2.1 Limits and Continuity: Numerically and Graphically

- One important aspect of the study of calculus is the analysis of how function values, or outputs, change when the inputs change. Suppose that the inputs get closer and closer to some number. If the corresponding outputs get closer and closer to a number, then that number is called a limit.

- A function \( f \) has the limit \( L \) as \( x \) approaches \( a \) from either side, written
  \[
  \lim_{x \to a} f(x) = L,
  \]
  if all values \( f(x) \) for \( x \) are close to \( L \) for values of \( x \) that are arbitrarily close, but not equal, to \( a \).

- A function \( f \) has the limit \( L \) as \( x \) approaches \( a \) if the limit from the left exists and the limit from the right exists and both limits are \( L \)—that is,
  \[
  \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L = \lim_{x \to a} f(x).
  \]

- Whether or not a limit exists at \( a \) has nothing to do with the function value \( f(a) \).

- When can we substitute to find a limit? The answer lies in the concept of continuity. When the limit of a function is the same as its function value, it satisfies a condition called continuity at a point.

- A function \( f \) is continuous at \( x = a \) if:
  1. \( f(a) \) exists,
  2. \( \lim_{x \to a} f(x) \) exists, and
  3. \( \lim_{x \to a} f(x) = f(a) \).

A function is continuous over an interval \( I \) if it is continuous at each point in \( I \).

- The following continuity principles allow us to determine whether a function is continuous.

C1. Any constant function is continuous.

C2. For any positive integer \( n \) and any continuous function \( f \), \( [f(x)]^n \) and \( \sqrt[n]{f(x)} \) are continuous.

C3. If \( f(x) \) and \( g(x) \) are continuous, then so are \( f(x) + g(x) \), \( f(x) - g(x) \), and \( f(x) \cdot g(x) \).

C4. If \( f(x) \) and \( g(x) \) are continuous, so is \( g(x)/f(x) \), so long as the inputs \( x \) do not yield outputs \( f(x) = 0 \).

2.2 Limits: Algebraically

- If a function is continuous at \( a \), we can substitute to find the limit.

- We can use Limit Principles to find limits when we are uncertain of the continuity of a function at a given point. If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \), then we have the following.

L1. \( \lim_{x \to a} c = c \).

L2. \( \lim_{x \to a} [f(x)]^n = \left[ \lim_{x \to a} f(x) \right]^n = L^n \),

\[
\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L},\]

for any positive integer \( n \).

When \( n \) is even, the inputs \( x \) in \( \sqrt[n]{f(x)} \) must be restricted to those inputs \( x \) for which \( f(x) \geq 0 \).
2.3 Average Rates of Change

- The average rate of change of \( y \) with respect to \( x \), as \( x \) changes from \( x_1 \) to \( x_2 \), is

\[
\frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_2 \neq x_1.
\]

[14a]

- Using a different notation \((x_1 = x \text{ and } x_2 = x + h)\), the average rate of change of \( f \) with respect to \( x \) is also called the difference quotient. It is given by

\[
\frac{f(x + h) - f(x)}{h}, \text{ where } h \neq 0.
\]

The difference quotient is equal to the slope of the line from a point \( P(x, f(x)) \) to a point \( Q(x + h, f(x + h)) \).

[6,8]

2.4 Differentiation Using Limits of Difference Quotients

- **Tangent line as the limit of Secant line:**

  slope of tangent line \( m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \)

- The derivative of a function \( y = f(x) \) at \( x \) is the function \( f' \) defined by

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

provided the limit exists. If \( f'(x) \) exists, then we say that \( f \) is differentiable at \( x \).

[6,20]

- \( f \) is not differentiable at \( a \) if
  1. \( f(a) \) is undefined or
  2. \( f \) is not continuous at \( a \) or
     (Note: If \( f(a) \) is defined and \( f \) is continuous at \( a \), \( f'(a) \) can still be undefined. For example, the function \( f(x) = |x| \) is not differentiable at 0. This means that “continuity does not imply differentiability.” On the other hand, it can be proved that “differentiability implies continuity.”)
  3. \( f \) has a “sharp point” or “corner” at \( a \), or
  4. \( f \) has a vertical tangent at \( a \).

2.5 Differentiation Techniques: The Power and Sum-Difference Rules

- **Notation.** \( f'(x) \) can be written as \( \frac{dy}{dx} \) if \( y \) is a function of \( x \), i.e. \( y = f(x) \). We can also write \( \frac{d}{dx} f(x) \). For example, if \( f \) is defined by \( f(x) = x^2 \), then we can write \( f'(x) \) as \( \frac{d}{dx} x^2 \).

