1. Compute the following matrix products, if possible.
   
a) \((7 \text{ pts.})\) \(AA^T\), where 
   \[
   A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}
   \]

b) \((5 \text{ pts.})\) \[
   \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 \end{bmatrix} = \]

2. \((4 \text{ pts. each})\) Let \(A, B\) and \(C\) be \(n \times n\) matrices. For each statement below, indicate whether it is always true or not always true. (No explanation needed.)
   
a) \((AB)C = A(BC)\)

b) \((A + B)^{-1} = A^{-1} + B^{-1}\) (Assume all inverses are defined.)

c) \((A + B)(A - B) = A^2 - B^2\)

d) \((A + B)^T = AT + BT\).

3. \((14 \text{ pts.})\) Find the inverse of the matrix \(A\). Or, if \(A\) is not invertible, explain why.
   \[
   A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 9 & 12 \\ 1 & 5 & 5 \end{bmatrix}
   \]

4. Let \(T\) be the transformation
   \[
   T(x_1, x_2, x_3) = (x_1 + 3x_2 - x_3, -x_1 + x_2 - 3x_3, 2x_1 + 5x_2 - x_3)
   \]
   
a) \((5 \text{ pts.})\) Find the standard matrix \(A\) of \(T\).

b) \((12 \text{ pts.})\) Determine whether \(T\) is one-to-one, onto, both, or neither. Show all of the details!

5. a) \((9 \text{ pts.})\) Find the standard matrix, \(A\), of the linear transformation \(T : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) which performs a reflection through the line \(x_2 = x_1\), followed by a rotation by 90° clockwise about the origin. Explain your method carefully! (A sketch could help.)

b) \((7 \text{ pts.})\) Let \(S : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) denote the mapping which reflects points through the line \(x_1 = 1\). Explain why \(S\) is NOT a linear transformation. (Show by a specific example which property of a linear transformation fails.)

6. \((12 \text{ pts.})\) Suppose the last column of \(AB\) is entirely zero but \(B\) itself has no column of zeros. What can you say about the columns of \(A\)? Explain \emph{as precisely as possible}, including appropriate equations.
7. (13 pts.) Suppose $A$ and $B$ are $n \times n$ matrices such that $I + AB$ is invertible. Solve the following system of matrix equations for the matrices $X$ and $Y$:

\[
\begin{align*}
AX + Y &= I \\
X - BY &= I
\end{align*}
\]

(Hint: This is not a routine $2 \times 2$ system of linear equations, because the “coefficients” $A$ and $B$ are matrices rather than numbers. You can NOT assume that $A$ and/or $B$ is invertible, so $A^{-1}$ and $B^{-1}$ may not exist! Furthermore, you can NOT divide by a matrix!)

8. Extra credit!!

a) (6 pts.) Suppose $A(A - 3I) = O$. Does this mean that $A = O$ or $A = 3I$? Give a proof or a counterexample.

b) (6 pts.) Find a natural condition under which $A(A - 3I) = O$ implies that $A = 3I$. Prove that your condition “works”.