# A NEW CHARACTERIZATION OF GIBBS MEASURES FOR UNBOUNDED LOCAL ENERGY FUNCTIONS ON $\mathbb{N}^{\mathbb{Z}^d}$.

STEPHEN MUIR

Abstract. A definition of Gibbs measure is introduced for a class of unbounded, regular functions on $\mathbb{N}^{\mathbb{Z}^d}$. This extends the definition introduced by Capocaccia ([3]) to a noncompact setting, using the "ergodic" approach of associating the measures to a function, rather than a specification or an interaction potential. The definition is checked to be compatible with Georgii's ([5]) probabilistic definition of Gibbs measure for a specification and, in the cases where the function is representable as the local energy of an absolutely summable interaction potential, with the standard DLR definition of Gibbs measure for an interaction potential.

## 1. Relation to the Literature.

The thermodynamic formalism of Gibbs measures in various settings has been well developed since the 1960's. Standard references on the physical side of the literature include Ruelle's [13] and it's precursor [12]; Simon's ([15]) and Israel's ([7]) functional analytic approaches; Georgii's probabilistic monograph [5]. Any of [15], the introduction to [7], or Section 2 of the long paper [16] provide thorough physical interpretations. The core of all these studies is a multidimensional integer lattice $E^{\mathbb{Z}^d}$ with an interaction potential. Gibbs measures are defined by the DLR equations (named for Dobrushin, Lanford, Ruelle), which specify conditional measure within a finite sublattice given a fixed tail configuration in terms of exponentiated interaction. Only [5] discusses the more general probabilistic theory of Gibbs measures for a product measure specification (or just "specification", though the term should not be confused with the concept of "dynamical systems with specification" as introduced in [2] and studied in [6], e.g.). Georgii's book [5] is also noteworthy for being the only among these standard references to deal with possibly noncompact alphabets such as our $E = \mathbb{N}$. For a discussion of existence of Gibbs measures, the reader may consult his Chapter 4.

In the present paper we diverge from all the above references in that we do not use an interaction potential or an explicit product measure specification in our definition, but rather only a *local energy function*. This approach is modeled after Chapter 5 of Keller's ergodic theory book [8], and should provide a more familiar setting for researchers in dynamical systems. Dynamicists rarely have occasion to study interaction potentials, which truly belong to mathematical physics, but frequently encounter 1D shift spaces (symbolic dynamics) with potential functions (Lyapunov exponents).

---

*Date*: December 17, 2010.

*Key words and phrases.* Gibbs measure, noncompact lattice model.

We show our definitions to be fully compatible with [5] in Observation 2.11, where we prove that all exp-summable, regular local energy functions define quasilocal specifications which are modifications of the infinite a priori counting measure on $\mathbb{N}$. In Section 3 we discuss compatibility with the most common "physical" setting for Gibbs measures, where the alphabet is a finite measure space $E, \mathcal{E}, \nu$ and the configuration space $E^{\mathbb{Z}^d}$ is equipped with an interaction potential $\Phi$ with convergent Hamiltonian series and finite partition functions.

The dynamic/ergodic approach of defining Gibbs measures for a function rather than an interaction or a specification has been extended to infinite alphabet (hence noncompact) one dimensional shifts in [9] and [14] and subsequent work of the same authors. As has been mentioned, it has also been presented in great detail ([8]) for higher dimensional shifts with finite alphabets. For existence of Gibbs measures we refer the reader to Our goal in the present paper, then, is to fill in the missing intersection of higher dimensional shifts with infinite alphabets. We note that we have also done the core theory of pressure, variational principle, and equilibrium states in a self contained way for the $\mathbb{N}^{\mathbb{Z}^d}$ setting, but in the interest of space we only submit the results relevant to characterization of Gibbs measures.

## 2. Terminology and Preliminary Estimates

### 2.1. Configuration Space $\mathbb{N}^{\mathbb{Z}^d}$ and Shift Action $T$.

Treat the lattice itself, $\mathbb{Z}^d$, with the norm

$$|\lambda| = \max\{|\lambda_j| : 1 \leq j \leq d\}.$$

Denote the rectangular open *box* about the origin of radius $n$ by

$$\Lambda_n \ \equiv \ \left\{\lambda \in \mathbb{Z}^d : |\lambda| < n\right\}.$$

As a topological product, the basic open sets in $\mathbb{N}^{\mathbb{Z}^d}$ are the *cylinder sets*

$$[\omega] \equiv \pi_\Lambda^{-1}(\omega) = \left\{x \in \mathbb{N}^{\mathbb{Z}^d} \ : \ x|_\Lambda = \omega\right\},$$

where $\Lambda \subset \mathbb{Z}^d$ is a finite set and $\omega = (\omega_\lambda)_{\lambda\in\Lambda} \in \mathbb{N}^\Lambda$ is a *finite configuration* (or *finite word*). In particular we denote *origin cylinders* by

$$[a] = \pi_0^{-1}(a) = \left\{x \in \mathbb{N}^{\mathbb{Z}^d} : x_0 = a\right\}$$

for any $a \in \mathbb{N}$. Given any subsets $\mathbb{Z}^d \supseteq \Lambda \supseteq \Lambda'$, and a configuration $\omega \in \mathbb{N}^\Lambda$, let

$$\omega|_{\Lambda'} \ = \ (\omega_{\lambda'})_{\lambda'\in\Lambda'} \ \in \mathbb{N}^{\Lambda'}$$

denote the *restriction* or *projection* of the larger configuration to the smaller. (It is only for aesthetic reasons that we choose to write $x|_{\Lambda'}$ rather than $\pi_{\Lambda'}(x)$.) We often speak of partial configurations of the form $\omega \in \mathbb{N}^\Lambda$, $\Lambda \subset \mathbb{Z}^d$. When adjoining/concatenating words, take care to indicate to which subset $\Lambda \subset \mathbb{Z}^d$ each sub-word is associated. Thus for $\Lambda, \Lambda' \subset \mathbb{Z}^d$ with $\Lambda \cap \Lambda' = \emptyset$, $\omega \in \mathbb{N}^\Lambda$, and $\omega' \in \mathbb{N}^{\Lambda'}$, there is no ambiguity in writing the configuration $(\omega, \omega') \in \mathbb{N}^{\Lambda\cup\Lambda'}$.

The action of $\mathbb{Z}^d$ on $\mathbb{N}^{\mathbb{Z}^d}$ is called the *shift*, or the group of *shift maps*. For each $\lambda \in \mathbb{Z}^d$ define the map $T^\lambda : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{N}^{\mathbb{Z}^d}$ by the coordinate-wise rule

$$\left(T^\lambda x\right)_{\lambda'} = x_{\lambda+\lambda'} \ \forall \ \lambda' \in \mathbb{Z}^d.$$

Indeed they form a group action on $\mathbb{N}^{\mathbb{Z}^d}$.

The product topology on $\mathbb{N}^{\mathbb{Z}^d}$ is metrizable with

$$\begin{aligned} d(x,y) &\equiv 2^{-n(x,y)} \\ &\equiv 2^{-\max\{n\geqslant 1:\ x|\Lambda_n = y|_{\Lambda_n}\}} \\ &= 2^{-\min\{\,|\lambda|:\ x_\lambda \neq y_\lambda\}}. \end{aligned}$$

This means for every $n \geq 1$,

$$B(x, 2^{-n}) = [x|_{\Lambda_{n+1}}] \quad \text{and} \quad \overline{B}(x, 2^{-n}) = [x|_{\Lambda_n}].$$

2.2. **Exp-summability.** Let $\mathcal{P}_F(\mathbb{Z}^d) = \{\Lambda \subset \mathbb{Z}^d : |\Lambda| < \infty\}$. If $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$, we define for every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ the *partition function*

$$Z_\Lambda(f) = \sum_{\omega \in \mathbb{N}^{\Lambda_n}} \sup f|_{[\omega]}.$$

In particular let

$$Z_n(f) \equiv Z_{\Lambda_n}(f)$$

for every $n \geq 1$.

