

Applications of Linear Algebra in Economics: Input-Output and
Inter-Industry Analysis.
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1. Introduction

In 1973 Wassily Leontiff won the Noble Prize in Economics for his work in input-output analysis. His seminal work allowed for a greater quantification of economic models. Input-output analysis, also called *Inter-Industry Analysis*, creates an environment where the user can predict the consumption and demand for a system. This system can be as small as a single business or as large as the greater global economy. It is interesting to note that the input-output analysis is robust enough deal with both closed systems, and ones where commodities are flowing into and out the systems (*i.e. imports, exports, taxes, etc...*). Regardless of the size or composition of the system which is being analyzed the procedure is essentially the same.

2. Input-Output Matrices

The procedure begins by looking at an input-output matrix of a given economy. An input-output matrix demonstrates how goods from one industry are consumed in other industries. The rows of the matrix represent the producing sector of the economy, while the columns of the matrix represent the consuming sector of the economy. A general example of the input-output matrix is in figure 1, where S_1 through S_n are sectors of the economy and where the the set of sectors $[S_1, S_n]$ is the entire producing economy.

$$\begin{array}{l}
 \text{Input - Output General Matrix} = \\
 \text{From \ To} \quad S_1 \quad S_2 \quad \dots \quad S_n \\
 S_1 \quad \left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{array} \right) \\
 S_2 \\
 \vdots \\
 S_n
 \end{array} \tag{1}$$

The entry a_{ij} represents the the amount of goods from sector i sector j demands. An example of a simplified input-output matrix is in Fig. 2.

$$\begin{array}{l}
 \text{Simplified Matrix } A = \\
 \text{From \ To} \quad \text{Agricultural Sector} \quad \text{Manufacturing Sector} \quad \text{Labor Sector} \\
 \text{Agricultural Sector} \quad \left(\begin{array}{ccc} 40 & 20 & 40 \\ 20 & 20 & 40 \\ 40 & 200 & 80 \end{array} \right) \\
 \text{Manufacturing Sector} \\
 \text{Labor Sector}
 \end{array} \tag{2}$$

In the simplified example entry A_{11} represents the amount product of the agricultural sector uses that is produced by itself, in this case forty units of goods. Likewise, entries A_{12} and A_{13} represent how much product produced

by the agricultural sector is used in the manufacturing sector and labor sector respectively. Similarly A_{21} is the amount of product produced by the manufacturing sector that is consumed by the agricultural sector and so on. It is interesting to note that this matrix is infinitely scalable, there can be any number of sectors or firms in this matrix. Also the input-output matrix is always square, which is quite helpful because we will be concerned with the inverse of this matrix later on.

The total internal demand for any economy is equal to the sum of the rows of the economies input-output matrix. The I_{th} entry of the internal demand vector is generated by the following:

$$a_{i1} + a_{i2} + \dots + a_{in} = d_i \quad (3)$$

This results in the internal demand vector.

$$\text{Internal Demand Vector} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_i \\ \vdots \\ d_n \end{bmatrix} \quad (4)$$

So for the previous example the input-output matrix and internal demand vector are:

Simplified Input – Output Matrix A =

From \ To	<i>Agg</i>	<i>Manu</i>	<i>Labor</i>	<i>Demand Vector</i>
<i>Agg</i>	40+	20+	40 =	100
<i>Manu</i>	20+	20+	40 =	80
<i>Labor</i>	40+	200+	80 =	320

(5)

These matrices shows the flow of goods between industries, however we are left with no information on the relative size of these industries in terms of currency. If we know the value or the average value of the goods that are being produced we can create a matrix that will transform the input-output matrix into one where the amount of good being consumed by each sector of the economy is measured in terms of currency instead of goods. To do this construct a diagonal matrix where the each entry is the price of the goods that the corresponding industry produces. For example the entry a_{11} is the price that the goods that industry S_1 produce are valued at.

