3.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

- A critical point of a function is an interior point $c$ of its domain at which the tangent line to the graph at $(c, f(c))$ is horizontal or at which the derivative does not exist.

- Thus, in the following graphs:

  1. $0$ in figure(a), $-1$ and $1$ in figure(b) are critical points because $f'(c) = 0$ for each point.

  2. $-2$ and $2$ in figure(a), $1$ in figure(c) are critical points because $f'(c)$ does not exist at each point.
• **Relative extrema.** Suppose that $f$ is a function whose value $f(c)$ exists at input $c$ in the domain of $f$. Then

  – $f(c)$ is a **relative minimum** if there exists an open interval $I_1$ containing $c$ in the domain such that $f(c) \leq f(x)$ for all $x$ in $I_1$;

  – $f(c)$ is a **relative maximum** if there exists an open interval $I_2$ containing $c$ in the domain such that $f(c) \geq f(x)$ for all $x$ in $I_2$.

• **Theorem:** If a function $f$ has a relative extreme value $f(c)$, then $c$ is a critical point. (i.e. $f'(c) = 0$ or $f'(c)$ does not exist.)

• **Warning:** The above theorem does not say that if a point is a critical point, its function value will necessarily be a relative maximum or minimum.
The First-Derivative Test for Relative Extrema. For any continuous function $f$ that has exactly one critical point $c$ in an open interval $(a, b)$:

**F1.** $f$ has a relative minimum at $c$ if $f'(x) < 0$ on $(a, c)$ and $f'(x) > 0$ on $(c, b)$. That is, $f$ is decreasing to the left of $c$ and increasing to the right of $c$.

**F2.** $f$ has a relative maximum at $c$ if $f'(x) > 0$ on $(a, c)$ and $f'(x) < 0$ on $(c, b)$. That is, $f$ is increasing to the left of $c$ and decreasing to the right of $c$.

**F3.** $f$ has neither a relative maximum nor a relative minimum at $c$ if $f'(x)$ has the same sign on $(a, c)$ as on $(c, b)$. [8,18,22]
3.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

• Suppose that \( f \) is a function whose derivative \( f' \) exists at every point in an open interval \( I \). Then

1. \( f \) is **concave up** on the interval \( I \) if \( f' \) is increasing over \( I \).

2. \( f \) is **concave down** on the interval \( I \) if \( f' \) is decreasing over \( I \).

• **A Test for Concavity.**

1. If \( f''(x) > 0 \) on an interval \( I \), then the graph of \( f \) is concave up on \( I \).

2. If \( f''(x) < 0 \) on an interval \( I \), then the graph of \( f \) is concave down on \( I \).
• **The Second-Derivative Test for Relative Extrema.** Suppose that \( f \) is a function for which \( f'(x) \) exists for every \( x \) in an open interval \((a, b)\) contained in its domain, and that there is a critical point \( c \) in \((a, b)\) for which \( f'(c) = 0 \). Then:

1. \( f(c) \) is a relative minimum if \( f''(c) > 0 \).

2. \( f(c) \) is a relative maximum if \( f''(c) < 0 \).

The test fails if \( f''(c) = 0 \). The First-Derivative Test would then have to be used.

• A **point of inflection**, or an **inflection point**, is a point across which the direction of concavity changes.

• **Finding Points of Inflection.** If a function \( f \) has a point of inflection, it occurs at a point \( x_0 \), where

\[
 f''(x_0) = 0 \text{ or } f''(x_0) \text{ does not exist.}
\]

[34,8,18]
3.3 Graph Sketching: Asymptotes and Rational Functions

- **Limits and Infinity.** Consider the graph of the rational function $F(x) = \frac{1}{x}$. We see that $\lim_{x \to 0} \frac{1}{x}$ does not exist. Note that as $x$ approaches 0 from the right, the outputs increase without bound, that is
  \[ \lim_{x \to 0^+} \frac{1}{x} = \infty. \]
  As $x$ approaches 0 from the left, the outputs become more and more negative without bound, that is
  \[ \lim_{x \to 0^-} \frac{1}{x} = -\infty. \]

