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(1) The equation $x^4 - x^2y^2 + y^4 = 16$ describes some kind of graph in the xy plane. Use implicit differentiation to find $\frac{dy}{dx}$ (a function of both x and y) and to find the equation of the line tangent to the graph at the point $(x = 2, y = -2)$.

$$4x^3 - \left(x^2 \cdot 2y \cdot \frac{dy}{dx} + 2x \cdot y^2\right) + 4y^3 \frac{dy}{dx} = 0$$

$$2x^3 - x^2y\frac{dy}{dx} - xy^2 + 2y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{xy^2 - 2x^3}{2y^3 - x^2y}$$

$$\left.\frac{dy}{dx}\right|_{(x=2,\, y=-2)} = \frac{8 - 16}{-16 + 8} = 1$$

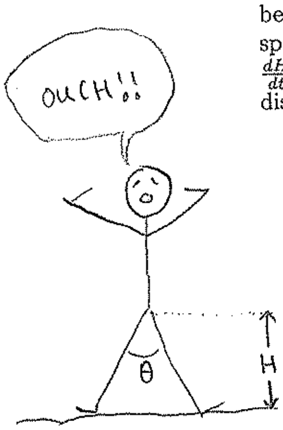
Equation of tangent:

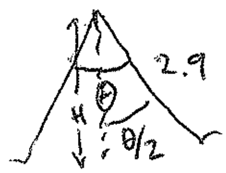
$$y + 2 = x - 2$$

$$y = x - 4$$

10

(2) Related Rates Problem. Walking across ice, you slip and do the splits. Suppose each of your legs is 2.9 feet tall. Let H represent the height of your crotch above the ground and let θ represent the angle of separation between your legs. As θ increases from 0^o (legs together) up to 180^o (full split) and H decreases from 2.9 feet to 0 feet, find an equation relating $\frac{dH}{dt}$ to $\frac{d\theta}{dt}$. Assume that your legs remain straight and are extended equal distances on either side of you as you fall.





$$\cos\left(\frac{\theta}{2}\right) = \frac{H}{2.9}$$

$$-\sin\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\cdot\frac{d\theta}{dt} = \frac{1}{2.9}\,\frac{dH}{dt}$$

$$\frac{d\theta}{dt} = \frac{-2}{2.9\sin\left(\frac{\theta}{2}\right)}\cdot\frac{dH}{dt}.$$

(3) Consider the polynomial $f(x) = x^3 - x^2 - x + 1$.

5

(a) Find $f'(x)$ and list all the critical points of f.

$f'(x) = 3x^2 - 2x - 1$

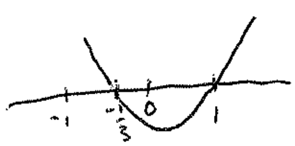
$= 0$ if $x = \dfrac{2 \pm \sqrt{4+12}}{6} = \dfrac{1}{3} \pm \dfrac{2}{3} = 1$ or $-\dfrac{1}{3}$.

Critical Points $x = 1$, $x = -\dfrac{1}{3}$.

5

(b) Use the "first derivative test" to decide on which intervals f is increasing and on which intervals f is decreasing. State your answer in interval notation.

Graph of $y = f'(x)$



$f'(x) > 0$ for $x < -\frac{1}{3}$, so $f(x)$ is increasing

$f'(x) < 0$ for $-\frac{1}{3} < x < 1$, so $f(x)$ is decreasing

$f'(x) > 0$ for $x > 1$, so $f(x)$ is increasing.

f is increasing on $(-\infty, -\frac{1}{3})$ and $(1, \infty)$; f is decreasing on $(-\frac{1}{3}, 1)$.

X-coords only

5

(c) State the coordinates of each local maximum and each local minimum of f.

Local max at $x = -\dfrac{1}{3}$,

Local min at $x = 1$.

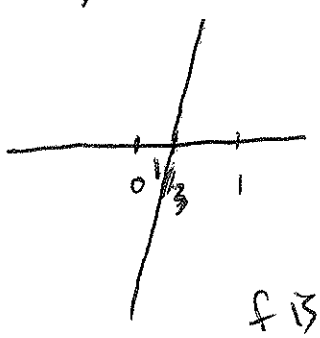
5 (d) Find $f''(x)$ and all of its zeros.

$$f''(x) = 6x - 2$$

$$= 0 \text{ if } \quad 6x = 2 \quad \text{i.e. } x = \frac{1}{3}.$$

5 (e) Use the "second derivative test" to decide on which intervals f is concave up and on which intervals f is concave down.

Sketch of $y = f''(x)$



$f''(x) > 0$ for $x > \frac{1}{3}$, so concave up

$f''(x) < 0$ ~~if~~ for $x < \frac{1}{3}$, so concave down

f is concave down on $(-\infty, \frac{1}{3})$;

f is concave up on $(\frac{1}{3}, \infty)$.

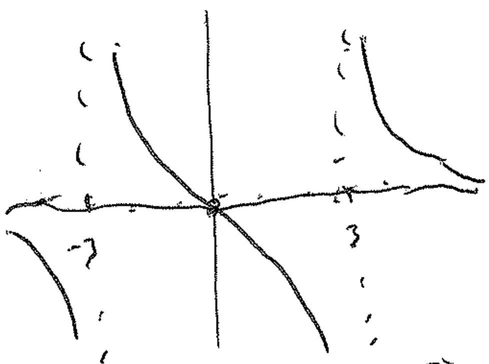
5 (f) State the coordinates of each inflection point of f.