$$\begin{array}{c}
\text{Price} \\
PS_1 \\
PS_2 \\
\vdots \\
PS_n
\end{array}
\begin{array}{cccc}
PS_1 & PS_2 & \dots & PS_n \\
\left(\begin{array}{cccc}
p_{11} & 0 & \dots & 0 \\
0 & p_{22} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & p_{nn}
\end{array} \right)
\end{array}
\tag{6}$$

After we have this matrix then multiply it by the input-output matrix.

$$\begin{bmatrix} p_{11} & 0 & \dots & 0 \\ 0 & p_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_{nn} \end{bmatrix}
\begin{bmatrix} a_{11} & a_{21} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}
=
\begin{bmatrix} p_{11}a_{11} & p_{11}a_{21} & \dots & p_{11}a_{1n} \\ p_{22}a_{21} & p_{22}a_{22} & \dots & p_{22}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{nn}a_{n1} & p_{nn}a_{n2} & \dots & p_{nn}a_{nn} \end{bmatrix}
\tag{7}$$

Now for the simplified example the price of one unit of agricultural goods is ten dollars, the price of one unit of manufacturing goods is twenty dollars, and finally the price of one unit of labor services is worth five dollars. So the price and input-output matrices for the example are:

$$\text{Value Matrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix}
\tag{8}$$

$$\text{Input - Output Matrix} = \begin{bmatrix} 40 & 20 & 40 \\ 20 & 20 & 40 \\ 40 & 200 & 80 \end{bmatrix}
\tag{9}$$

Now we multiple the price matrix by the input-output matrix.

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix}
\begin{bmatrix} 40 & 20 & 40 \\ 20 & 20 & 40 \\ 40 & 200 & 80 \end{bmatrix}
=
\begin{bmatrix} 400 & 200 & 400 \\ 400 & 400 & 800 \\ 200 & 1000 & 400 \end{bmatrix}
\tag{10}$$

$$\text{Value Input - Output Matrix} = \begin{bmatrix} 400 & 200 & 400 \\ 400 & 400 & 800 \\ 200 & 1000 & 400 \end{bmatrix}
\tag{11}$$

Now the internal demand vector in terms of currency instead of goods.

$$\begin{array}{c}
\text{From \ To} \\
\text{Agg} \\
\text{Manu} \\
\text{Labor}
\end{array}
\begin{array}{c}
\text{Agg} \\
\text{Manu} \\
\text{Labor}
\end{array}
\begin{array}{c}
\text{Manu} \\
\text{Labor}
\end{array}
\begin{array}{c}
\text{Internal Demand Vector} \\
= \\
= \\
=
\end{array}
\begin{array}{c}
1000 \\
1600 \\
1600
\end{array}
\tag{12}$$

So while in terms of the number of goods sold the agricultural and manufacturing look relatively similar, in terms of the amount of value the manufacturing sector is a good deal larger than the agricultural sector. In recent years the number of sectors that define the sectors in the economy has increased exponentially. As of 2006 more detailed tables had over one thousand different sectors making well over one million entries.

Suppose in addition to the internal demand, which models the flow of goods and services exchanges in between the industries, there is also a sector of the economy called does not produce anything but consumes good and services from all sectors. This sector is called the open sector. Let some vector f be called a final demand vector, and let it be equal to the total demand of the nonproductive sectors of the economy. Also lets define an vector x is the total amount produced. Then it stands that:

$$\begin{bmatrix} \textit{Amount} \\ \textit{Produced} \\ x \end{bmatrix} = \begin{bmatrix} \textit{Internal} \\ \textit{Demand} \end{bmatrix} + \begin{bmatrix} \textit{Final} \\ \textit{Demand} \\ f \end{bmatrix} \quad (13)$$

For the example suppose a final demand vector of :

$$\textit{FinalDemand} = \begin{bmatrix} 600 \\ 800 \\ 400 \end{bmatrix} \quad (14)$$

Then the totals for this economy would be

$$\begin{bmatrix} \textit{Amount} \\ \textit{Produced} \\ x \end{bmatrix} = \begin{bmatrix} 1000 \\ 1600 \\ 1600 \end{bmatrix} + \begin{bmatrix} 600 \\ 800 \\ 400 \end{bmatrix} \quad (15)$$

$$= \begin{bmatrix} 1600 \\ 2400 \\ 2000 \end{bmatrix} \quad (16)$$

More generally these amounts are x_1, x_2, \dots, x_n . Where the x_j corresponds to industry S_j .

