Research Statement

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My research areas are descriptive set theory and mathematics education. Descriptive set theory combines techniques from analysis, topology, algebra, computability theory, and set theory to investigate the properties of definable objects. In my dissertation, I studied general topological aspects and definable equivalence relations. My results are contained in the papers [4, 5, 6], all accepted for publication.

In mathematics education my research interests are related to the training of teachers. Moreover, I am interested in the application of item response theory to mathematics instruments.

Descriptive set theory

Polish spaces (separable, completely metrizable spaces) are the natural setting for analysis and descriptive set theory. Therefore, we may want to know when a separable metrizable space is completely metrizable. In particular, consider the following situation. Let X be a Polish space, Y a separable metrizable space, and $f\colon X \to Y$ a continuous surjection. Which conditions on the map f imply that Y is completely metrizable?

Classical results by Sierpiński [19] and Vainštein [21, 22] show that Y is completely metrizable when f is an open map or a closed map, respectively. More recently, Ostrovsky [16] proved that if the image of every open set or every closed set is the union of an open and a closed set, then Y is completely metrizable. He raised the question whether the same is true when each image is the intersection of an open and a closed set.

The lowest two levels of the Hausdorff differency hierarchy consist of the open sets and intersections of an open set and a closed set, together with their complements, the closed sets and the unions of an open set and a closed set. More complicated combinations of open and closed sets occur at higher levels; collectively, the sets in the difference hierarchy are known as resolvable sets. Using this notion, we proved the following theorem in [6]:

Theorem. *Let X be a Polish space, Y a separable metrizable space, and $f\colon X \to Y$ a continuous surjection. If the image under f of every open set or every closed set is resolvable, then Y is completely metrizable.*

This theorem generalizes all earlier results. In a precise sense, it is the strongest result possible along these lines. A related question is extent of the difference hierarchy. Lavrentiev showed that in Polish spaces there are ω_1 distinct levels, but his method does not work in more general spaces. Using an inductive construction, we can show that for the rationals the hierachy also has ω_1 levels.

Theorem. *The difference hierarchy of the rationals has ω_1 distinct levels.*

Moreover, this implies that the difference hierarchy of any uncountable separable metrizable space as length ω_1.

The order topology generated by the subbase of open rays forms a natural topology on ordinals. A complete set of homeomorphism invariants for these ordinal topologies can be computed from their Cantor normal form ([1], we gave an independent proof in [12]). A related problem is the classification of ordinal topologies up to

Borel isomorphism. Since (for example) all countable ordinals are Borel isomorphic but not all are homeomorphic, this is a genuinely different notion of equivalence.

In [5] we provided a complete classification of ordinals up to Borel isomorphism. To state our Borel isomorphism invariants precisely, define for every ordinal α a cardinal $\kappa(\alpha)$ as follows. Let $\kappa(\alpha) = 0$ if $|\alpha|$ is singular or countable; otherwise, let $\kappa(\alpha)$ be the largest cardinal such that $|\alpha| \cdot \kappa(\alpha) \leq \alpha$.

Theorem. *Two ordinals α and β are Borel isomorphic if and only if $|\alpha| = |\beta|$ and $\kappa(\alpha) = \kappa(\beta)$.*

This classification depends on the axiom of choice; we only have the following partial result under the axiom of determinacy:

Theorem. *Under the axiom of determinacy, two ordinals $\alpha, \beta \leq \omega_2$ are Borel isomorphic if and only if $|\alpha| = |\beta|$.*

In future work, I intend to use Jackson's measure analysis [10] to extend this theorem to larger ordinals.

An active area within contemporary descriptive set theory is the study of definable equivalence relations. Over the years, a complexity theory of equivalence relations and their associated classification problems has been developed. Often, classification problems in different fields of mathematics can be compared. For example, classifying countable graphs up to isomorphism is strictly more complicated (in a technical sense) than classifying complex matrices up to conjugacy. Part of this complexity theory deals with possible cardinalities of definable sets.

