E_V$ such that $E_V/F_V$ is generated by a “Borel” free action of a countable $H \leq G$. 
§4. Coordinatewise decomposition

We ensure also that $E_V, F_V$ share a common ergodic probability measure.

This allows us to prove a measure-theoretic strengthening of the inexistence of decompositions.

To see that if $E_S$ is smooth, then $S$ admits Borel decompositions, we use a result of Hjorth:

**Theorem.** Suppose that $E$ is a Borel equivalence relation whose classes are the connected components of an acyclic Borel graph. Then $E$ is smooth if and only if $E$ admits a Borel transversal.

As $G_S$ is acyclic and $E_S$ is smooth, it follows that $E_S$ admits a Borel transversal.

We have already seen that this implies that $S$ admits Borel decompositions.  \[\square\]
§4. Coordinatewise decomposition

There is a much finer question to be asked:

**Question.** Under what circumstances does a given \( f : S \to G \) admit a Borel decomposition?

When \( G = \langle \mathbb{R}, + \rangle \), the answer depends entirely on the presence of certain measures on \( Z_S \).

We will work with \( \langle (0, \infty), \cdot \rangle \) instead of \( \langle \mathbb{R}, + \rangle \).

As before, it is elementary to see that the existence of a decomposition for \( f : S \to (0, \infty) \) is equivalent to a simple combinatorial property of \( G_S \) and \( f \).

For \( (x, y) \in S \), set \( \varphi_f(x, y) = f(x, y) \).

Extend \( \varphi_f \) to the space of all paths \( \gamma \) through \( G_S \), by insisting that \( \varphi_f(\gamma^{-1}) = \varphi_f(\gamma)^{-1} \) and \( \varphi_f(\gamma_1 \gamma_2) = \varphi_f(\gamma_1) \varphi_f(\gamma_2) \).
Claim. The following are equivalent:

1. $f$ admits a decomposition;

2. For every loop $\gamma$ through $\mathcal{G}_S$, $\varphi_f(\gamma) = 1$.

If $f$ admits a decomposition, we can therefore define $\rho_f : E_S \to (0, \infty)$ by $\rho_f(x, y) = \varphi_f(\gamma)$, where $\gamma$ is any path from $x$ to $y$.

Then $\rho_f$ is a Borel cocycle, i.e.,

$$\forall xE_S yE_S z (\rho_f(x, z) = \rho_f(x, y)\rho_f(y, z)).$$

Definition. A cocycle $\rho : E \to (0, \infty)$ is a (Borel) coboundary if there is a Borel function $w : Z \to (0, \infty)$ such that

$$\forall z_1 E z_2 (\rho(z_1, z_2) = w(z_1)/w(z_2)).$$
Claim. Suppose that $f : S \to (0, \infty)$ admits a decomposition. Then the following are equivalent:

1. $f$ admits a Borel decomposition;
2. $\rho_f$ is a coboundary.

When every horizontal and vertical section of $S$ is countable, the existence of a Borel decomposition is therefore a special case of the following:

Question. Suppose that $E$ is a countable Borel equivalence relation and $\rho : E \to (0, \infty)$ is a Borel cocycle. Under what circumstances is $\rho$ a coboundary?
§5. Some remarks about measures

**Definition.** The *(Borel) full group* of $E$ is the group $[E]$ of Borel automorphisms $f : Z \to Z$ such that $\text{graph}(f) \subseteq E$.

**Definition.** A measure $\mu$ is *$E$-quasi-invariant* if every $f \in [E]$ sends $\mu$-null sets to $\mu$-null sets.

**Definition.** A set $B \subseteq Z$ is an *(E-complete section)* if $\forall x \in X \exists y \in B \ (x E y)$.

**Fact.** For every $\sigma$-finite measure $\mu$ on $Z$, there is a $\mu$-conull Borel $E$-complete section $B \subseteq Z$ such that $\mu|B$ is $(E|B)$-quasi-invariant.

**Remark.** When studying probability measures and countable Borel equivalence relations, this essentially allows us to assume quasi-invariance.
§5. Some remarks about measures

Definition. A measure $\mu$ is $\rho$-invariant if, for every Borel set $B \subseteq Z$ and every $f \in [E],$

$$\mu(f^{-1}(B)) = \int_B \rho(f^{-1}(x), x) \, d\mu(x).$$

Definition. A measure $\mu$ is $E$-invariant, if it is invariant with respect to the trivial cocycle $\rho \equiv 1,$ or equivalently, if every element of $[E]$ is $\mu$-measure preserving.

Fact. Suppose that $\mu$ is an $E$-quasi-invariant, $\sigma$-finite measure. Then there is a Borel cocycle $\rho : Z \to (0, \infty)$ such that $\mu$ is $\rho$-invariant. Moreover, this cocycle is unique $\mu$-almost everywhere.
Theorem. Suppose that $\rho : E \to (0, \infty)$ is a Borel cocycle. Then the following are equivalent:

1. $\rho$ is a coboundary;

2. The following conditions hold:
   
   (a) Every $E$-invariant, $\sigma$-finite measure is equivalent to a $\rho$-invariant, $\sigma$-finite measure;
   
   (b) Every $\rho$-invariant, $\sigma$-finite measure is equivalent to an $E$-invariant, $\sigma$-finite measure.
§6. Borel cocycles

The proof uses 3 Glimm-Effros style dichotomies:

1. The original Glimm-Effros dichotomy, which can be viewed as a characterization of the existence of ergodic measures of type II;

2. A characterization of the existence of $\rho$-invariant measures of type II;

3. A characterization of the existence of $\rho$-invariant measures of type III.

All of the details can be found at:

http://www.math.ucla.edu/~bdm