3. Technical Coefficients and Structural and Consumption Matrices

A structural matrix is one composed of technical coefficients. Technical coefficients demonstrate the amount of a good i per unit of good j produced that is consumed. A general example of a technical coefficient c_{ij} is computed by dividing the a_{ij} entry in the input-output matrix (fig 1) by the total of the good demanded, expressed as x_j (fig 16). This is also known as the input coefficient.

$$c_{ij} = \frac{a_{ij}}{x_j} \quad (17)$$

So the structural matrix of the input-output matrix of (fig 2) follows:

$$\begin{array}{r}
\text{SimplifiedStructuralMatrix}A = \\
\text{From \ To} \quad \text{Agg} \quad \text{Manu} \quad \text{Labor} \\
\text{Agg} \quad \left(\begin{array}{ccc} 40/160 & 20/120 & 40/400 \end{array} \right) \\
\text{Manu} \quad \left(\begin{array}{ccc} 20/160 & 20/120 & 40/400 \end{array} \right) \\
\text{Labor} \quad \left(\begin{array}{ccc} 40/160 & 200/120 & 80/400 \end{array} \right)
\end{array}
\tag{18}$$

$$\begin{array}{r}
\text{SimplifiedStructuralMatrix}A = \\
\text{From \ To} \quad \text{Agg} \quad \text{Manu} \quad \text{Labor} \\
\text{Agg} \quad \left(\begin{array}{ccc} .25 & .167 & .1 \end{array} \right) \\
\text{Manu} \quad \left(\begin{array}{ccc} .125 & .167 & .1 \end{array} \right) \\
\text{Labor} \quad \left(\begin{array}{ccc} .25 & 1.667 & .2 \end{array} \right)
\end{array}
\tag{19}$$

In practice most structural matrices are constructed in terms of dollars rather than of goods. However it is important to remember the basis for this analysis is founded on the flow of goods through an economy, not the flow of money. When the structural matrix is composed of technical coefficients derived from the monetary value ratio rather than the ratio of goods the matrix is often called a consumption matrix. An example of the consumption matrix is below.

$$\begin{array}{r}
\text{SimplifiedConsumptionMatrix}A = \\
\text{From \ To} \quad \text{Agg} \quad \text{Manu} \quad \text{Labor} \\
\text{Agg} \quad \left(\begin{array}{ccc} .25 & .083 & .2 \end{array} \right) \\
\text{Manu} \quad \left(\begin{array}{ccc} .25 & .167 & .4 \end{array} \right) \\
\text{Labor} \quad \left(\begin{array}{ccc} .125 & .4167 & .2 \end{array} \right)
\end{array}
\tag{20}$$

The sum of the columns of the consumption matrix can display certain traits of the economy in which the system is operating. If the sum of the columns is less than one this means that one unit of the good takes costs less to produce and sell than the value it sells for. Thus the company making the product is profiting. If the sum of the columns equals one that means the final product takes exactly the amount of currency it takes to produce that the product sells for. Therefore the company is breaking even. If the sum of the columns is greater than one this means that it costs more to produce one unit of the good than it does to make. Thus the producing company is incurring a loss.

4. Final Demand, Amount Produced, and Unit Consumption Vectors.

Going back to figure 13.

$$\begin{bmatrix} \text{Amount} \\ \text{Produced} \\ x \end{bmatrix} = \begin{bmatrix} \text{Internal} \\ \text{Demand} \end{bmatrix} + \begin{bmatrix} \text{Final} \\ \text{Demand} \\ f \end{bmatrix} \quad (21)$$

As was noted previously, each column in A represents the the amount of goods consumed by a sector. Let each column in the consumption matrix be called a unit consumption vector. Thus unit consumption vector also represents the the amount of inputs needed to produce on unit of a good. Denote these vectors as $c_1, c_2, \dots c_n$.