- To determine limits when the inputs get larger and larger without bound is to find **limits at infinity**. Such a limit is expressed as
  \[ \lim_{x \to \infty} f(x). \]
  One way to find such a limit is to use an input-output table. Another way to find this limit is to use some algebra and the fact that as $x \to \infty$, $\frac{1}{x^n} \to 0$ for any positive integer $n$. [4,12,14]
• The line $x = a$ is a **vertical asymptote** if any of the following limit statements is true:

$$\lim_{x \to a^-} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = -\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = -\infty.$$  

• The line $y = b$ is a **horizontal asymptote** if either or both of the following limit statements is true:

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b.$$

• The line $y = mx + b$ is an **oblique asymptote** of the rational function $f(x) = P(x)/Q(x)$ if $f(x)$ can be expressed as

$$f(x) = (mx + b) + g(x),$$

where $g(x)$ approaches 0 as $|x|$ approaches $\infty$. Oblique asymptotes occur when the degree of the numerator is exactly 1 more than the degree of the denominator. We can find it by division.
• **Analytic Strategy for Sketching Graphs.**

1. Find all intercepts.

2. Find all asymptotes.

3. Find $f'(x)$ and $f''(x)$.

4. Find the domain and critical points of $f$.

5. Use second and/or first derivative test to determine the relative extrema.

6. Find all inflection points.

7. Use the above information to sketch the graph, plotting extra points as needed.  

[34,38]
3.4 Using Derivatives to Find Absolute Maximum and Minimum Values

- Suppose that \( f \) is a function whose value \( f(c) \) exists at input \( c \) in an interval \( I \) in the domain of \( f \). Then:

  \[
  f(c) \text{ is an } \textbf{absolute minimum} \text{ if } f(c) \leq f(x) \text{ for all } x \text{ in } I.
  \]

  \[
  f(c) \text{ is an } \textbf{absolute maximum} \text{ if } f(c) \geq f(x) \text{ for all } x \text{ in } I.
  \]
• **Maximum-Minimum Principle 1.** Suppose that $f$ is a continuous function over a closed interval $[a, b]$. To find the absolute maximum and minimum values of the function over $[a, b]$:

1. Find $f'(x)$.

2. Determine the critical points of $f$ in $[a, b]$.

3. Find $f(a)$, $f(b)$, and all $f(c)$’s where $c$ is a critical point. The largest of these is the **absolute maximum** of $f$ over the interval $[a, b]$. The smallest of these is the **absolute minimum** of $f$ over the interval $[a, b]$. \[8,20\]

• **Maximum-Minimum Principle 2.** Suppose that $f$ is a function such that $f'(x)$ exists for every $x$ in an interval $I$, and there is exactly one (critical) point $c$, interior to $I$, for which $f'(c) = 0$. Then

\[
f(c) \text{ is the absolute maximum value over } I \text{ if } f''(c) < 0, \text{ or}
\]

\[
f(c) \text{ is the absolute minimum value over } I \text{ if } f''(c) > 0.\[56,16,74,66\]

3.5 Maximum-Minimum Problems; Business and Economics Applications

- **Max-Min Problems:** [14,20,28]

- **Theorem:** Maximum profit is achieved when \( R'(x) = C'(x) \) and \( R''(x) < C''(x) \). (This is due to the fact that \( P(x) = R(x) - C(x) \) and \( P \) is maximized when \( P'(x) = 0 \) and \( P''(x) < 0 \).) [32]
- Minimizing Inventory Costs. We are trying to minimize the total inventory costs given by

\[ C(x) = \text{(Yearly carrying costs)} + \text{(Yearly reorder costs)} \]

where \( x \) is the lot size, the largest amount ordered each reordering period. If \( x \) is ordered each period, then during that time there is somewhere between 0 and \( x \) units in stock. To have a representative expression for the amount in stock at any one time in the period, we can use the average, \( x/2 \). This represents the average amount held in stock over the course of the year. Thus

\[
\text{Yearly carrying costs} = \left( \text{Yearly cost per item} \right) \cdot \left( \text{Average number of items} \right)
\]

and

\[
\text{Yearly reorder costs} = \left( \text{Cost of each order} \right) \cdot \left( \text{Number of reorders} \right).
\]

[40]
3.7 Implicit Differentiation and Related Rates

- *Implicit Differentiation*: a method to find $dy/dx$ without solving for $y$ by using the Chain Rule (or Extended Power Rule) and treating $y$ as a function of $x$. [6,26,22]