**Definition 2.1.** *We say a function $f$ is* exp-summable *if $Z_1(f) < \infty$.*

Exp-summability has important consequences for the pressure function, the variational principle, and the existence of Gibbs states, but we do not discuss these varied issues here. For now we only need note that if $Z_1(f) < \infty$, then for every $n \geq 1$

$$Z_n(f) \leq |\Lambda_n| Z_1(f) < \infty,$$

too.

2.3. **Smoothness Criteria for Real Valued Functions of $\mathbb{N}^{\mathbb{Z}^d}$.**

Denote by $C\left(\mathbb{N}^{\mathbb{Z}^d}\right)$ the vector space of continuous real valued functions on $\mathbb{N}^{\mathbb{Z}^d}$, by $UC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$ the vector subspace of uniformly continuous functions, and by $BC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$ the Banach space of bounded continuous real valued functions with the supremum norm

$$\|f\|_\infty = \sup\left\{|f(x)| \ :\ x \in \mathbb{N}^{\mathbb{Z}^d}\right\}.$$

As $\mathbb{N}^{\mathbb{Z}^d}$ is not compact, $UC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$ intersects but does not contain $BC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$. (Nor does $BC \supseteq UC$.) Let $BUC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$ be the Banach subspace $BC\left(\mathbb{N}^{\mathbb{Z}^d}\right) \cap UC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$ of $BC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$. For a function $f \in C(\mathbb{N}^{\mathbb{Z}^d})$ define the *oscillation*

$$\delta_n(f) \ \equiv\ \sup\left\{|f(x) - f(y)| \ :\ x|_{\Lambda_n} = y|_{\Lambda_n}\right\},$$

for any $n \geq 1$. Throughout this paper we will consider only functions with finite oscillation on origin cylinders,

$$\delta_1(f) \ = \sup\{|f(x) - f(y)| \ : \ x_0 = y_0\} < \infty.$$

We note that Sarig and collaborators have studied Gibbs measures for functions with infinite oscillation on origin cylinders on a $\mathbb{Z}$-action subshifts with a countable alphabet, e.g. [14]. In terms of oscillations, note that

(1) $f \in UC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$ if and only if $\delta_n(f) \to 0$ as $n \to \infty$, i.e. if and only if $f$ is *quasilocal.*

and

(2) $f$ is Hölder continuous if and only if $\delta_n(f) \to 0$ exponentially fast as $n \to \infty$.

It is important to realize that with our discrete alphabet the uniformly continuous functions are exactly the *quasilocal* functions (in the language of [5], e.g.).

Of special interest for our theorems will be the *regular* functions:

$$\text{Reg} \equiv \left\{ f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R} \ : \ \sum_{n=1}^{\infty} n^{d-1} \delta_n(f) \ < \infty \right\}.$$

The hierarchy of smoothness, all defined in terms of the oscillations $\delta_n$, is

$$\text{Hölder} \subsetneq \text{Reg} \subsetneq UC.$$

The ergodic *spatial averages* are denoted

$$\frac{1}{|\Lambda|} f_\Lambda(x) \ \equiv \ \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} f \circ T^\lambda,$$

and in particular

$$f_n(x) \equiv f_{\Lambda_n}(x).$$

We find it helpful to have a special symbol for the oscillations of the (unnormalized) spatial averages. So let

$$\Delta_n(f) \ \equiv \ \delta_n(f_n) \ = \sup\left\{|f_n(x) - f_n(y)| \ : \ x|_{\Lambda_n} = y|_{\Lambda_n}\right\}.$$

**Lemma 2.2.** *A commonly occurring error term: "stratified" summation over* $\mathbb{Z}^d$. *For every* $n_0 \geq 1$ *there is a constant* $B_{n_0} > 0$ *such that for all* $n \geq 1$

$$|\Lambda_{n+n_0} \setminus \Lambda_{n+n_0-1}| \leq B_{n_0} n^{d-1}$$

*and hence for any* $f \in Reg\left(\mathbb{N}^{\mathbb{Z}^d}\right)$

$$\sum_{n=1}^{\infty} |\Lambda_{n+n_0} \setminus \Lambda_{n+n_0-1}| \, \delta_n(f) \leq B_{n_0} \, Reg(f).$$

*Proof.* First of all observe that $|\Lambda_n| = (2n-1)^d$. Applying the mean value theorem to the function $x^d$ on the interval $(2(n+n_0-1)-1, 2(n+n_0)-1)$ we find there is a number $c$ in the same interval satisfying

$$\begin{aligned}(2(n+n_0)-1)^d-(2(n+n_0-1)-1)^d &= dc^{d-1}\left((2(n+n_0)-1)-(2(n+n_0-1)-1)\right)\\ &< 2d(2(n+n_0)-1)^{d-1}\\ &\leq 2d((2n_0+1)n)^{d-1}.\end{aligned}$$

So one may use

$$B_{n_0} = 2d(2n_0+1)^{d-1}.$$

□

**Lemma 2.3.** *On the asymptotic oscillations of spatially averaged functions:*
*If* $f \in UC(\mathbb{N}^{\mathbb{Z}^d})$ *and* $\delta_1(f) < +\infty$ *then* $\Delta_n(f) = o(|\Lambda_n|)$.
*If* $f$ *is regular, then* $\Delta_n(f) = O(n^{d-1})$.

*Proof.* If $x \in \mathbb{N}^{\mathbb{Z}^d}$ and $y \in \mathbb{N}^{\mathbb{Z}^d}$ with $x|_{\Lambda_n} = y|_{\Lambda_n}$ and $\lambda \in \Lambda_n$ then $T^\lambda(x)|_{\Lambda_{n-|\lambda|}} = T^\lambda(y)|_{\Lambda_{n-|\lambda|}}$. This is true because $\Lambda_{n-|\lambda|} \subset \Lambda_n \cap (\lambda + \Lambda_n)$. So, as a general rule,

$$\sup\{|f\circ T^\lambda(x) - f\circ T^\lambda(y)| : d(x,y) \leq 2^{-n}\} \leq \delta_{n-|\lambda|}$$

holds for any $f \in \mathbb{R}^{\mathbb{N}^{\mathbb{Z}^d}}$, $n \geq 1$, and $\lambda \in \Lambda_n$.
Let $\epsilon > 0$. Because $f$ is here assumed uniformly continuous, there is some index $n_\epsilon \geq 1$ for which $\delta_n(f) \leq \epsilon$ for every $n \geq n_\epsilon$. If $n \geq n_\epsilon$ and $x|_{\Lambda_n} = y|_{\Lambda_n}$ then $n - |\lambda| \geq n_\epsilon$ if and only if $\lambda \in \Lambda_{n-n_\epsilon+1}$, and so

$$\begin{aligned}|f_n(x) - f_n(y)| &\leq \sum_{\lambda\in\Lambda_n} |f\circ T^\lambda(x) - f\circ T^\lambda(y)|\\ &\leq \sum_{\lambda\in\Lambda_n} \delta_{n-|\lambda|}(f)\\ &\leq \sum_{\lambda\in\Lambda_{n-n_\epsilon+1}} \epsilon + \sum_{\lambda\in\Lambda_n\setminus\Lambda_{n-n_\epsilon+1}} \delta_1(f).\end{aligned}$$

Thus $\Delta_n(f) \leq |\Lambda_{n-n_\epsilon+1}|\epsilon + |\Lambda_n \setminus \Lambda_{n-n_\epsilon+1}|\delta_1(f)$. Although it won't be so used, this estimate makes sense even when $n < n_\epsilon$ if $\Lambda_k$ is understood to be the empty set for $k \leq 0$, in which case it reduces to the trivial observation $\Delta_n(f) \leq |\Lambda_n|\delta_1(f)$.