- **Power Rule.** For any real number \( k \), \( \frac{d}{dx} x^k = k \cdot x^{k-1} \).

[2, \frac{d}{dx} x, 16, 12]

- **Derivative of a constant function.** \( \frac{d}{dx} c = 0 \) where \( c \) is a constant.

- **Derivative of a constant times a function.** \( \frac{d}{dx} [c \cdot f(x)] = c \cdot \frac{d}{dx} f(x) \).

[10,14]

- **Sum-Difference Rule.** \( \frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x) \).

[18,44,60,72]
2.6 Instantaneous Rates of Change

- If \( y \) is a function of \( x \), i.e. \( y = f(x) \), then the (instantaneous) rate of change of \( y \) with respect to \( x \) is
  \[
  \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
  \]

- The marginal cost, denoted by \( C'(x) \), is the rate of change of the total cost with respect to the number of units, \( x \), produced.

- The marginal revenue \( R'(x) \) and the marginal profit \( P'(x) \) are defined in the similar way. \[8\]

- The average cost \( A(x) \) of producing \( x \) units is defined as
  \[
  A(x) = \frac{C(x)}{x}.
  \]

2.7 Differentiation Techniques: The Product and Quotient Rules

- The derivative of a product is not the same as the product of derivatives. For example,
  \[
  \frac{d}{dx} \left( x^2 \cdot x^5 \right) \neq \frac{d}{dx} x^2 \cdot \frac{d}{dx} x^5.
  \]

- The Product Rule. Suppose \( F(x) = f(x) \cdot g(x) \). Then
  \[
  F'(x) = \frac{d}{dx} \left[ f(x) \cdot g(x) \right] = f(x) \cdot \left[ \frac{d}{dx} g(x) \right] + \left[ \frac{d}{dx} f(x) \right] \cdot g(x).
  \]

- The Quotient Rule. If
  \[
  Q(x) = \frac{N(x)}{D(x)},
  \]
  then
  \[
  Q'(x) = \frac{D(x) \cdot N'(x) - D'(x) \cdot N(x)}{[D(x)]^2}.
  \]

- An Application: Finding Marginal Revenue with respect to price. The demand function is \( x = D(p) \). The total revenue as a function of price would be
  \[
  R(p) = x \cdot p = D(p) \cdot p = pD(p).
  \]
  This gives
  \[
  R'(p) = pD'(p) + 1 \cdot D(p) = pD'(p) + D(p)
  \]
  by applying the product rule.
2.8 The Chain Rule

- If $y = (1 + x^3)^3$, then what is $\frac{dy}{dx}$? We can try expanding $y$ and differentiate to see what the rule should be.

- **The Extended Power Rule.** For any real number $k$,
  \[ \frac{d}{dx} [g(x)]^k = k [g(x)]^{k-1} \cdot \frac{d}{dx} g(x). \]

- **Composition of Functions.** If $g(x) = \frac{5}{9} (x - 32)$ is the function for converting temperature from Fahrenheit to Celsius and $f(x) = x + 273$ is the function for converting temperature from Celsius to Kelvin, then what is the function for converting temperature from Fahrenheit to Kelvin? If we denote this function as $h$, then $h$ will be called the composition of $f$ and $g$. The notation is $h = f \circ g$, and $f \circ g$ is defined as
  \[ (f \circ g)(x) = f(g(x)). \]
  So in our example, the formula for converting temperature from Fahrenheit to Kelvin is
  \[ h(x) = (f \circ g)(x) = f(g(x)) = f \left( \frac{5}{9} (x - 32) \right) = \frac{5}{9} (x - 32) + 273. \]
  With the concept of composition of functions, we can state the next rule.

- **The Chain Rule.** The derivative of the composition $f \circ g$ is given by
  \[ \frac{d}{dx} [(f \circ g)(x)] = \frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot \frac{d}{dx} g(x). \]

- **Alternate form of chain rule.** Suppose $y = f(u)$ and $u = g(x)$. Then
  \[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \]

2.9 Higher-Order Derivatives

- The notation $f''$ means $(f')'$, i.e. the derivative of the derivative of $f$. We call $f''$ the second derivative of $f$.

- Similarly, $f'''$ means $((f')')'$ and we call it the third derivative of $f$.

- In general, $f^{(n)}$ is called the $n$th derivative of $f$.

- If the function is written in the form $y = f(x)$, then the notation for the $n$th derivative of $y$ with respect to $x$ is
  \[ \frac{d^n y}{dx^n} = \frac{d}{dx} \left( \frac{d^{n-1} y}{dx^{n-1}} \right) \text{ or } \frac{d^n}{dx^n} f(x). \]