Cantor's famous continuum hypothesis states that a subset of a Polish space has either at most countably many elements or else continuum many elements. Alexandrov, Hausdorf, and Luzin–Suslin showed that this holds for Borel and analytic sets. These sets are either countable or else contain a perfect set and hence have continuum many elements. A theorem of Silver [20] extends this to definable equivalence relations: a coanalytic equivalence relation either has at most countably many equivalence classes or else there is a perfect set of pairwise inequivalent elements.

This body of results has been extended to limits superior of sequences of sets in the following way. Let $[\omega]^\omega$ denote the collection of all infinite subsets of the natural numbers. Komjáth [13], building on the work of Laczkovich [14], showed that for a sequence $(A_n)_{n\in\omega}$ of analytic subsets of a Polish space, if $\limsup_{n\in H} A_n$ is uncountable for every $H \in [\omega]^\omega$, then there is an $H \in [\omega]^\omega$ such that $\bigcap_{n\in H} A_n$ contains a perfect set. Balcerzak and Głab [2] proved the corresponding theorem for equivalence relations which can be written as a countable intersection of closed sets (that is, F_σ equivalence relations). In [4] we generalized this result to coanalytic equivalence relations:

Theorem. *Let E be a coanalytic equivalence relation on a Polish space X and $(A_n)_{n\in\omega}$ a sequence of analytic subsets of X. If $\limsup_{n\in H} A_n$ meets uncountably many E-equivalence classes for every $H \in [\omega]^\omega$, then there is an $H \in [\omega]^\omega$ such that $\bigcap_{n\in H} A_n$ contains a perfect set of pairwise E-inequivalent elements.*

In contrast to the purely topological methods of earlier researchers, our proof uses techniques from computability theory. Currently, I am adapting the methods of Harrington–Shelah [7] to generalize the above result to co-κ-Suslin equivalence relations in settings without the axiom of choice. In the future I want to study other classes of equivalence relations, like those equivalence relations which are induced by a Borel action of a Polish group.

Although much of modern descriptive set theory uses advanced techniques, there are still interesting open questions accessible with a moderate background. For example, is there a theory of definable equivalence relations on ordinal spaces? What is the complexity of certain specific, concrete equivalence relations? Other interesting topics are natural examples of descriptive set-theoretic phenomena (e.g. natural non-Borel analytic sets), special subsets of the real line, and large cardinal properties without the axiom of choice (see for example my Master's thesis [11]). These topics would be suitable an undergraduate research-like experience or a thesis.

Mathematics education

My research in mathematics education relates to mathematics teachers. In particular, I am interested in how content knowledge, technological proficiency, teacher efficacy, and beliefs about mathematics as a discipline combine to enable teaching for understanding (see for example [23]).

At the moment, I participate in two research projects related to the mathematics content knowledge of elementary pre-service teachers. The goal of one project is to construct an instrument to measure content knowledge growth in pre-service elementary teachers. The other project is specifically about proportional reasoning. In both projects, I use item response theory which is a powerful refinement of classical test theory. Item response theory can be used to analyse a wide range of instruments, from content assessments to attitude scales. One advantage of item response theory is that trait levels and item parameters are measured on the same scale and can therefore be compared. I believe that this comparison will enhance the understanding of the underlying constructs.

A potential obstacle to the use of item response theory in mathematics education is the sample size and test length needed to accurately estimate model parameters. Currently, my focus is the graded response model [18] which can be used to analyse Likert-scale items. Only two studies [15, 17] have considered parameter recovery for this model, both using maximum likelihood methods. I am conducting a simulation study to assess parameter recovery using Markov chain Monte Carlo methods and investigate techniques to improve parameter estimation for smaller sample sizes and test lengths. An intented application is the analysis of the Mathematics Teachers' Efficacy Beliefs Instrument [8].

There remains a need of item response analyses of instruments using sufficiently large samples in future projects. Moreover, I want to combine item response theory with qualitative theories of mathematical understanding like the Van Hiele theory in geometry [9] or the APOS theory in undergraduate mathematics education [3].

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