Now the "little oh" becomes apparent as

$$\begin{aligned}\overline{\lim_{n\to\infty}}\frac{\Delta_n(f)}{|\Lambda_n|} &\leq \lim_{n\to\infty}\frac{|\Lambda_{n-n_\epsilon+1}|\epsilon + |\Lambda_n\setminus\Lambda_{n-n_\epsilon+1}|\delta_1(f)}{|\Lambda_n|}\\ &= 1\cdot\epsilon + 0\cdot\delta_1(f).\end{aligned}$$

To be sure, the ratios of cardinalities are

$$\frac{|\Lambda_{n-n_\epsilon+1}|}{|\Lambda_n|} = \frac{(2(n-n_\epsilon)+1)^d}{(2n-1)^d} \to 1 \text{ as } n \to \infty$$

and

$$\frac{|\Lambda_n \setminus \Lambda_{n-n_\epsilon+1}|}{|\Lambda_n|} = 1 - \frac{|\Lambda_{n-n_\epsilon+1}|}{|\Lambda_n|} \to 0 \text{ as } n \to \infty.$$

For the second claim, let $f$ be regular, and let $x, y \in \mathbb{N}^{\mathbb{Z}^d}$ with $x|_{\Lambda_n} = y|_{\Lambda_n}$. Then

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq \sum_{l=0}^{n-1} \sum_{\lambda \in \Lambda_{l+1} \setminus \Lambda_l} |f(T^\lambda x) - f(T^\lambda y)| \\ &\leq \sum_{l=0}^{n-1} |\Lambda_{l+1} \setminus \Lambda_l| \delta_{n-l}(f) \\ &\leq \sum_{l=0}^{n-1} B_1 l^{d-1} \delta_{n-l}(f) \\ \text{(letting } j := n-l) \quad &= B_1 \sum_{j=1}^{n} \left(\frac{n-j}{j}\right)^{d-1} j^{d-1} \delta_j(f) \\ &\leq B_1 (n-1)^{d-1} \mathrm{Reg}(f). \end{aligned}$$

This proves $\Delta_n(f) \leq B_1 \,\mathrm{Reg}(f)(n-1)^{d-1}$. □

2.4. **Coordinate-wise Permutation within $\Lambda$ and the associated Infinite Volume Energy Loss.** Suppose that for each $\lambda \in \mathbb{Z}^d$, $\tau_\lambda \in \mathfrak{S}_{\mathbb{N}}$, the set of all permutations of $\mathbb{N}$. Then the map

$$\tau : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{N}^{\mathbb{Z}^d}, \quad (\tau(x))_\lambda = \tau_\lambda(\omega_\lambda)$$

is a bijection, and if $\tau_\lambda$ is the identity for all sufficiently large $|\lambda|$ it is also a homeomorphism. Classify these homeomorphisms according to where they act:

**Definition 2.4.** *For* $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ *let*

$$\mathcal{E}_\Lambda = \left\{ \tau = (\tau_\lambda)_{\lambda \in \mathbb{Z}^d} \ : \ \forall \ \lambda \in \mathbb{Z}^d \ \ \tau_\lambda \in S_{\mathbb{N}}, \ \ \forall \ \lambda \in \Lambda^C \ \ \tau_\lambda = Id \right\}.$$

*These are* coordinate-wise permutations of finitely many coordinates *or* local permutations. *Note they are* conjugating homeomorphisms *of* $\mathbb{N}^{\mathbb{Z}^d}$ *with the shift action* $T$*, in the language of* [3]*, e.g.*

In particular let $\mathcal{E}_n \equiv \mathcal{E}_{\Lambda_n}$. We may further specify by letting, for each sub-alphabet $I \subseteq \mathbb{N}$,

$$\mathcal{E}_{\Lambda,I} = \{ \ \tau \in \mathcal{E}_\Lambda \ : \ \forall \ \lambda \in \Lambda \ \ \tau_\lambda|_{I^C} = \mathrm{Id} \ \}.$$

In defining Gibbs measures it will suffice to consider the cases where $I$ is a finite subset of $\mathbb{N}$. In progressing to the DLR equations in Theorem 2.1 we shall want to distinguish a family of these permutations.

**Definition 2.5.** *For each* $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ *and* $\omega \in \mathbb{N}^\Lambda$ *let* $\tau_\omega \in \mathcal{E}_\Lambda$ *by defined by*

$$(\tau_\omega)_\lambda = (\omega_\lambda 1) \in \mathfrak{S}_{\mathbb{N}} \ \forall\ \lambda \in \Lambda,$$

*wherein appears the cycle notation for the transposition of the letters* $\omega_\lambda$ *and* 1*.*

The preceding definition is made exactly so that for any $t \in \mathbb{N}^{\Lambda^C}$

$$\tau_\omega(\omega, t) = (1^\Lambda, t) \qquad \text{and} \qquad \tau_\omega(1^\Lambda, t) = (\omega, t).$$

**Theorem 2.6.** *Fix some* $n_0 \geq 1$ *and* $\tau \in \mathcal{E}_{n_0}$*. Together with a regular local energy function* $f$*,* $\tau$ *defines a uniformly continuous* infinite volume energy loss

$$f_\tau \ \equiv \ \|\ \|_\infty \lim_{n\to\infty} (f_n \circ \tau^{-1} - f_n).$$

*Proof.* For any integers $m > n \geq n_0$,

$$\begin{aligned}
&\left\|(f_m \circ \tau^{-1} - f_m) - (f_n \circ \tau^{-1} - f_n)\right\|_\infty = \\
&= \left\| \sum_{\lambda \in \Lambda_m} (f \circ T^\lambda \circ \tau^{-1} - f \circ T^\lambda) - \sum_{\lambda' \in \Lambda_n} (f \circ T^\lambda \circ \tau^{-1} - f \circ T^\lambda) \right\|_\infty \\
&= \left\| \sum_{\lambda \in \Lambda_m \setminus \Lambda_n} f \circ T^\lambda \circ \tau^{-1} - f \circ T^\lambda \right\|_\infty \\
&\leq \sum_{l=n}^{m-1} \sum_{|\lambda|=l} \left\| f \circ T^\lambda \circ \tau^{-1} - f \circ T^\lambda \right\|_\infty \\
&\leq \sum_{l=n}^{m-1} |\Lambda_{l+1} \setminus \Lambda_l| \ \delta_{l-n_0+1}(f) \\
\left(\begin{array}{c}\text{letting} \\ k=l-n_0+1\end{array}\right) \quad &= \sum_{k=n-n_0+1}^{m-n_0} |\Lambda_{k+n_0} \setminus \Lambda_{k+n_0-1}| \delta_k(f) \\
&\leq B_{n_0} \sum_{k=n-n_0+1}^{m-n_0} (k)^{d-1} \delta_k(f).
\end{aligned}$$

Since it was assumed $Reg(f) < \infty$, the sum in the lower right corner, which is the difference between $Reg(f)$ and a partial sum of $Reg(f)$, must shrink to 0 as $n \to \infty$.
Thus the sequence $\{f_n \circ \tau^{-1} - f_n\}_{n=1}^\infty$ is $\|\cdot\|_\infty$ Cauchy, and moreover the rate of convergence obtained above does not depend on the choice of $\tau \in \mathcal{E}_{n_0}$. The standard "$\frac{\epsilon}{3}$" argument shows that if each $f_n$ is uniformly continuous (which is true as long as $f$ is uniformly continuous), then there is a uniformly continuous limit $f_\tau$ under $\|\cdot\|_\infty$.

