1. (due Jan. 28, 2010; done) Prove that the Bernoulli measure generated by a probability measure on a finite set is ergodic with respect to the shift map. What about if the generating set is infinite?

2. (due Jan. 28, 2010; done) Show that any factor of an ergodic measure preserving dynamical system is ergodic.

3. (due Feb. 02, 2010) Exercises 6.5.11-6.5.14 from the textbook.

4. (due Feb. 02, 2010; done) For every topological dynamical system $T : X \to X$ let

$$\deg |T| = \min_{x \in X} \# T^{-1}(x).$$

Show that if T is a local homeomorphism, then $\mathrm{h_{top}}(T) \geq \log \deg |T|$.

5.(due Feb. 02, 2010) Let X be a countable compact metric space and $T : X \to X$, a topological dynamical system. Show that $\mathrm{h_{top}}(T) = 0$.

6. (due Feb. 02, 2010; done) Let $T : [0, 1] \to [0, 1]$ be the Gauss map, that is given by the formula

$$T(x) = \left\{ \frac{1}{x} \right\}.$$

Let μ be the Borel probability measure on $[0, 1]$ determined by the condition that

$$\frac{d\mu}{dl}(x) = \frac{1}{\ln 2} \frac{1}{1 + x}.$$

Show that μ is T-invariant.

7. (due Feb. 02, 2010; done) Show that the measure μ from the Problem 6 is ergodic.

8. (due Feb. 02, 2010; done) Let $T : X \to X$ be a measurable dynamical system preserving a probability measure μ. Let A be a measurable set with $\mu(A) > 0$ and let μ_A will be a probability measure on A given by the formula

$$\mu_A(B) = \mu(B \cap A).$$

Show that $\mu_A \circ T_A^{-1} = \mu_A A$, where $T_A : A_\infty \to A_\infty$ is the first return map.

9. (due Feb. 02, 2010; done) With the setting of Problem 8 show that if T is ergodic, then so is T_A.

10. (due Feb. 04, 2010; done) Show that each product (Bernoulli) measure on a shift space $E^{\boldsymbol{Z}}$ is ergodic.

11. (due Feb. 04, 2010; done) Show that each ergodic translation of a torus is uniquely ergodic.

12. (due Feb. 02, 2010; done) Prove that the Cartesian product of two ergodic maps is ergodic if and only if the only common eigenvalue of their induced isometries (onL^2 spaces) is the number 1.

13. (due Feb. 02, 2010; done) Given an example of a minimal system which is not uniquely ergodic.

14. (due Feb. 23, 2010; done) With the setting of Problem 8, let $\tau_A : A \to \{1, 2, \ldots\}$ be the first return time to A. Show that $\int_A \tau_A d\mu_A = \frac{1}{\mu(A)}$.

15. (due Feb. 25, 2010; done) Let d be the function defined on the set of all countable measurable partitions of a probability space (X, μ) by the following formula:

$$d(\mathcal{A}, \mathcal{B}) = \mathrm{H}_\mu(\mathcal{A}|\mathcal{B}) + \mathrm{H}_\mu(\mathcal{B}|\mathcal{A}).$$

Show that d is a metric.

16. (due Feb. 25, 2010; done) Let $T : X \to X$ be a measurable dynamical system preserving a probability measure μ. Prove that you will get the same number for the entropy of T regardless whether you take the supremum over finite partitions or countable partitions with finite entropy.

17. (due March 09, 2010; done) Let E be a finite set and let $P : E \to [0, 1]$ be a probability vector on E. Let μ_P be the infinite product measure (Bernoulli measure) of P on $E^{\mathbb{N}}$. Let $\mathcal{A}$ be the partition of $E^{\mathbb{N}}$ into initial cylinders of length 1. Calculate $\mathrm{h}_{\mu_P}(\sigma, \mathcal{A})$.

18. (due March 09, 2010; done) Let $R : S^1 \to S^1$ be a rotation of the circle and let $\mathcal{A}$ be any finite partition of S^1 into its subarcs. Show that $\mathrm{h}_\lambda(R, \mathcal{A}) = 0$, where λ is Lebesgue measure on S^1.

19. (due March 09, 2010; done) Show that if S is a factor of T, then the entropy of S does not exceed the entropy of T.

20. (due March 09, 2010; done) Show that the entropy of the Cartesian product of two maps is equal to the sum of the entropies of these maps.

21. (due March 30, 2010; done) Show that if $\mathrm{h}_\mu(T, \mathcal{A}) > 0$, then the number of elements of partitions $\mathcal{A}^n$ grows exponentially fast.

22. (due March 30, 2010) Show that if μ is an Borel probability measure invariant and ergodic with respect to the maps $x \mapsto 2x (mod 1)$ and $x \mapsto 3x (mod 1)$, $x \in [0, 1]$, then μ is Lebesgue measure on the interval $[0, 1]$.

23. (due March 30, 2010) Prove versions of the Problem 22 as indicated by Tushar in the classroom.

24. (due April 15, 2010) Prove that $\mathrm{P}(\phi + c) = \mathrm{P}(\phi) + c$.

25. (due April 15, 2010) Suppose that $T : X \to X$ is a topological dynamical system. Prove that the following three conditions are equivalent.

