Important Theorems & Proofs for Cal I

Rolle's Theorem states that if a function $f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the interior (a,b) and $f(a) = f(b)$ then there is a point c in the interval (a,b) at which $f'(c) = 0$.

The proof is that

- $f(x)$ attains both an absolute max and an absolute min on $[a,b]$, by the "extreme Values theorem".
- if neither the absolute max nor the absolute min occur in the interior (a,b) then $f(x)$ is constant and $f'(x) = 0$ for every x in (a,b).
- If one of the absolute max or absolute min occur in the interior then the derivative at that point is 0 by local extremum theorem.

The **Mean Value Theorem** states more generally that if a function $f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the interior (a,b) then there is a point c in the interior (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, i.e. the average rate of change of $f(x)$ on the interval $[a,b]$ is the instantaneous rate of change of $f(x)$ at the point c.

The proof is to consider the function $d(x) = f(x) - \left(f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)\right)$, notice $d(a) = d(b) = 0$ and apply Rolle's theorem to $d(x)$. Then notice

$$0 = d'(c) = f'(c) - \left(\frac{f(b)-f(a)}{b-a}\right).$$

An important corollary of the mean value theorem is that if a function $f(x)$ is differentiable on an open interval (a,b) and moreover $f'(x)=0$ for every x in (a,b), then $f(x)$ is a constant function, i.e. $f(x)=y_0$ for some constant y_0, for every x in (a,b).

The proof is simply that if there were points x_1 and x_2 with $f(x_1)\neq f(x_2)$ then there would have to be a point c between x_1 and x_2 for which $f'(c)=\dfrac{f(x_2)-f(x_1)}{x_2-x_1}\neq 0$.

This corollary can be rephrased as follows: if $f(x)$ and $g(x)$ are both differentiable on an open interval (a,b) and moreover $f'(x)=g'(x)$ for every x in (a,b), then $f(x)-g(x)$ is a constant function. The proof is that $(f(x)-g(x))' = f'(x)-g'(x) = 0$ for every x in (a,b).

This corollary allows a thorough solution of initial value problems:

A classical initial value problem is to solve for an unknown position function $s(t)$ throughout an open time interval (a,b), given only the velocity $v(t)$ and an initial position $s(t_0)$ at some initial time t_0 in (a,b). The only work is to find an antiderivative for $v(t)$, because once we've found a function $g(t)$ such that $g'(t) = v(t)$ we know by the corollary to the mean value theorem that $\left(\begin{matrix}\text{because}\\ s'(t)=g'(t)\end{matrix}\right)$ $s(t) - g(t)$ is constant for all t in (a,b) and therefore $s(t) - g(t) = s(t_0) - g(t_0)$

i.e. $s(t) = g(t) + s(t_0) - g(t_0)$ for every t in (a,b), and an explicit formula for $s(t)$ has been found.

The Fundamental Theorem of Calculus Part 1, establishes that if a function $f(x)$ is continuous on an interval $[a,b]$ then the area function $A(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

(Here capital "A" is intended to stand for "area under the graph of $f(x)$".
It is a happy coincidence that it turns out to also be an Antiderivative for $f(x)$.)

Part 2 of the Fundamental Theorem of Calculus is another initial value problem, which shows that if any formula F(x) is found for an antiderivative of f(x), then in fact (using initial value at a")

$$A(x) = F(x) + \underbrace{A(a)}_{0} - F(a) = F(x) - F(a).$$

In symbols, $\int_a^x f(t)\,dt = F(x) - F(a)$, where $F(x)$ is any antiderivative of f(x).