□

To see in Fact 2.13 that the specification associated to our Gibbs measures is quasilocal in the sense of [5], we'll also need the following equicontinuity result:

**Lemma 2.7.** *For every regular, exp-summable function $f$ and every $n_0 \geq 1$*

$$\sup_{\tau \in \mathcal{E}_{n_0}} \delta_n(f_\tau) \to 0 \text{ as } n \to \infty.$$

*Proof.* Let $n_0 \geq 1$, $\tau \in \mathcal{E}_{n_0}$, $n > n_0$, and $x, y \in \mathbb{N}^{\mathbb{Z}^d}$ with $x|_{\Lambda_n} = y|_{\Lambda_n}$. We have

$$|f_\tau(x) - f_\tau(y)| = \lim_{m \to \infty} |f_m(\tau^{-1}x) - f_m(x) - f_m(\tau^{-1}y) + f_m(y)|,$$

but there is some question in how to pair the terms. Working with $m > n$, we have the decomposition

$$f_m = \sum_{\lambda \in \Lambda_{\lfloor \frac{n+n_0}{2} \rfloor + 1}} f \circ T^\lambda + \sum_{\lambda \in \Lambda_m \setminus \Lambda_{\lfloor \frac{n+n_0}{2} \rfloor + 1}} f \circ T^\lambda.$$

In the first sum, we pair $x$ with $y$ and $\tau^{-1}x$ with $\tau^{-1}y$ because $x|_{\Lambda_n} = y|_{\Lambda_n}$ and $(\tau^{-1}x)|_{\Lambda_n} = (\tau^{-1}y)|_{\Lambda_n}$. In the second sum we pair $x$ with $\tau^{-1}x$ and $y$ with $\tau^{-1}y$ because $x|_{\Lambda_{n_0}^C} = (\tau^{-1}x)|_{\Lambda_{n_0}^C}$ and $y|_{\Lambda_{n_0}^C} = (\tau^{-1}y)|_{\Lambda_{n_0}^C}$. This yields

$$\begin{aligned}
&|f_m(\tau^{-1}x) - f_m(x) - f_m(\tau^{-1}y) + f_m(y)| \leq \\
\leq\ & \sum_{\lambda \in \Lambda_{\lfloor \frac{n+n_0}{2} \rfloor + 1}} 2\delta_{n-|\lambda|}(f) + \sum_{\lambda \in \Lambda_m \setminus \Lambda_{\lfloor \frac{n+n_0}{2} \rfloor + 1}} 2\delta_{|\lambda| - n_0}(f) \\
\leq\ & 2B \left( \sum_{l=0}^{\lfloor \frac{n+n_0}{2} \rfloor} \delta_{n-l}(f) l^{d-1} + \sum_{l=\lfloor \frac{n+n_0}{2} \rfloor + 1}^{m-1} \delta_{l-n_0}(f) l^{d-1} \right) \\
& \Big(\text{reindex the first sum with } k=n-l, \text{ note } n-\lfloor \tfrac{n+n_0}{2} \rfloor = \lceil \tfrac{n-n_0}{2} \rceil \Big) \\
=\ & 2B \left( \sum_{k=\lceil \frac{n-n_0}{2} \rceil}^{n} \delta_k(f) k^{d-1} \left( \frac{n-k}{k} \right)^{d-1} + \sum_{l=\lfloor \frac{n+n_0}{2} \rfloor + 1}^{m-1} \delta_{l-n_0}(f)(l-n_0)^{d-1} \left( \frac{l}{l-n_0} \right)^{d-1} \right) \\
\leq\ & 2B \left( \left( \frac{n - \lceil \frac{n-n_0}{2} \rceil}{\lceil \frac{n-n_0}{2} \rceil} \right)^{d-1} \sum_{k=\lceil \frac{n-n_0}{2} \rceil}^{n} \delta_k(f) k^{d-1} + \right. \\
& \left. + \left( \frac{\lfloor \frac{n+n_0}{2} \rfloor + 1}{\lfloor \frac{n+n_0}{2} \rfloor + 1 - n_0} \right)^{d-1} \sum_{l=\lfloor \frac{n+n_0}{2} \rfloor + 1}^{m-1} \delta_{l-n_0}(f)(l-n_0)^{d-1} \right).
\end{aligned}$$

Taking $m \to \infty$ the right hand sum remains finite by regularity, and we obtain the common upper bound on $\delta_n(f_\tau)$, $\tau \in \mathcal{E}_{n_0}$. The floor/ceiling coefficients on both sums converge to 1 as $n \to \infty$ (note for the left one that $n - \lceil \frac{n-n_0}{2} \rceil = \lfloor \frac{n+n_0}{2} \rfloor$). By regularity of $f$ both sums shrink to 0 as $n \to \infty$. Thus the lemma is proved. □

2.5. **Gibbs Measures.** Because we use the local energy $f$ without reference to an interaction, the traditional physical (DLR) definition of a Gibbs measure must be modified.

Let $\mathcal{P}_F(\mathbb{N}) = \{F \subset \mathbb{N} : |F| < \infty\}$.

**Definition 2.8.** *A* Gibbs state *for a regular, exp-summable local energy function* $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ *is a Borel probability measure for which* $\mu \circ \tau^{-1} \ll \mu$ *with Radon-Nikodym derivative*

$$\frac{d\mu \circ \tau^{-1}}{d\mu} = e^{f_\tau}$$

*for every* $F \in \mathcal{P}_F(\mathbb{N})$, $n \geq 1$, *and* $\tau \in \mathcal{E}_{n,F}$. *For short we write* $\mu \in G.S.(f)$.

This definition of Gibbs state by transfer of measure under local permutations of the configuration space is motivated by Keller's definition for finite alphabet lattice systems in [8], which is in turn motivated by Capocaccia's definition in [3] of Gibbs states for an expansive $\mathbb{Z}^d$ action on a general compact metric space. To extend Keller/Capocaccia's definition to the non-compact space $\mathbb{N}^{\mathbb{Z}^d}$ we have simply required the measure to respond correctly to local permutations of finitely many letters.

We record an obvious algebraic and topological consequence of this definition.

**Observation 2.9.** *The set of Gibbs states for a regular, exp-summable function is convex and weak closed.*

*Proof.* In the defining equation $\mu\circ\tau^{-1} = e^{f_\tau}\mu$, both sides are affine in $\mu$. Thus for $0 < \alpha < 1$ and $\{\mu, \nu\} \subseteq \text{G.S.}(f)$ it follows that

$$(\alpha\mu + (1-\alpha)\nu) \circ \tau^{-1} = e^{f_\tau} (\alpha\mu + (1-\alpha)\nu),$$

i.e. $(\alpha\mu + (1-\alpha)\nu) \in \text{G.S.}(f)$, too. For weak closedness, let $\{\mu_j\}_{j=1}^{\infty} \subseteq \text{G.S.}f$ and let $\mu$ be a weak limit of the sequence. Let $g \in B.C.(\mathbb{N}^{\mathbb{Z}^d})$ and $\tau \in \mathcal{E}_{n,F}$ as before. Then

$$\mu \circ \tau^{-1} \;=\; {}^{w}\lim_{j\to\infty} \mu_j \circ \tau^{-1} \;=\; {}^{w}\lim_{j\to\infty} e^{f_\tau}\, \mu_j \;=\; e^{f_\tau}\, \mu.$$

and this proves $\mu$ is a Gibbs state for $f$, too. □

2.6. **DLR Characterization, Specifications.**

We check that our definition is equivalent to satisfying a local energy form of the DLR equations. These equations are also significant because they constitute a quasilocal specification without reference to an interaction potential $\Phi$. The approach is modeled after [8]'s Chapter 5.

**Theorem 2.10.** *A Borel probability measure on* $\mathbb{N}^{\mathbb{Z}^d}$ *is a Gibbs state for the regular, exp-summable function* $f$ *if, and only if, for every* $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$, $\xi \in \mathbb{N}^\Lambda$ *and* $x \in \mathbb{N}^{\mathbb{Z}^d}$, *it has the conditional measures*

$$\begin{aligned} \mu([\xi]|\mathcal{A}_{\Lambda^C})(x) &= \frac{\exp f_{\tau_\xi}\left(1^\Lambda, x|_{\Lambda^C}\right)}{\sum_{\omega\in\mathbb{N}^\Lambda} \exp f_{\tau_\omega}\left(1^\Lambda, x|_{\Lambda^C}\right)} \\ &= \lim_{m\to\infty} \frac{\exp f_m(\xi, x|_{\Lambda^C})}{\sum_{\omega\in\mathbb{N}^\Lambda} \exp f_m(\omega, x|_{\Lambda^C})}. \qquad (2.1) \end{aligned}$$

*Proof.* First suppose $\mu$ is a Gibbs state. If $A \in \mathcal{A}_{\Lambda^C}$ and $(\omega, x) \in A$ for some $\omega \in \mathbb{N}^\Lambda$, $x \in \mathbb{N}^{\Lambda^C}$, then $(\omega', x) \in A$ for every $\omega' \in \mathbb{N}^\Lambda$. This means $\mathcal{A}_{\Lambda^C}$ measurable sets are preserved by all the maps in $\mathcal{E}_\Lambda$. In particular, if $\tau \in \mathcal{E}_\Lambda$, and $\alpha, \omega \in \mathbb{N}^\Lambda$ with $\tau^{-1}[\alpha] = [\omega]$ then in fact for any $A \in \mathcal{A}_{\Lambda^C}$ $\tau^{-1}([\alpha] \cap A) = [\omega] \cap A$. Using some basic properties of conditional expectation, we see that for all $A \in \mathcal{A}_{\Lambda^C}$

$$\begin{aligned}
\int_A \mathbb{1}_{[\omega]} d\mu &= \int_A \mathbb{1}_{[\alpha]} d(\mu \circ \tau^{-1}) \\
&= \int_A \mathbb{1}_{[\alpha]} \exp f_\tau d\mu \\
&= \int_A \mathbb{1}_{[\alpha]}(x) \exp f_\tau(\alpha, x|_{\Lambda^C}) d\mu(x) \\
&= \int_A E_\mu \left[ \mathbb{1}_{[\alpha]} \exp f_\tau(\alpha, \cdot|_{\Lambda^C}) \middle| \mathcal{A}_{\Lambda^C} \right](x) d\mu(x) \\
&= \int_A E_\mu \left[ \mathbb{1}_{[\alpha]} \middle| \mathcal{A}_{\Lambda^C} \right](x) \exp f_\tau(\alpha, x|_{\Lambda^C}) d\mu(x).
\end{aligned}$$

This proves that

$$E_\mu \left[ \mathbb{1}_{[\alpha]} \middle| \mathcal{A}_{\Lambda^C} \right](x) \exp f_\tau(\alpha, x|_{\Lambda^C}) = E_\mu \left[ \mathbb{1}_{[\omega]} \middle| \mathcal{A}_{\Lambda^C} \right](x). \tag{2.2}$$

Now let $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$, $F \in \mathcal{P}_F(\mathbb{N})$ with $1 \in F$, and $\omega \in F^\Lambda$. Recall Definition 2.5 of $\tau_\omega \in \mathcal{E}_{\Lambda,F}$, for which $\tau_\omega([\omega]) = [1^\Lambda]$. Taking $\tau = \tau_\omega$ and $\alpha = 1^\Lambda$ in Equation 2.2 yields

$$E_\mu \left[ \mathbb{1}_{[1^\Lambda]} \middle| \mathcal{A}_{\Lambda^C} \right](x) \exp f_{\tau_\omega}(1^\Lambda, x|_{\Lambda^C}) = E_\mu \left[ \mathbb{1}_{[\omega]} \middle| \mathcal{A}_{\Lambda^C} \right](x),$$

and summing over all $\omega \in F^\Lambda$ yields (by linearity of conditional expectation)

$$E_\mu \left[ \mathbb{1}_{F^\Lambda \times \mathbb{N}^{\Lambda^C}} \middle| \mathcal{A}_{\Lambda^C} \right](\cdot) = \sum_{\omega \in F^\Lambda} E_\mu \left[ \mathbb{1}_{[\omega]} \middle| \mathcal{A}_{\Lambda^C} \right](\cdot) = E_\mu \left[ \mathbb{1}_{[1^\Lambda]} \middle| \mathcal{A}_{\Lambda^C} \right] \sum_{\omega \in F^\Lambda} \exp f_{\tau_\omega}(1^\Lambda, \cdot|_{\Lambda^C}).$$

Because $\bigcap_{F \in \mathcal{P}_F(\mathbb{N})} \left( F^\Lambda \times \mathbb{N}^{\Lambda^C} \right)^C = \varnothing$, the "continuity" property of countably additive measures implies

$$\int |\mathbb{1} - \mathbb{1}_{F^\Lambda \times \mathbb{N}^{\Lambda^C}}|^2 d\mu = \int |\mathbb{1} - \mathbb{1}_{F^\Lambda \times \mathbb{N}^{\Lambda^C}}| d\mu = \mu((F^\Lambda \times \mathbb{N}^{\Lambda^C})^C) \to 0$$

as $F \to \mathbb{N}$. Since conditional expectation given $\mathcal{A}_{\Lambda^C}$ is a continuous linear operator on $L^2(\mu)$ (namely, orthogonal projection onto the closed subspace of $\mathcal{A}_{\Lambda^C}$ measurable square integrable functions),

$$E_\mu \left[ \mathbb{1}_{F^\Lambda \times \mathbb{N}^{\Lambda^C}} \middle| \mathcal{A}_{\Lambda^C} \right] \to E_\mu \left[ \mathbb{1} \middle| \mathcal{A}_{\Lambda^C} \right]$$

in $L^2(\mu)$ as $F \to \mathbb{N}$. Now by general measure theory there is a subsequence $\{F_m\}_{m \geq 1} \subset \mathcal{P}_F(\mathbb{N})$ for which $E_\mu \left[ \mathbb{1}_{F_m^\Lambda \times \mathbb{N}^{\Lambda^C}} \middle| \mathcal{A}_{\Lambda^C} \right]$ converges $\mu - a.e.$ to $E_\mu \left[ \mathbb{1} \middle| \mathcal{A}_{\Lambda^C} \right]$. The constant function $\mathbb{1} \equiv \mathbb{1}_{\mathbb{N}^{\mathbb{Z}^d}}$ is a version of $E_\mu\left[\mathbb{1}\middle|\mathcal{A}_{\Lambda^C}\right]$, so for $\mu - a.e.x \in \mathbb{N}^{\mathbb{Z}^d}$

$$\begin{aligned} 1 &= \lim_{m\to\infty} E_\mu\left[\mathbb{1}_{F_m^\Lambda\times\mathbb{N}^{\Lambda^C}}\middle|\mathcal{A}_{\Lambda^C}\right](x) \\ &= \lim_{m\to\infty} E_\mu\left[\mathbb{1}_{[1^\Lambda]}\middle|\mathcal{A}_{\Lambda^C}\right](x)\sum_{\omega\in F_m^\Lambda}\exp f_{\tau_\omega}(1^\Lambda, x|_{\Lambda^C}) \\ &= E_\mu\left[\mathbb{1}_{[1^\Lambda]}\middle|\mathcal{A}_{\Lambda^C}\right](x)\sum_{\omega\in\mathbb{N}^\Lambda}\exp f_{\tau_\omega}(1^\Lambda, x|_{\Lambda^C}). \end{aligned} \tag{2.3}$$