Simplified Consumption Matrix $A =$

From \ To	<i>Agg</i>	<i>Manu</i>	<i>Labor</i>	
<i>Agg</i>	(.25	.083	.2
<i>Manu</i>		.25	.167	.4
<i>Labor</i>		.125	.4167	.2
		↑	↑	↑

(22)

$$c_1 \quad c_2 \quad c_3 \quad (23)$$

The market changes and this year the Manufacturing Sector decides to produce 2600 dollars worth of goods, lets call this x . Then using the manufacturing unit consumption vector we get the amount of inputs needed.

$$x_2 c_2 = 2400 \begin{bmatrix} .083 \\ .167 \\ .4167 \end{bmatrix} = \begin{bmatrix} 200 \\ 400 \\ 1000 \end{bmatrix} \quad (24)$$

So this tells us from the consumption matrix how many inputs are needed for the manufacturing sector of this economy to produce 2400 dollars worth of goods. This matches above.

let x , as mentioned above be the total amount of all goods demands. Therefore if each sector of the economy, $S_1 \dots S_n$, decide to produce $x_1 \dots x_n$ amount of good than the internal demand is given by.

$$\text{Internal Demand} = x_1 c_1 + x_2 c_2 + \dots + x_n c_n \quad (25)$$

Where the consumption matrix = $C = [c_1, c_2, \dots, c_n]$

These equations yield Leontiff's Input-Output Model. Note that this is the same value as the Value Demand Vector as before, however we will rarely use the previous notation.

5. The Leontiff Input-Output Model, or Production Equation

$$\begin{bmatrix} \textit{Amount} \\ \textit{Produced} \\ x \end{bmatrix} = [Cx] + \begin{bmatrix} \textit{Final} \\ \textit{Demand} \\ f \end{bmatrix} \quad (26)$$

Therefore

$$x = Cx + f \quad (27)$$

Using the algebraic properties of R^n

$$Ix = Cx + f \quad (28)$$

$$Ix - Cx = f \quad (29)$$

$$(I - C)x = f \quad (30)$$

A consumption matrix is considered economically feasible if the sum of any column of C is less than 1. Then the next theorem follows:

Let C be the consumption matrix for an economy, and let f the final demand. If C and f have nonnegative entries, and if C is economically feasible, then the inverse of the matrix $(I-C)$ exists and the production vector:

$$x = (I - C)^{-1}f \quad (31)$$

has nonnegative entries and is the unique solution of

$$x = Cx + f \quad (32)$$

To see why this is true, let the final demand presented to companies at the beginning of the year be f . Thus the industries will respond by setting their output, x , at:

$$x = f \quad (33)$$

This will exactly meet final demand.

To produce the final demand, each industry places out orders for raw materials, labor, or other inputs. This creates the internal demand. The additional demand is Cf , and thus will need additional inputs given by

$$C(Cf) = C^2f \quad (34)$$

This creates a second round of internal demand, so industries buy more, which creates more internal demand:

$$C(C^2f) = C^3f \quad (35)$$

And so on. While it would seem that this process would continue indefinitely in reel life C^m approaches the 0 matrix quickly. We can model this process by the equation equation:

$$x = f + Cf + C^2f + C^2f + C^2f + \dots + C^{m-1}f + C^m f \quad (36)$$

Thus

$$x = (I + CI + C^2I + C^2I + C^2I + \dots + C^{m-1}I + C^mI)f \quad (37)$$

The we use the algebraic identity:

$$(I - C)(I + C + C^2 + C^3 + \dots + C^m) = I - C^{m+1} \quad (38)$$

Since C^m approaches the zero matrix we know $(I - C^{m+1})$ approaches I as m get arbitrarily large. Therefore

$$(I - C)^{-1} = (I + C + C^2 + C^3 + \dots + C^m) \quad (39)$$

Therefore

$$x = (I - C)^{-1}f \quad (40)$$

6. The Importance $(I - C)^{-1}$ $(I - C)^{-1}$ is very important because its entries provide information on how the production, x, will change when the final demand, f, changes. For example take our simplified economy from the previous example. We have the consumption matrix give by figure (20). Therefore (I-C) is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .25 & .083 & .2 \\ .25 & .167 & .4 \\ .125 & .4167 & .2 \end{bmatrix} = \begin{bmatrix} .75 & -.083 & -.2 \\ -.25 & .833 & -.4 \\ -.125 & -.4167 & .8 \end{bmatrix} \quad (41)$$