$x = \frac{1}{3}$ is an inflection point.

(4) Let $f(x) = \dfrac{x}{x^2 - 9}$.

5 (a) Find the critical points of f.

"$x = \pm 3$" is ok, because $f'(x)$ undefined there.



also – "No critical points" is ok because $f(\pm 3)$ is also not defined

4 (b) On which intervals is f increasing? On which intervals is f decreasing?

$$f'(x) = \frac{x^2 - 9 - 2x^2}{(x^2-9)^2} = \frac{-x^2 - 9}{(x^2-9)^2}$$

$= 0$ if $-x^2 - 9 = 0$, which never happens.

$f'(x)$ is < 0 where it is defined (all x except ± 3)

Therefore $f(x)$ is decreasing everywhere, on $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

3 (c) Does f have any local extrema? (Yes or no). No

3 (d) Does f have any absolute extrema? (Yes or no). No

3

(5) True or False?
If $f'(x_0) = 0$, then $f(x)$ must have either a local max or a local min at x_0.

False, consider $f(x) = x^3$ with $x_0 = 0$.

3

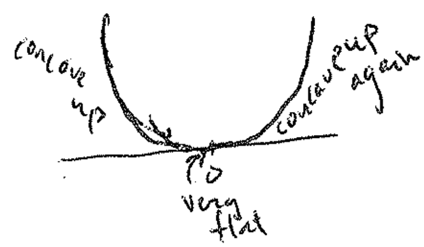
(6) Does the graph $y = x^3$ have an inflection point at $x = 0$?

Yes: $y'' = 6x = \begin{cases} = 0 & \text{if } x = 0 \\ < 0 & \text{if } x < 0 \\ > 0 & \text{if } x > 0 \end{cases}$

3

(7) Does the graph $y = x^4$ have an inflection point at $x = 0$?

No: $y'' = 12x^2 = \begin{cases} = 0 & \text{if } x = 0 \\ > 0 & \text{if } x < 0 \\ > 0 & \text{if } x > 0 \end{cases}$



The concavity doesn't change at $x = 0$

3

(8) True or False? If a function is differentiable on a closed interval then it must have an absolute maximum on that interval.

Yes - differentiable $\Rightarrow$ continuous.

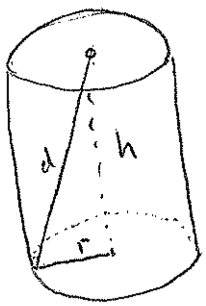
3

(9) True or False? The absolute maximum of a differentiable function on a closed interval must occur at a critical point.

False, let $f(x) = x$, interval $[0, 1]$.

10

(10) Optimization problem. In a cylinder let d represent the length of the diagonal line from the top center to the bottom perimeter, as pictured. Of all the combinations of radius and height that yield $d = 10$, which yields the greatest volume?



$$100 = d^2 = h^2 + r^2$$

$$V = \pi r^2 \cdot h$$

Solve for r^2 and substitute:

$$r^2 = 100 - h^2$$

$$V = \pi(100 - h^2) \cdot h$$
$$= \pi(100h - h^3)$$

$$\frac{dV}{dh} = \pi(100 - 3h^2) = \begin{cases} = 0 & \text{if } h = \pm\frac{10}{\sqrt{3}} \\ > 0 & \text{if } h < \frac{10}{\sqrt{3}} \\ < 0 & \text{if } h > 10/\sqrt{3} \end{cases}$$

So $V(h)$ increases for $h < \frac{10}{\sqrt{3}}$ and decreases for $h > \frac{10}{\sqrt{3}}$, so $h = \frac{10}{\sqrt{3}}$ gives a local max, which is an absolute max because there are no other critical points.

Answer: $h = \frac{10}{\sqrt{3}}$, $r = \sqrt{100 - \frac{100}{3}} = 10\sqrt{\frac{2}{3}}$

10

Typo

(11) Let $f(x) = 2\cos(3x) + 3\sin(2x)$. Find the linearization (i.e. best linear approximation) of $f(x)$ at the point $x = \frac{\pi}{3}$. Then use it to approximate the numerical value of $2\cos(1.01\pi) + 3\cancel{\cos}\left(\frac{2.02\pi}{3}\right)$, which is $f\left(\frac{1.01\pi}{3}\right)$.

sin

- $f(\pi/3) =$

$= 2\cos(\pi) + 3\sin\left(\frac{2\pi}{3}\right)$

$= -2 + \frac{3\sqrt{3}}{2}$.

- $f'(x) = -6\sin(3x) + 6\cos(2x)$

- $f'\left(\frac{\pi}{3}\right) = -6\sin(\pi) + 6\cos\left(\frac{2\pi}{3}\right)$

$= 6\left(\frac{-1}{2}\right) = -3$.

- Thus $L(x) = \frac{3\sqrt{3}}{2} - 2 + (-3)\left(x - \frac{\pi}{3}\right)$.

$f\left(\frac{1.01\pi}{3}\right) \approx L\left(\frac{1.01\pi}{3}\right) = \frac{3\sqrt{3}}{2} - 2 - 3\left(\frac{.01\pi}{3}\right)$

$= \frac{3\sqrt{3}}{2} - 2 - 0.01\pi$.

This is true because $\frac{1.01\pi}{3}$ is close to $\frac{\pi}{3}$, where the linearization is centered