HOMEWORK 1 SELECTED SOLUTIONS

1. (a)

Proof. Using only axioms (1)-(10) in our definition of a vector space, we have

\[ \vec{w} + (\vec{v} + (-\vec{w})) = \vec{w} + (-\vec{w} + \vec{v}) \]

By axiom (2) (commutativity of addition)

\[ = (\vec{w} + (-\vec{w})) + \vec{v} \]

By axiom (3) (associativity of addition)

\[ = \vec{0} + \vec{v} \]

By axiom (5) (additive inverses)

\[ = \vec{v} \]

By Example 2 (1) in the notes.

□

Some things to watch for:

1) Some students will use a “subtraction” operation in their solution, i.e. they will write \( \vec{w} + (\vec{v} - \vec{w}) = \vec{w} + (\vec{v} + (-\vec{w})) \) instead of \( \vec{w} + (\vec{v} + (-\vec{w})) \). They should lose some points for this on the first homework assignment, since I told them to pay attention to order of operations, and omitting the parentheses obfuscates the importance of axiom (3) in the definition of a vector space.

(b)

Proof. Since \( 0 = 0 + 0 \), we have

\[ 0\vec{v} = (0 + 0)\vec{v} \]

By axiom (8) (distributivity)

\[ = 0\vec{v} + 0\vec{v} \]

Now, axiom (5) guarantees that \( 0\vec{v} \) has an additive inverse \(-0\vec{v} \in V\). Let’s add the latter vector to the right of both sides:

\[ 0\vec{v} + (-0\vec{v}) = (0\vec{v} + 0\vec{v}) + (-0\vec{v}) \]

By axiom (5) (additive inverses)

\[ \vec{0} = (0\vec{v} + 0\vec{v}) + (-0\vec{v}) \]

By axiom (3) (associativity of addition- watch for this!)

\[ \vec{0} = 0\vec{v} + \vec{0} \]

By axiom (5) (additive inverses)

\[ \vec{0} = 0\vec{v} \]

By axiom (4) (additive identity)

This completes the proof.

□

(c)

Proof 1. Since \( \vec{0} + \vec{0} = \vec{0} \) by axiom (4), we have
\[ \mathbf{0} = c(\mathbf{0} + \mathbf{0}) \]
\[ = c\mathbf{0} + c\mathbf{0} \quad \text{By axiom (7) (distributivity).} \]

Again, use axiom (5) to find an additive inverse \(-c\mathbf{0}\), and add it to both sides on the right.

\[ c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0}) \]
\[ = \mathbf{0} + (c\mathbf{0} + (-c\mathbf{0})) \quad \text{By axiom (5) (additive inverses)} \]
\[ = \mathbf{0} + \mathbf{0} \quad \text{By axiom (3) (associativity of addition)} \]
\[ = \mathbf{0} \quad \text{By axiom (5) (additive inverses)} \]
\[ \mathbf{0} = c\mathbf{0} + (-c\mathbf{0}) \quad \text{By axiom (4) (additive identity)} \]

Again the proof is complete. \(\square\)

**Proof 2.** By part (b) above, we know that \(0 \cdot \mathbf{0} = \mathbf{0}\). So we have

\[ c\mathbf{0} = c(0 \cdot \mathbf{0}) \]
\[ = (c \cdot 0) \cdot \mathbf{0} \quad \text{By axiom (9) (associativity of multiplication)} \]
\[ = 0 \cdot \mathbf{0} \quad \text{By part (b) above.} \]
\[ \mathbf{0} \]

\(\text{(d)}\)

**Proof.** Note that

\[ \mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} \quad \text{By axiom (10)} \]
\[ = (1 + (-1))\mathbf{v} \quad \text{By axiom (8)} \]
\[ = 0\mathbf{v} \]
\[ = \mathbf{0} \quad \text{By part (b) above.} \]

It follows that \((-1)\mathbf{v}\) is an additive inverse of \(\mathbf{v}\), and hence by the uniqueness of the additive inverse (Example 2(3) in our notes), we have \(-\mathbf{v} = (-1)\mathbf{v}\). \(\square\)

3.

**Proof.** By Theorem 1 in the notes, it suffices to show that \(\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in V\), and that \(V\) is closed under addition and scalar multiplication.

To see that \(\mathbf{0} \in V\), simply observe that \(0 + 2(0) - 10(0) = 0\), and hence \(\mathbf{0}\) satisfies the definition of \(V\). So \(\mathbf{0} \in V\).

To see that \(V\) is closed under addition, let \(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\) and \(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\) be two vectors in \(V\). Since both are in \(V\), we know that \(x_1 + 2y_1 - 10z_1 = 0\) and \(x_2 + 2y_2 - 10z_2 = 0\). Adding the two equations together, we get \((x_1 + x_2) + 2(y_1 + y_2) - 10(z_1 + z_2) = 0 + 0 = 0\). So the vector \(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\) is in \(V\) by the definition
of \( V \), and hence \( V \) is closed under addition.

Lastly, to see that \( V \) is closed under scalar multiplication, let \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V \) and let \( c \in \mathbb{R} \) be arbitrary. Since \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V \), we know \( x + 2y - 10z = 0 \). Multiplying both sides by \( c \), we get that \( cx + 2(cy) - 10(cz) = c0 = 0 \). In other words, the vector \( \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} = c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) is also in \( V \). So \( V \) is closed under scalar multiplication. Now Theorem 1 implies that \( V \) is a vector subspace of \( \mathbb{R}^3 \).

4. These two problems should be done by counterexample. That is, the students wish to demonstrate that some axiom of being a vector space fails for these examples. In order to show this, they must give a concrete example of a vector or vectors and scalar or scalars for which the axiom fails: they should not receive full credit for a vague justification of why an axiom fails. Here are some possible counterexamples, although there are infinitely many that the student may find.

(a) Proof. The set \( A \) is not a vector subspace of \( \mathbb{R}^2 \) because it fails axiom (1) in the definition of a vector space. To see this, consider the vector \( \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \) which is an element of \( A \). Adding the vector to itself yields:

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}
\]

But the latter vector is not an element of \( A \), since \( 1^2 + 0^2 \) is not less than 1. So \( A \) is not closed under addition, and hence cannot be a vector space.

(b) Proof. \( B \) is not a vector subspace of \( \mathbb{R}^2 \) because it does not contain the zero vector \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) (because \( 0 \neq 3(0) + 1 \), so axiom (4) fails).

5. The students should be able to show that this is NOT a vector space: it fails axioms (4) and (10), at the very least.