(a) $\mathrm{h}_{\mathrm{top}}(T) < +\infty$.
(b) There exists $\phi \in C(X)$ such that $\mathrm{P}(\phi) < +\infty$.
(c) $\mathrm{P}(\phi) < +\infty$ for every function $\phi \in C(X)$.

26. (due April 15, 2010) Show that if $\mathrm{h}_{\mathrm{top}}(T) < +\infty$, then the topological pressure function $\mathrm{P} : C(X) \to \mathbb{R}$ is Lipschitz continuous. Identify the smallest Lipschitz constant of P.

27. (due April 15, 2010) Show that $\mathrm{P}(\phi + u - u \circ T) = \mathrm{P}(\phi)$.

28. (due April 20, 2010) Let E be a finite set and $\overline{\phi} : E^2 \to I\!R$, an arbitrary function. Define $\phi : E^\infty \to I\!R$ by setting

$$\phi(\omega) = \overline{\phi}(\omega_0\omega_1).$$

Prove that the topological pressure $\mathrm{P}(\phi)$ is equal to the spectral radius of the matrix $(\overline{\phi})_{1\le i,j\le \#E}$.

29. (due April 20, 2010)If $R_\alpha : S^1 \to S^1$ is a rotation about an irrational angle and $\phi : S^1 \to I\!R$ is a continuous function, then

$$\mathrm{P}(R_\alpha, \phi) = \int_{S61} \phi dl,$$

where l is the normalized Lebesgue measure on the circle S^1.

30. (due April 20, 2010) Show that the pressure function $\mathrm{P} : C(X) \to I\!R$ is convex, meaning that

$$\mathrm{P}(\alpha\phi + (1-\alpha)\psi) \le \alpha\mathrm{P}(\phi) + (1-\alpha)\mathrm{P}(\psi)'$$

$\alpha \in [0, 1]$.

31. (due April 20, 2010) Suppose that $T : S^1 \to S^1$ is a C^2-expanding map of the unit circle. Put

$$\phi = -\log |T'|.$$

Show that

[(a)] the function $I\!R \ni t \mapsto \mathrm{P}(t\phi) \in I\!R$ is convex,
[(b)] the function $I\!R \ni t \mapsto \mathrm{P}(t\phi) \in I\!R$ is continuous,
[(c)] the function $I\!R \ni t \mapsto \mathrm{P}(t\phi) \in I\!R$ is strictly decreasing,
[(d)] the function $I\!R \ni t \mapsto \mathrm{P}(t\phi) \in I\!R$ is real-analytic,
[(e)]

$$\lim_{t\to-\infty} \mathrm{P}(t\phi) = +\infty \quad \text{and} \quad \lim_{t\to+\infty} \mathrm{P}(t\phi) = -\infty,$$

[(f)] $\mathrm{P}(\phi) = 0$.

32. (due April 20, 2010) Generalize Problem 31 to all C^2-expanding maps of compact Riemannian manifolds.

33. (due April 20, 2010) Suppose that $T : X \to X$ is a topological dynamical system with finite entropy and $\phi \in C(X)$. Define, for complex numbers s,

$$\zeta_\phi(s) = \sum_{n=1}^{\infty} \sum_{z\in Fix(T^n)} \frac{s^n}{n} \exp\big(S_n(\phi(z))\big).$$

[(a)] Find T and ϕ such that the radius of convergence of ζ_ϕ is equal to 0.
[(b)] If $\sigma : E^\infty \to E^\infty$ is a full shift, then the radius of convergence of ζ_ϕ is equal to $\exp(-\mathrm{P}(\phi))$.
[(c)] If, as in (b), ϕ is in addition, Hölder continuous, then ζ_ϕ can be extended meromorphically to an open disk $D(0, r)$ with $r > 0 \exp(-\mathrm{P}(\phi))$ and the only pole of ζ_ϕ is located at the point $\exp(-\mathrm{P}(\phi))$.

34. (due April 29, 2010) Show that the Variational Principle remains true if invariant measures are replaced by ergodic invariant measures.

35. (due April 29, 2010) Show that if $T : X \to X$ is a topological dynamical system and X is a countable compact space, then $\mathrm{h}_{\mathrm{top}}(T) = 0$.

36. (due May 06, 2010) Show that for every $t \geq 0$ there exists a topological dynamical system whose topological entropy is equal to t.

37. (due May 06, 2010) Find a topological dynamical system with no measure of maximal entropy.

38. (due May 06, 2010) Find a transitive topological dynamical system with no measure of maximal entropy.

39. (due May 06, 2010) Find a topological dynamical system with more than one measure of maximal entropy.

40. (due May 06, 2010) Find a transitive topological dynamical system with more than one measure of maximal entropy.

41. (due May 06, 2010) Show that no continuous self-map of the interval [0.1] is expansive.

42. (due May 06, 2010) Let $T : [0, 1] \to [0, 1]$ be a piecewise monotone continuous map of the interval [0.1]. Prove that the map $\mu \mapsto \mathrm{h}_\mu(T)$ is upper semi continuous.

43. (due May 06, 2010) Show that in the formulation of the variational principle Borel probability invariant measures can be replaced by Borel probability ergodic invariant measures.

44. (due May 06, 2010) Show that equilibrium states form a compact convex set.

45. (due May 06, 2010) Show that if an equilibrium state exists, then there exists an ergodic equilibrium state.