Math 1720 Class Notes

Section 8.3 - Infinite Series
• **Definition.** Given a sequence of numbers \( \{a_n\} \), an expression of the form

\[
a_1 + a_2 + a_3 + \cdots + a_n + \cdots
\]

is an **infinite series**. The number \( a_n \) is the \( n\text{th} \) **term** of the series. The **partial sums** of the series form a sequence

\[
\begin{align*}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= \sum_{k=1}^{n} a_k \\
&\vdots
\end{align*}
\]

of real numbers, each defined as a finite sum. If the sequence of partial sums has a limit \( S \) as \( n \to \infty \), we say that the series **converges** to the sum \( S \), and we write

\[
a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k = S.
\]

Otherwise, we say that the series **diverges**.
• **Geometric series** are series of the form

\[ a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \]

in which \(a\) and \(r\) are fixed numbers and \(a \neq 0\). The ratio \(r\) can be positive or negative.

If \(|r| \neq 1\), we can show that the \(n\)th partial sum is

\[ s_n = \frac{a(1 - r^n)}{1 - r}, \quad (r \neq 1). \]

If \(|r| < 1\), then \(r^n \to 0\) as \(n \to \infty\) and \(s_n \to a/(1 - r)\). If \(|r| > 1\), then \(|r^n| \to \infty\) and the series diverges.

If \(r = 1\), the \(n\)th partial sum of the geometric series is

\[ s_n = a + a(1) + a(1)^2 + \cdots a(1)^{n-1} = na, \]

and the series diverges. If \(r = -1\), the series diverges because the \(n\)th partial sums alternate between \(a\) and 0. [2,8,20,42,6,Example 6]
• **Theorem 6: Limit of the $n$th Term of a Convergent Series.** If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

**Proof.** Let $S$ represent the series’ sum and $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ the $n$th partial sum. When $n$ is large, both $s_n$ and $s_{n-1}$ are close to $S$, so

$$a_n = s_n - s_{n-1} \to S - S = 0.$$

• **$n$th-Term Test for Divergence.** $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \to \infty} a_n$ fails to exist or is different from zero. [Example 8, 9]

• **Theorem 7: Properties of Convergent Series.** If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

  1. **Sum Rule:** $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
  2. **Difference Rule:** $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
  3. **Constant Multiple Rule:** $\sum ka_n = k \sum a_n = kA$ (any number $k$).

[10]