Thus $(I - C)^{-1}$ is:

$$\begin{bmatrix} 1.6 & .479 & .6396 \\ .8 & 1.841 & 1.121 \\ .667 & 1.03 & 1.9335 \end{bmatrix} \quad (42)$$

Suppose the final demand from the open sector increases is the same as before. This gives us a demand of:

$$FinalDemand = \begin{bmatrix} 800 \\ 1000 \\ 600 \end{bmatrix} \quad (43)$$

So the final production, x, is for this economy is:

$$x = (I - C)^{-1}f \quad (44)$$

$$x = \begin{bmatrix} 1.6 & .479 & .6396 \\ .8 & 1.841 & 1.121 \\ .667 & 1.03 & 1.9335 \end{bmatrix} \begin{bmatrix} 600 \\ 800 \\ 400 \end{bmatrix} \quad (45)$$

$$x = \begin{bmatrix} 1600 \\ 2400 \\ 2000. \end{bmatrix} \quad (46)$$

So to satisfy both final and internal demand for this economy the agricultural sector would have to produce 1600 units of currency worth of goods. The manufacturing sector would have to produce 2400 unit of currency worth of goods. Finally the labor sector would have to produce 2000 units of currency worth of goods.

From this point is it very easy to get back to internal demand. We use the equation given in figure 21:

$$\begin{bmatrix} \textit{Amount} \\ \textit{Produced} \\ x \end{bmatrix} = \begin{bmatrix} \textit{Internal} \\ \textit{Demand} \end{bmatrix} + \begin{bmatrix} \textit{Final} \\ \textit{Demand} \\ f \end{bmatrix} \quad (47)$$

Therefore:

$$\begin{bmatrix} \textit{Internal} \\ \textit{Demand} \end{bmatrix} = \begin{bmatrix} \textit{Amount} \\ \textit{Produced} \\ x \end{bmatrix} - \begin{bmatrix} \textit{Final} \\ \textit{Demand} \\ f \end{bmatrix} \quad (48)$$

$$\begin{bmatrix} \textit{Internal} \\ \textit{Demand} \end{bmatrix} = \begin{bmatrix} 1600 \\ 2400 \\ 2000 \end{bmatrix} - \begin{bmatrix} 600 \\ 800 \\ 400 \end{bmatrix} \quad (49)$$

$$\begin{bmatrix} \textit{Internal} \\ \textit{Demand} \end{bmatrix} = \begin{bmatrix} 1000 \\ 1600 \\ 1600 \end{bmatrix} \quad (50)$$

This shows the $(I - C)^{-1}$ provides us with an unique solution to the problem of finding the total amount produced if the consumption matrix and final demand of an economy is already given.

7. In Closing

There are other ways then those described above to reach certain values. For example the consumption matrix can be measured sector by sector and assembled instead of using technical coefficients. Similarly the value that economists are interested in is not always final amount of production. This basic input-output analysis however is a very powerful tool. As shown above it can predict what happens to an economy when final demand changes. By changing the consumption matrix this can represent what happens to an economy when the relative cost in terms of other goods (a change in one or more entries in internal demand) of producing one good can change both internal and final demand.

Also this can model what happens when companies decide to increase or decrease the price of a good in terms of currency. Another useful application of this set of equations is figuring out the GDP of an economy. The total demand, x , of all the sectors of an economy of a nation is equal to the GDP. Imports and export are handled by changing the final demand vector. In Closing this application of very basic concepts in linear algebra creates an incredibly powerful tool for the analysis of economies. Wassily Leontiff won the Nobel Prize in 1973 for his work in modernizing economics using these concepts.

Work Cited

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