6.5 Linear First-Order Differential Equations

- A first-order differential equation that can be written in the form

\[ \frac{dy}{dx} + P(x) y = Q(x), \]

where \( P \) and \( Q \) are functions of \( x \), is a linear first-order equation of standard form.

- We solve the above equation by multiplying both sides by a positive function \( v(x) \) that transforms the left-hand side into the derivative of the product \( v(x) \cdot y \). This will require \( v \) to satisfy

\[ \frac{d}{dx}(vy) = v \frac{dy}{dx} + Pvy. \]

This equation holds if

\[ \frac{dv}{dx} = P v. \]

By separating the variables, we obtain

\[ \ln v = \int P \, dx. \]

Notice that since \( v > 0 \), we do not need absolute value signs in \( \ln v \). Finally, solving for \( v \) gives

\[ v = e^{\int P \, dx}. \]  \hspace{1cm} \text{(Integrating factor)}

- With the above construction of \( v \), we can state the solution of the linear equation as

\[ y = \frac{1}{v(x)} \int v(x) Q(x) \, dx, \]

where

\[ v(x) = e^{\int P(x) \, dx}. \]

In the formula for \( v \), we do not need the most general antiderivative of \( P(x) \). \hspace{1cm} \text{[2,16]}

- **Mixture Problems.** A chemical in a liquid solution runs into a container holding the liquid with a specified amount of the chemical dissolved. The mixture is kept uniform by stirring and flows out of the container at a known rate. The differential equation describing the process is based on the formula

\[ \text{Rate of change} \ \text{of chemical} = \left( \text{rate at which chemical arrives} \right) - \left( \text{rate at which chemical departs.} \right) \]

If \( y(t) \) is the amount of chemical in the container at time \( t \) and \( V(t) \) is the total volume of liquid in the container at time \( t \), then the equation is

\[ \frac{dy}{dt} = (\text{chemical’s arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}). \]  \hspace{1cm} \text{[26]}