Math 1710 Class Notes

Chapter 5
5.1 Volumes by Slicing and Rotation About an Axis

- The **volume** of a solid of known integrable cross-section area $A(x)$ from $x = a$ to $x = b$ is the integral of $A$ from $a$ to $b$,

$$V = \int_{a}^{b} A(x) \, dx.$$  

(2b)

- Solid of Revolution: Circular Cross Sections  

(18,24,14)

- Solid of Revolution: Washer Cross Sections  

(34,40)
5.2 Modeling Volume Using Cylindrical Shells

- Volume by Cylindrical Shells (2,4)
5.3 Lengths of Plane Curves

- A function with a continuous first derivative is **smooth**, and its graph is a **smooth curve**.

Suppose that the graph of a smooth function \( f \) begins at the point \((a, c)\) and ends at \((b, d)\). We partition the interval \( a \leq x \leq b \) into subintervals so short that the arcs of the curve above them are nearly straight. The length of the segment approximating the arc above the subinterval \([x_{k-1}, x_k]\) is \( \sqrt{\Delta x_k^2 + \Delta y_k^2} \). The sum \( \sum \sqrt{\Delta x_k^2 + \Delta y_k^2} \) approximates the length of the curve. We apply the Mean Value Theorem to \( f \) on each subinterval to rewrite the sum as a Riemann sum,

\[
\sum \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sum \sqrt{1 + \left( \frac{\Delta y_k}{\Delta x_k} \right)^2} \Delta x_k \\
= \sum \sqrt{1 + \left( f'(c_k) \right)^2} \Delta x_k
\]
for some point $c_k$ in $(x_{k-1}, x_k)$. By the continuity of the first derivative of $f$ (since $f$ is smooth), the function $\sqrt{1 + (f'(x))^2}$ is integrable. (Theorem 1 of 4.4) Thus passing to the limit as the norms of the subdivisions go to zero gives the length of the curves as

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$ 

Similarly, we could derive another formula for the length of the curves as

$$L = \int_c^d \sqrt{1 + (g'(y))^2} \, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy.$$

(2)

- **Dealing with Discontinuities in $dy/dx$.** At a point on a curve where $dy/dx$ fails to exist, $dx/dy$ may exist and we may be able to find the curve’s length by expressing $x$ as a function of $y$. 


Another way to deal with discontinuities in $\frac{dy}{dx}$ is to use a parametrization for the plane curve. If a curve $C$ is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where $f'$ and $g'$ are continuous and not simultaneously zero on $[\alpha, \beta]$, and if $C$ is traversed exactly once as $t$ increases from $\alpha$ to $\beta$, then the length of $C$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$ 

What if there are two different parametrizations for a curve whose length we want to find; does it matter which one we use? The answer, from advanced calculus, is no, as long as the parametrization we choose meets the conditions preceding the above formula for $L$. \hfill (12)
5.4 Springs, Pumping, and Lifting

- When a body moves a distance \(d\) along a straight line as a result of being acted on by a force of constant magnitude \(F\) in the direction of motion, we calculate the work \(W\) done by the force on the body with the formula

\[
W = Fd \quad \text{(constant – force formula for work).}
\]

- In SI units, the unit of force is a newton, the unit of distance is a meter, and the unit of work is a newton-meter (N·m). This combination appears so often it has a special name, the joule. In the British system, the unit of work is the foot-pound, a unit frequently used by engineers.

- The work done by a variable force \(F(x)\) directed along the \(x\)-axis from \(x = a\) to \(x = b\) is

\[
W = \int_{a}^{b} F(x) \, dx. \quad \text{(4)}
\]
• **Hooke’s law** says that the force it takes to stretch or compress a spring $x$ length units from its natural (unstressed) length is proportional to $x$. In symbols,

$$F = kx.$$  

The constant $k$, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or spring constant) of the spring. Hooke’s law gives good results as long as the force doesn’t distort the metal in the spring. We assume that the forces in this section are too small to do that. (10)

• **Pumping Liquids from Containers.** To find out the work to pump all or part of the liquid from a container, we imagine lifting the liquid out one thin horizontal slab at a time and applying the equation $W = Fd$ to each slab. We then evaluate the integral this leads to as the slabs become thinner and more numerous. (16)
5.5 Fluid Forces

- **The Pressure-Depth Equation.** In a fluid that is standing still, the pressure $p$ at depth $h$ is the fluid’s weighth-density $w$ times $h$:
  
  $$p = wh.$$ 

- **Fluid Force on a Constant-Depth Surface.** $F = pA = whA$

- **The Integral for Fluid Force Against a Vertical Flat Plate.** Suppose that a plate submerged vertically in fluid of weight-density $w$ runs from $y = a$ to $y = b$ on the $y$-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level $y$. Then the force exerted by the fluid against one side of the plate is

  $$F = \int_a^b w \cdot \text{(strip depth)} \cdot L(y) \, dy.$$  

  (2)
5.6 Moments and Centers of Mass

- Many structures and mechanical systems behave as if their masses were concentrated at a single point, called the center of mass.