Before proceeding, we want to say something about the countable sum in the last line. Let $n_0$ be the least index such that $\Lambda \subseteq \Lambda_{n_0}$, and let

$$D_\omega = \sum_{\lambda\in\Lambda_{n_0}\setminus\Lambda} |f\circ T^\lambda(\omega, x|_{\Lambda^C}) - f\circ T^\lambda(1^\Lambda, x|_{\Lambda^C})| \;\leq |\Lambda_{n_0}\setminus\Lambda|\delta_1(f).$$

Then for every $n > n_0$

$$\begin{aligned} &\sum_{\omega\in\mathbb{N}^\Lambda}\exp f_{\tau_\omega}^n(1^\Lambda, x|_{\Lambda^C}) \\ \leq\; &\sum_{\omega\in\mathbb{N}^\Lambda}\exp\left(f_\Lambda(\omega, x|_{\Lambda^C}) - f_\Lambda(1^\Lambda, x|_{\Lambda^C}) + D_\omega + \sum_{l=1}^{n-n_0}|\Lambda_{n_0+l}\setminus\Lambda_{n_0+l-1}|\delta_l(f)\right) \\ \leq\; &\exp\left(|\Lambda_{n_0}\setminus\Lambda|\delta_1(f) + B_{n_0}\,Reg(f) - f_\Lambda(1^\Lambda, x|_{\Lambda^C})\right) Z_\Lambda^x(f) < \infty. \end{aligned}$$

This shows that each of the functions

$$\mathbb{N}^\Lambda \ni \omega \mapsto \exp\left(f_{\tau_\omega}^n(1^\Lambda, x|_{\Lambda^C})\right),\; \Lambda_n \supseteq \Lambda$$

is integrable by the counting measure on $\mathbb{N}^\Lambda$ and moreover the whole sequence is dominated by the integrable function

$$\omega \mapsto \exp\left(f_\Lambda(\omega, x|_{\Lambda^C}) - f_\Lambda(1^\Lambda, x|_{\Lambda^C}) + |\Lambda_{n_0}\setminus\Lambda|\delta_1(f) + B_{n_0}\,Reg(f)\right).$$

With the Lebesgue dominated convergence theorem, this shows that the limit function $\omega \mapsto e^{f_{\tau_\omega}(1^\Lambda, x|_{\Lambda^C})}$ is integrable and justifies the coming exchange of sum and limit.

Moving on, fix any one $\xi \in \mathbb{N}^\Lambda$ and again use $\eta = 1^\Lambda$ and $\tau = \tau_\xi$ in Equation 2.2. Substitute the expression this provides for $E_\mu\left[\mathbb{1}_{[1^\Lambda]}\middle|\mathcal{A}_{\Lambda^C}\right](x)$ into Equation 2.3 to obtain (for $\mu - a.e.x \in \mathbb{N}^{\mathbb{Z}^d}$),

$$1 = E_\mu\left[\mathbb{1}_{[\xi]}\middle|\mathcal{A}_{\Lambda^C}\right](x)\exp\left(-f_{\tau_\xi}\left(1^\Lambda, x|_{\Lambda^C}\right)\right)\sum_{\omega\in\mathbb{N}^\Lambda}\exp f_{\tau_\omega}\left(1^\Lambda, x|_{\Lambda^C}\right),$$

and so

$$E_\mu\left[\mathbb{1}_{[\xi]}\middle|\mathcal{A}_{\Lambda^C}\right](x) \;=\; \frac{\exp f_{\tau_\xi}\left(1^\Lambda, x|_{\Lambda^C}\right)}{\sum_{\omega\in\mathbb{N}^\Lambda}\exp f_{\tau_\omega}\left(1^\Lambda, x|_{\Lambda^C}\right)}.$$

This formula, the first line of Equation 2.1, is derived by Keller for finite alphabet lattice systems in Chapter 5 of [8]. To obtain the formula in the second line of Equation 2.1, write out the $f_\tau$ as limits and cancel the common factor.

$$\begin{aligned}\frac{\exp f_{\tau_\xi}\left(1^\Lambda, x|_{\Lambda^C}\right)}{\sum_{\omega\in\mathbb{N}^\Lambda}\exp f_{\tau_\omega}\left(1^\Lambda, x|_{\Lambda^C}\right)} &= \frac{\exp\lim_{m\to\infty}\left(f_m\circ\tau_\xi^{-1}-f_m\right)\left(1^\Lambda, x|_{\Lambda^C}\right)}{\sum_{\omega\in\mathbb{N}^\Lambda}\exp\lim_{m\to\infty}\left(f_m\circ\tau_\omega^{-1}-f_m\right)\left(1^\Lambda, x|_{\Lambda^C}\right)}\\ &= \frac{\exp\lim_{m\to\infty}\left(f_m\left(\xi, x|_{\Lambda^C}\right)-f_m\left(1^\Lambda, x|_{\Lambda^C}\right)\right)}{\sum_{\omega\in\mathbb{N}^\Lambda}\exp\lim_{m\to\infty}\left(f_m\left(\omega, x|_{\Lambda^C}\right)-f_m\left(1^\Lambda, x|_{\Lambda^C}\right)\right)}\\ &= \lim_{m\to\infty}\frac{\exp\left(f_m\left(\xi, x|_{\Lambda^C}\right)-f_m\left(1^\Lambda, x|_{\Lambda^C}\right)\right)}{\sum_{\omega\in\mathbb{N}^\Lambda}\exp\left(f_m\left(\omega, x|_{\Lambda^C}\right)-f_m\left(1^\Lambda, x|_{\Lambda^C}\right)\right)}\\ &= \lim_{m\to\infty}\frac{\exp f_m\left(\xi, x|_{\Lambda^C}\right)}{\sum_{\omega\in\mathbb{N}^\Lambda}\exp f_m\left(\omega, x|_{\Lambda^C}\right)}.\end{aligned}$$

The exchange of limit and sum in the denominator from the second line to the third is allowed by the Lebesgue dominated convergence theorem, as mentioned above. This completes the proof that Gibbs states for a regular exp-summable local energy $f$ must have the DLR-type conditional expectations of Equation 2.1.

