(10)

(1) Determine if the following functions are even, odd, or neither.

(a) $x^5 + 5$ Neither

(b) $x^2 - 7$ even

(c) $x^{11} - x^5$ odd

(d) $\cos(x)$ even

(e) $\sin(x)$ odd

(f) $\tan^2(3x)$ even

Remember: $f(x)$ is even if $f(x) = f(-x)$ and odd if $f(-x) = -f(x)$.

(2) Calculate the following integrals (please note whether they are definite or indefinite). For definite integrals you may use the method of the second part of the fundamental theorem of calculus, or you may use your knowledge of the geometry/symmetry of the integrand.

(5)

(a) $\int_{-10}^{10} x^{11} - x^7 \, dx$

odd integrand; symmetric interval $[-10, 10]$.

$\therefore$ integral $= 0$.

(5)

(b) $\int_{-1}^{1} \sin(\pi x) \, dx$

odd integrand, symmetric interval $[-1, 1]$.

$\therefore$ integral $= 0$.

(c) $\int_{-1}^{1} \cos(\pi x)\, dx$

$$= \left[\frac{1}{\pi}\sin(\pi x)\right]_{-1}^{1} = \frac{1}{\pi}\left(\sin\pi - \sin(-\pi)\right)$$

$$= \frac{1}{\pi}(0 - 0) = 0.$$

(d) $\int 2x^7 - 3x^8\, dx$

$$\frac{1}{4}x^8 - \frac{1}{3}x^9 + C$$

(e) $\int_a^b \sqrt{2x}\, dx$

$$= \sqrt{2}\int_a^b x^{1/2}\, dx = \sqrt{2}\left[x^{3/2} \cdot \frac{2}{3}\right]_a^b$$

$$= \frac{2\sqrt{2}}{3}\left(b^{3/2} - a^{3/2}\right).$$

(15)

(f) $\int \tan^5(x)\sec^2(x)\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$\int u^5\,du$

$\frac{1}{6}u^6 + C$

$\boxed{\frac{1}{6}\tan^6(x) + C}$.

(g) $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$

$u = x^2$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}du$

x	u
0	0
$\sqrt{\pi}$	π

$= \int_0^{\pi} \sin(u)\cdot\frac{1}{2}\,du$

$= \frac{1}{2}\left[-\cos u\right]_0^{\pi}$

$= \frac{1}{2}\left(-(-1) - (-1)\right)$

$= \boxed{1.}$

(h) $\int_0^1 x^{\frac{1}{9}}\,dx$

$= \left[\frac{9}{10}x^{10/9}\right]_0^1$

$= \boxed{\frac{9}{10}.}$

15

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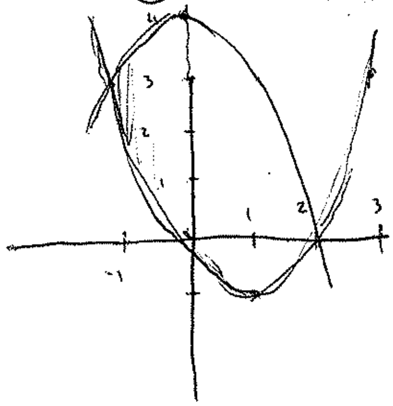
(3) Find the area of the region enclosed between the two parabolas $y = (x-1)^2 - 1$ and $y = 4 - x^2$. Also make a sketch of the region and label the intersection points of the two curves.



$$\int_{-1}^{2} 4 - x^2 - ((x-1)^2 - 1)\, dx$$

$$= \int_{-1}^{2} 4 - x^2 + 1 - x^2 - 1 + 2x \quad dx$$

$$= \int_{-1}^{2} 4 + 2x - 2x^2 = \left[4x + x^2 - \frac{2}{3}x^3 \right]_{-1}^{2} =$$

$$= 8 + 4 - \frac{16}{3} - \left(-4 + 1 + \frac{2}{3}\right) = 9.$$

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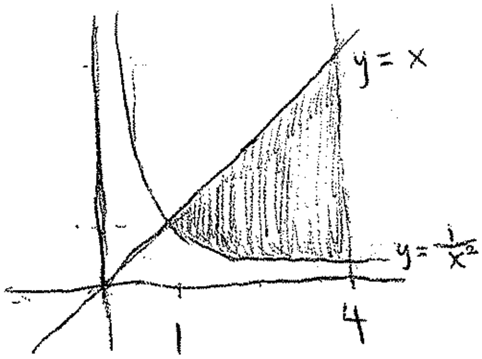
(4) Find the area of the shaded region.



$$\int_{1}^{4} x - \frac{1}{x^2}\, dx = \int_{1}^{4} x - x^{-2}\, dx =$$

$$= \left[\frac{1}{2}x^2 + x^{-1} \right]_{1}^{4}$$

$$= 8 + \frac{1}{4} - \left(\frac{1}{2} + 1\right)$$

$$= 6.75.$$

(10)

(5) Initial value problem: an object moves up and down, experiencing a variable acceleration of $a(t) = 3\cos(t)$ meters per seconds squared.

(a) If the initial velocity is $v(0) = 0$ meters per second and the initial position is $s(0) = -3$ meters, find a formula for $s(t)$, the position at any further time t.

$v(t) - v(0) = \int_0^t a(\tau)\,d\tau = [3\sin\tau]_0^t = 3\sin t$

$v(t) = 3\sin t$.

$s(t) - s(0) = \int_0^t v(\tau)\,d\tau = [-3\cos\tau]_0^t = -3\cos t + 3$.

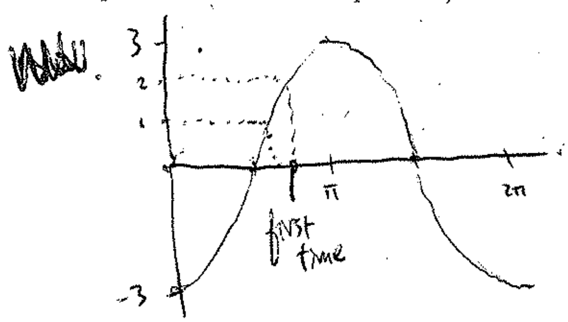
$s(t) = -3\cos t$

(b) Does the object at any time achieve a *displacement* of 5 meters? If it does, state the first time. (Recall that the displacement over a time interval is the integral of the velocity, which is also the difference between the final position and the initial position.)

$-3\cos t = 2$

$\cos t = \frac{-2}{3}$

$t = \cos^{-1}\left(\frac{-2}{3}\right)$



(c) At what time does the *distance traveled* by the object equal 5 meters? (Recall that the distance traveled over a time interval is the integral of the absolute value of the velocity.)

$t = \cos^{-1}\left(\frac{-2}{3}\right)$

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(6) Rewrite the following sums with "sigma" notation:

(a) $1 + 3 + 5 + 7 + \cdots + 51$

$$\sum_{k=1}^{26} 2k-1$$

OR

$$\sum_{k=0}^{25} 2k+1$$

(b) $8 + 10 + 12 + 14 + 16 + 18 + \cdots + 50$

$$\sum_{k=4}^{25} 2k$$

(c) $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100$

$$\sum_{k=1}^{10} k^2$$

10

(7) Evaluate the following sums:

(a) $\sum_{k=1}^{50} k$

$$\frac{(50)(51)}{2} = 25 \cdot 51 = 1275$$

(b) $\sum_{k=50}^{100} k$

$$\frac{(100)(101)}{2} - \frac{(50)(49)}{2} = 25(202 - 49) = (25)(153)$$

$$= 3825$$

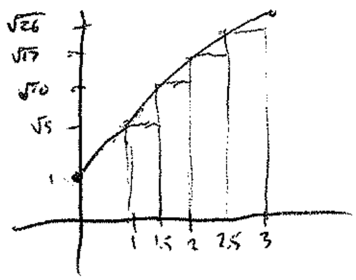
(c) $\sum_{k=0}^{4} \sin \frac{k\pi}{6}$

$$= \sin 0 + \sin\frac{\pi}{6} + \sin\frac{2\pi}{6} + \sin\frac{3\pi}{6} + \sin\frac{4\pi}{6}$$

$$= 0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} = \frac{3}{2} + \sqrt{3}.$$

10

(8) Use a left Riemann sum with 4 subintervals to approximate the area under the curve $y = \sqrt{1+4x^2}$ on the interval $[1, 3]$. Draw a sketch of the graph and include the rectangles whose areas you are summing. Show your calculations clearly for full credit.



$$\frac{1}{2}\cdot\left(\sqrt{5}+\sqrt{10}+\sqrt{17}+\sqrt{26}\right) = \approx 7.31$$

(9) (Bonus)

(a) Fill in the missing boundaries and coefficient in the right hand integral of this generalized change of variables (i.e. U-substitution) formula:

$$\int_a^b f(c+dg(x))g'(x)\,dx = \int_{c+d\cdot g(a)}^{c+d\cdot g(b)} \frac{1}{d} f(u)\,du. \Big\} \int_{c+d\cdot g(a)}^{c+d\cdot g(b)} \frac{1}{d} f(u)\,du .$$

(b) Letting $F(u)$ be any antiderivative for $f(u)$, use the second part of the fundamental theorem of calculus to prove the formula. (Hint: apply F.T.C.II to left and right hand side separately, and show that the results coincide.)

If $F'(u) = f(u)$, then by the chain rule $\frac{d}{dx}\left(F(c+d\cdot g(x))\right) = f(c+d\cdot g(x))\cdot d\cdot g'(x)$

so $\frac{1}{d}\cdot F(c+d\cdot g(x))$ is an antiderivative of $f(c+d\cdot g(x))\cdot g'(x)$, so by FTC pt II

$$\int_a^b f(c+d\cdot g(x))\cdot g'(x) = \frac{1}{d}\left(F(c+d\cdot g(b)) - F(c+d\cdot g(a))\right).$$

Again by FTC pt II $\displaystyle\int_{c+d\cdot g(a)}^{c+d\cdot g(b)} \frac{1}{d}\cdot f(u)\,du = \frac{1}{d}\int_{c+dg(a)}^{c+dg(b)} f(u)\,du = \frac{1}{d}\left(F(c+dg(b)) - F(c+dg(a))\right)$

(just because $F(u)$ is an antideriv. of $f(u)$)