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MIDDLE TERM TEST

1. Which of the following matrices are in reduced eschelon form, which are in eschelon form but not in reduced eschelon form, and which are not in eschelon form?

$$A = \begin{bmatrix} 2 & 0 & 3 & \pi \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 5 & 2 & 3 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 & -8 \\ 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 2 & -\pi \end{bmatrix};$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 5 & 3 & 3 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix}$$

A - eschelon form; not reduced eschelon form

B - eschelon form; not reduced eschelon form.

C - reduced eschelon form

D - not eschelon form

2. Write the vector equation and the matrix-vector equation that are respectively equivalent to the following system of linear equations.

$$\begin{aligned} 2x_1 + 3x_3 - x_4 &= 7 \\ 5x_1 + 4x_2 + 3x_3 + 2x_4 &= 1 \\ x_3 - x_4 &= 0. \end{aligned} \tag{0.1}$$

and

$$x_1 + x_2 + x_3 + x_4 = 5. \tag{0.2}$$

(0.1) $x_1 \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix}$ - vector equation form

$\begin{bmatrix} 2 & 0 & 3 & -1 \\ 5 & 4 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix}$ - matrix-vector form.

(0.2) $x_1 [1] + x_2 [1] + x_3 [1] + x_4 [1] = [5]$ - vector form

$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = [5]$ - matrix-vector form

3. Calculate

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

4. Does there exist a vector $\mathbf{x}$ in $\mathbb{R}^3$ such that

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

Justify your answer.

The augmented matrix of this equation is

$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 3 & 7 \end{bmatrix}$. We need to transform it to an eschelon form.

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$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -3 & 3 & 9 \end{bmatrix}$ - this is an eschelon form. Since there is no row of the form $[0\,0\,0\,b]$ with $b \neq 0$, the matrix is consistent, and our equation has a solution.

5. Are the following indexed sets of vectors linearly independent?

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 9 \end{bmatrix} \right\}$$

and

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Justify your answer.

Since $\begin{bmatrix} -3 \\ 3 \\ 9 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$ the first indexed set of vectors is linearly dependent.

In the second set of vectors the number of vectors $= 5$ is greater than the number of coordinates $= 3$, so this indexed set of vectors is linearly dependent.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map. Suppose that

$$T(\mathbf{u}) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{v}) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

find $T(\mathbf{u} - \mathbf{v})$ and $T(2\mathbf{u} + 5\mathbf{v})$.

$$T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v}) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$T(2\vec{u} + 5\vec{v}) = 2T(\vec{u}) + 5T(\vec{v}) =$$

$$= 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 20 \end{bmatrix} = \begin{bmatrix} 13 \\ 22 \end{bmatrix}.$$