For the converse take some $n \geq 1$, $F \in \mathcal{P}_F(\mathbb{N})$, and $\tau \in \mathcal{E}_{n,F}$. Take any $\xi \in \mathbb{N}^{\Lambda_n}$ and let $\omega = \tau|_{\Lambda_n^{-1}}(\xi) \in \mathbb{N}^{\Lambda_n}$. Assuming the conditional expectations of equation 2.1,

$$\begin{aligned}\mu\circ\tau^{-1}\left([\xi]\right) = \mu\left([\omega]\right) = \int \mathbb{1}_{[\omega]}d\mu = \int \mu\left([\omega]|\mathcal{A}_{\Lambda_n^C}\right)(x)d\mu(x) = \\ = \int \exp f_\tau\left(\xi, x|_{\Lambda_n^C}\right)\mu\left([\xi]|\mathcal{A}_{\Lambda_n^C}\right)(x)d\mu(x)\\ = \int E_\mu\left[\exp f_\tau\left(\xi, \cdot|_{\Lambda_n^C}\right)\mathbb{1}_{[\xi]}|\mathcal{A}_{\Lambda_n^C}\right](x)d\mu(x)\\ = \int \exp f_\tau\left(\xi, x|_{\Lambda_n^C}\right)\mathbb{1}_{[\xi]}(x)d\mu(x) = \int_{[\xi]}\exp f_\tau d\mu.\end{aligned}$$

This proves $\mu\circ\tau^{-1} = e^{f_\tau}\mu$. □

Now we turn to the compatibility of our definition of Gibbs measure for a regular,exp-summable function of $\mathbb{N}^{\mathbb{Z}^d}$ with the probabilistic theory of Gibbs measures for a product measure specification as presented in [5]. With the local energy DLR 2.1 characterization of G.S.$(f)$, we can check that it consists precisely of the Gibbs measures for a certain quasilocal specification $\gamma$ on the measurable space $\left(\mathbb{N}^{\mathbb{Z}^d}, \text{Borel}\right)$. The details follow.

**Observation 2.11.**

$$\rho_\Lambda(x) \equiv \frac{\exp f_{\tau_{x|_\Lambda}}(1^\Lambda, x|_{\Lambda^C})}{\sum_{\omega \in \mathbb{N}^\Lambda} \exp f_{\tau_\omega}(1^\Lambda, x|_{\Lambda^C})}$$

*constitutes a positive $\chi$-modification, where $\chi$ is the counting measure on $\mathbb{N}$.*

*Proof.* For normalization recall from the proof of Theorem 2.10 that the denominator of $\rho_\Lambda(x)$ is finite for any $x \in \mathbb{N}^{\mathbb{Z}^d}$. Thus

$$\rho_\Lambda \chi_\Lambda(\mathbb{N}^{\mathbb{Z}^d}|x) = \sum_{y \in \mathbb{N}^{\mathbb{Z}^d} : y|_{\Lambda^C} = x|_{\Lambda^C}} \rho_\Lambda(x) = 1.$$

For the consistency condition we use the equivalent formulation in the second line of Equation 2.1 and criterion (c) of Proposition (1.30) in [5]: if $\Lambda \subset \Lambda' \in \mathcal{P}_F(\mathbb{Z}^d)$, $a, b \in \mathbb{N}^\Lambda$ and $t \in \mathbb{N}^{\Lambda^C}$, then

$$\rho_{\Lambda'}(a,t)\rho_\Lambda(b,t) = \lim_{m\to\infty} \frac{\exp f_m(a,t)}{\sum_{\omega' \in \mathbb{N}^{\Lambda'}} \exp f_m(\omega', t|_{\Lambda'^C})} \frac{\exp f_m(b,t)}{\sum_{\omega \in \mathbb{N}^\Lambda} \exp f_m(\omega, t|_{\Lambda^C})} = \rho_{\Lambda'}(b,t)\rho_\Lambda(a,t).$$

$\square$

With this modification $\rho$ of the a priori counting measure $\chi$, we get the specification $\gamma = \rho\chi$ defined by

$$\gamma_\Lambda([\xi]|x) = \frac{\exp f_{\tau_\xi(1^\Lambda, x|_{\Lambda^C})}}{\sum_{\omega \in \mathbb{N}^\Lambda} \exp f_{\tau_\omega}(1^\Lambda, x|_{\Lambda^C})} = \rho_\Lambda(\xi, x|_{\Lambda^C}). \tag{2.4}$$

for any $\xi \in \mathbb{N}^\Lambda$ and $x \in \mathbb{N}^{\mathbb{Z}^d}$, and more generally

$$\gamma_\Lambda(B|x) = \frac{\sum_{\xi \in \mathbb{N}^\Lambda : (\xi, x|_{\Lambda^C}) \in B} \exp f_{\tau_\xi(1^\Lambda, x|_{\Lambda^C})}}{\sum_{\omega \in \mathbb{N}^\Lambda} \exp f_{\tau_\omega}(1^\Lambda, x|_{\Lambda^C})} \tag{2.5}$$

for any Borel subset $B$ of $\mathbb{N}^{\mathbb{Z}^d}$. We recall from [5] that the Gibbs measures $\mathfrak{G}_\gamma$ for a specification $\gamma$ are those probability measures for which

$$\mu(B|\mathcal{A}_{\Lambda^C})(x) = \gamma_\Lambda(B|x) \tag{2.6}$$

holds for every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$, $B \in \text{Borel}\left(\mathbb{N}^{\mathbb{Z}^d}\right)$, and $\mu$-a.e. $x \in \mathbb{N}^{\mathbb{Z}^d}$.

**Observation 2.12.** *For a regular, exp-summable function $f$ on $\mathbb{N}^{\mathbb{Z}^d}$, the Gibbs measures for $f$ (G.S.($f$), Definition 2.8) conicide with the Gibbs measures for the specification $\gamma$ ($\mathfrak{G}_\gamma$, Definitions 2.5,2.6).*

*Proof.* Using the local energy DLR characterization theorem 2.10, it is clear from the special case 2.4 of equation 2.5 that any Gibbs measure for the specification $\gamma$ is an element of G.S.$(f)$. Conversely, if $\mu \in$G.S.$(f)$ then we know

$$\mu([\omega]|\mathcal{A}_{\Lambda^C})(x) = \gamma_\Lambda([\omega]|x)$$

for every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$, $\omega \in \mathbb{N}^\Lambda$, and $\mu$-a.e. $x \in \mathbb{N}^{\mathbb{Z}^d}$. In other words, if $\mu \in$ G.S.$(f)$ then for each $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ the class of sets

$$\left\{ S \in \text{Borel}\left(\mathbb{N}^{\mathbb{Z}^d}\right) \,:\, \mu(S|\mathcal{A}_{\Lambda^C})(x) = \gamma_\Lambda(S|x) \text{ for } \mu-\text{a.e.} x \in \mathbb{N}^{\mathbb{Z}^d} \right\}$$

contains all cylinders $[\omega]$ where $\omega \in \mathbb{N}^\Lambda$. It follows from the basic properties of conditional expectation that this class is closed under complementation and further contains all "rectangle" sets of the form $A \times B \in \text{Borel}\left(\mathbb{N}^\Lambda\right) \times \text{Borel}\left(\mathbb{N}^{\Lambda^C}\right)$ (here we mean the Cartesian product of the $\sigma$-algebras). Then it follows from basic measure theory and appropriate use of the Lebesgue dominated convergence theorem that this class is closed under countable unions. Thus it is a $\sigma$-algebra, and contains the algebra of finite unions of cylinders. So in fact $\mu \in$ G.S.$(f)$ implies that for every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ and every $S \in \text{Borel}\left(\mathbb{N}^{\mathbb{Z}^d}\right)$ the function $\gamma_\Lambda(S|x)$ is a version of $\mu(S|\mathcal{A}_{\Lambda^C})(x)$, i.e. $\mu \in \mathfrak{G}_\gamma$. □

It does not follow directly from [5] that $\gamma$ is a quasilocal specification, so we prove it here.