- Masses Along a Line.

Imagine masses on a rigid $x$-axis supported by a fulcrum at the origin. Each mass $m_k$ exerts a downward force $m_k g$. Each of these forces has a tendency to turn the axis about the origin. This turning effect, called a torque, is measured by multiplying the force $m_k g$ by the signed distance $x_k$ from the point of application to the origin. The sum of the torques measures the tendency of a system to rotate about the origin. This sum is called the system torque.

$$\text{System torque} = \sum m_k g x_k = g \sum m_k x_k$$
Thus, the torque is the product of the gravitational acceleration $g$, which is a feature of the environment in which the system happens to reside, and the number $\sum m_kx_k$, which is a feature of the system itself, a constant that stays the same no matter where the system is placed. The number $M_0 = \sum m_kx_k$ is called the **moment of the system about the origin**. We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torques add to zero. So we need to solve the equation $\sum (x_k - \bar{x}) m_k g = 0$ for $\bar{x}$. The solution is

$$\bar{x} = \frac{\sum m_kx_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}.$$

The point $\bar{x}$ is called the system’s **center of mass**.
• **Wires and Thin Rods.** Imagine a long, thin strip lying along the \( x \)-axis from \( x = a \) to \( x = b \) and cut into small pieces of mass \( \Delta m_k \) by a partition of the interval \([a, b]\). The \( k \)th piece is \( \Delta x_k \) units long and lies approximately \( x_k \) units from the origin. Then

\[
\bar{x} \approx \frac{\text{system moment}}{\text{system mass}} 
\approx \frac{\sum x_k \Delta m_k}{\sum \Delta m_k} 
\approx \frac{\sum x_k \delta(x_k) \Delta x_k}{\sum \delta(x_k) \Delta x_k}
\]

where \( \delta(x_k) \) is the density of the strip at \( x_k \), expressed in terms of mass per unit length and if \( \delta \) is continuous, then \( \Delta m_k \) is approximately equal to \( \delta(x_k) \Delta x_k \).

• **Moment, Mass, and Center of Mass of a Thin Rod or Strip Along the \( x \)-axis with Density Function \( \delta(x) \)**

Moment about the origin: \( M_0 = \int_{a}^{b} x \delta(x) \, dx \)

Mass: \( M = \int_{a}^{b} \delta(x) \, dx \)

Center of mass: \( \bar{x} = \frac{M_0}{M} \) \hspace{1cm} (8)
Masses Distributed over a Plane Region. Suppose that we have a finite collection of masses located in the plane, with mass $m_k$ at the point $(x_k, y_k)$. The mass of the system is

$$\text{System mass } M = \sum m_k.$$  

The moments of the entire system about the two axes are

- Moment about $x$-axis: $M_x = \sum m_k y_k$,
- Moment about $y$-axis: $M_y = \sum m_k x_k$.

The $x$-coordinate of the system’s center of mass is defined to be

$$\bar{x} = \frac{M_y}{M}.$$  

With this choice of $\bar{x}$, the system balances about the line $x = \bar{x}$. The $y$-coordinate of the system’s center of mass is defined to be

$$\bar{y} = \frac{M_x}{M}.$$  

With this choice of $\bar{y}$, the system balances about the line $y = \bar{y}$ as well. We call the point $(\bar{x}, \bar{y})$ the system’s center of mass.
• **Thin, Flat Plates.** Imagine the plate occupying a region in the \(xy\)-plane, cut into thin strips parallel to one of the axes. The center of mass of a typical strip is \((\tilde{x}, \tilde{y})\). We treat the strip’s mass \(\Delta m\) as if it were concentrated at \((\tilde{x}, \tilde{y})\). Then

\[
\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x} \Delta m}{\sum \Delta m}, \quad \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y} \Delta m}{\sum \Delta m}.
\]

• **Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the \(xy\)-plane.**

Moment about the \(x\)-axis: \(M_x = \int \tilde{y}dm\)

Moment about the \(y\)-axis: \(M_y = \int \tilde{x}dm\)

Mass: \(M = \int dm\)

Center of mass: \(\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}\)
To evaluate these integrals, we picture the plate in the coordinate plane and sketch a strip of mass parallel to one of the coordinates axes. We then express the strip's mass $dm$ and the coordinates $(\tilde{x}, \tilde{y})$ of the strip's center of mass in terms of $x$ and $y$. Finally, we integrate $\tilde{y}dm$, $\tilde{x}dm$, and $dm$ between limits of integration determined by the plate's location in the plane. If the distribution of mass in a thin, flat plate has an axis of symmetry, the center of mass will lie on this axis. This fact often help to simplify our work. (14,26,36)

- When the density function is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape.