Math 1710 Class Notes

Chapter 1
1.1 Rates of Change and Limits

- Informal Definition of Limit: Let \( f(x) \) be defined on an open interval about \( x_0 \), except possibly at \( x_0 \) itself. If \( f(x) \) gets arbitrarily close to \( L \) for all \( x \) sufficiently close to \( x_0 \), we say that \( f \) approaches the limit \( L \) as \( x \) approaches \( x_0 \), and we write

\[
\lim_{x \to x_0} f(x) = L.
\]

\((f(x) = \frac{x^2-1}{x-1}, \lim_{x \to 1} f(x) = ?)\)

- The limit value does not depend on how the function is defined at \( x_0 \). \((f(x_0)\) may be undefined, may be different from \( L \), or may be equal to \( L \).)
Formal Definition of Limit: Let $f(x)$ be defined on an open interval about $x_0$, except possibly at $x_0$ itself. Then

$$\lim_{x \to x_0} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $x$,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon.$$ 

(32: with an arbitrary $\epsilon$ first, then do it with the specific $\epsilon$; use graph to illustrate the mechanism) (34)
1.2 Finding Limits and One-Sided Limits

- Limit Rules \( \Rightarrow \) Limits of Polynomials Can Be Found by Substitution and Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero. (12d, 14a)

- Sandwich Theorem: Suppose that \( g(x) \leq f(x) \leq h(x) \) for all \( x \) in some open interval containing \( c \), except possibly at \( x = c \) itself. Suppose also that

\[
\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.
\]

Then \( \lim_{x \to c} f(x) = L. \)
• A function $f(x)$ has a limit as $x$ approaches $c$ if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$  

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• $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ (\(\theta\) in radians) (proved by using Sandwich Theorem, some trigonometry on the first quadrant of the unit circle and the fact that $\frac{\sin \theta}{\theta}$ is even.)
1.3 Limits Involving Infinity

- \( \lim_{x \to \infty} f(x) = L \) means as \( x \) moves increasingly far from the origin in the positive direction, \( f(x) \) gets arbitrarily close to \( L \). (\( \lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0 \); \( \lim_{x \to \infty} k = \lim_{x \to -\infty} k = k \).) (8,10,12)

- A line \( y = b \) is a **horizontal asymptote** of the graph of a function \( y = f(x) \) if either

  \[
  \lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.
  \]

  A line \( x = a \) is a **vertical asymptote** of the graph if either

  \[
  \lim_{x \to a^+} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm \infty.
  \]

  (eg. \( y = \frac{x+3}{x+2}, f(x) = -\frac{8}{x^2-4} \)).
• The function \( g \) is

1. a **right end behavior model** for \( f \) if and only if

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.
\]

2. a **left end behavior model** for \( f \) if and only if

\[
\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 1.
\]

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• Oblique Asymptote

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1.4 Continuity

- Continuity at a Point

  - **Interior point**: A function $y = f(x)$ is **continuous at an interior point** $c$ of its domain if

    $$\lim_{x \to c} f(x) = f(c).$$

  - **Endpoint**: A function $y = f(x)$ is **continuous at a left endpoint** $a$ or is **continuous at a right endpoint** $b$ of its domain if

    $$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b),$$

    respectively.

- If a function $f$ is not continuous at a point $c$, we say that $f$ is **discontinuous** at $c$ and $c$ is a **point of discontinuity** of $f$. Note that $c$ need not be in the domain of $f$. 

A function \( f \) is **right-continuous** (continuous from the right) at a point \( x = c \) in its domain if \( \lim_{x \to c^+} f(x) = f(c) \). It is **left-continuous** (continuous from the left) at \( c \) if \( \lim_{x \to c^-} f(x) = f(c) \).

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. \( (y = 1/x \) is a continuous function, even though it has a point of discontinuity at \( x = 0 \))

The following types of functions are continuous at every point in their domains: polynomial, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions.
• The sums, differences, products, and quotients of functions that are continuous at $x = c$ are continuous at $x = c$. The same goes with constant multiples of functions.

• If $f$ is continuous at $c$ and $g$ is continuous at $f(c)$, then the composite $g \circ f$ is continuous at $c$.

• The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if $y_0$ is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c$ in $[a, b]$. (Application: root-finding)  

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1.5 Tangent Lines

- The tangent to the curve at $P$ is the line through $P$ whose slope is the limit of the secant slopes as $Q \to P$ from either side. (10,18)

- The expression
  \[
  \frac{f(x_0 + h) - f(x_0)}{h}
  \]
  is called the difference quotient of $f$ at $x_0$ with increment $h$. If the difference quotient has a limit as $h$ approaches zero, that limit is called the derivative of $f$ at $x_0$. 