**Fact 2.13.** *The specification defined in Equation 2.4 is quasilocal for any regular, exp-summable local energy function $f$.*

*Proof.* We need to show that for each $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ and bounded local function $g$ that $\gamma_\Lambda g$ is a quasilocal i.e. uniformly continuous function. Recall

$$\gamma_\Lambda g(x) = \sum_{\xi \in \mathbb{N}^\Lambda} \gamma_\Lambda([\xi]|x) g(\xi, x|_{\Lambda^C}).$$

If $g$ only depends on the coordinates in $\Lambda_j$, $j \geq 1$, then let $n \geq j$ and $x, y \in \mathbb{N}^{\mathbb{Z}^d}$ with $x|_{\Lambda_n} = y|_{\Lambda_n}$. Using normalization of $\gamma_\Lambda(\ |x)$ and the fact that $g(\xi, x|_{\Lambda^C}) = g(\xi, y|_{\Lambda^C})$ for every $\xi \in \mathbb{N}^\Lambda$, we compute

$$\begin{aligned} |\gamma_\Lambda g(x) - \gamma_\Lambda g(y)| &= |\sum_{\xi \in \mathbb{N}^\Lambda} \gamma_\Lambda([\xi]|x) g(\xi, x|_{\Lambda^C}) - \gamma_\Lambda([\xi]|y) g(\xi, y|_{\Lambda^C})| \\ &= |\sum_{\xi \in \mathbb{N}^\Lambda} g(\xi, x|_{\Lambda^C}) \left(\gamma_\Lambda([\xi]|x) - \gamma_\Lambda([\xi]|y)\right)| \\ &\leq \|g\|_\infty \max_{\xi \in \mathbb{N}^\Lambda} \left| 1 - \frac{\gamma_\Lambda([\xi]|y)}{\gamma_\Lambda([\xi]|x)} \right|. \end{aligned}$$

But it is clear from the middle term in equation 2.4 that, because $x|_{\Lambda_n} = y|_{\Lambda_n}$

$$\exp\left(-2 \sup_{\tau \in \mathcal{E}_\Lambda} \delta_n(f_\tau)\right) \leq \frac{\gamma_\Lambda([\xi]|y)}{\gamma_\Lambda([\xi]|x)} \leq \exp\left(2 \sup_{\tau \in \mathcal{E}_\Lambda} \delta_n(f_\tau)\right)$$

holds for every $\xi \in \mathbb{N}^\Lambda$, and hence (with the mean value theorem)

$$\left|1 - \frac{\gamma_\Lambda([\xi]|y)}{\gamma_\Lambda([\xi]|x)}\right| \leq 2 \sup_{\tau \in \mathcal{E}_\Lambda} \delta_n(f_\tau) \exp\left(2 \sup_{\tau \in \mathcal{E}_\Lambda} \delta_n(f_\tau)\right).$$

Thus

$$\delta_n(\gamma_\Lambda g) \leq 2\|g\|_\infty \sup_{\tau \in \mathcal{E}_\Lambda} \delta_n(f_\tau) \exp\left(2 \sup_{\tau \in \mathcal{E}_\Lambda} \delta_n(f_\tau)\right),$$

which shrinks to 0 as $n \to \infty$ by Lemma 2.7. □

## 3. COMPATIBILITY WITH THE “INTERACTION” FORMULATION OF A LATTICE MODEL.

We present the preliminary Propositions 3.1, 3.2 of this section in a more general setting that takes a measurable space $E, \mathcal{E}$ for an alphabet. (Note: in [5], the lattice $\mathbb{Z}^d$ is also generalized to an arbitrary countable set, but we see no need to go so far here because our notion of “spatial averaging” doesn't produce useful results with a structureless indexing set for the product.) Then in Theorem 3.4 we compare our notion of Gibbs measures on $\mathbb{N}^{\mathbb{Z}^d}$ (Definition 2.8, Theorem 2.10) to the established notion of Gibbs measures in this general setting.

In the physical theory an *interaction* (variously called an interaction potential, or just a potential) is an element

$$\Phi = (\Phi_\Lambda)_{\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)} \in \prod_{\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)} \mathrm{C}_\Lambda\left(\mathbb{N}^{\mathbb{Z}^d}\right),$$

where we've labeled the $\Lambda$-*local functions* by

$$\mathrm{C}_\Lambda\left(\mathbb{N}^{\mathbb{Z}^d}\right) = \left\{f \in \mathrm{C}\left(\mathbb{N}^{\mathbb{Z}^d}\right) : \ x|_\Lambda = y|_\Lambda \ \Rightarrow \ f(x) = f(y)\right\}.$$

In [13],[15],and [7], the largest class of interactions studied is the “Big” Banach space of interactions $\Phi$ with norm

$$\|\Phi\|_B = \sum_{0 \in \Lambda \in \mathcal{P}_F(\mathbb{Z}^d)} |\Lambda|^{-1} \|\Phi_\Lambda\|_\infty < \infty.$$

The “Small” Banach space, for comparison, has the norm

$$\|\Phi\|_S = \sum_{0 \in \Lambda \in \mathcal{P}_F(\mathbb{Z}^d)} \|\Phi_\Lambda\|_\infty < \infty.$$

In [5], the only interactions considered are those for which all the *Hamiltonian* series

$$H_\Lambda(\Phi) = \sum_{\substack{\Lambda' \in \mathcal{P}_F(\mathbb{Z}^d): \\ \Lambda \cap \Lambda' \neq \emptyset}} \Phi_{\Lambda'}$$

at least converge pointwise, i.e. configurationwise. (Note for $\Phi \in \mathcal{S}$ the Hamiltonians converge absolutely, uniformly, but $\Phi \in \mathcal{B}$ neither implies nor is implied by the Hamiltonians' pointwise convergence.) An interaction in [5] is further called *admissible* by an a priori measure $\nu$ on the alphabet $E$ if all the Hamiltonian series converge pointwise and for all $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ and all "tail configurations" $t \in E^{\Lambda^C}$ the "partition function"

$$Z_\Lambda^t(\Phi) = \int_{E^\Lambda} e^{-H_\Lambda(\Phi)(a,t)} d\nu^\Lambda(a) = \int e^{-H_\Lambda(\Phi)} d\left(\nu^\Lambda \times \delta_t\right)$$

is finite. An admissible interaction $\Phi$ defines a product measure specification by modification of $\nu^{\mathbb{Z}^d}$. Specifically, for every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$, $B \in \mathcal{E}^{\mathbb{Z}^d}$, and $x \in E^{\mathbb{Z}^d}$,

$$\eta_\Lambda(B|x) = \eta_\Lambda^{\Phi,\nu}(B|x) = \left(Z_\Lambda^{x|_{\Lambda^C}}(\Phi)\right)^{-1} \int_{E^\Lambda} \mathbb{1}_B(\omega, x|_{\Lambda^C}) e^{-H_\Lambda(\Phi)\left(\omega, x|_{\Lambda^C}\right)} d\nu^\Lambda. \tag{3.1}$$

An important feature of product measure specifications is that for each fixed $B$ the function $x \mapsto \eta_\Lambda(B|x)$ is $\mathcal{A}_{\Lambda^C}$ measurable, i.e. does not depend on the coordinates $x|_\Lambda$. For convenience in what follows, this allows us to use the symbol

$$\eta_\Lambda(B| \cdot t),$$

where $t \in E^{\Lambda^C}$, to stand for the common value of $\eta_\Lambda(B|(\omega, t))$ for all $\omega \in E^\Lambda$. In [5] a *Gibbs measure* $\mu$ for an interaction $\Phi$ admissible by some a priori measure $\nu$ is defined as a Gibbs measure for the specification $\eta$, i.e. one satisfying

$$\mu\left(B|\mathcal{A}_{\Lambda^C}\right)(x) = \eta_\Lambda^{\Phi,\nu}(B|x) \tag{3.2}$$

for every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$, $B \in \mathcal{E}^{\mathbb{Z}^d}$, and $x \in E^{\mathbb{Z}^d}$. In [13],[15], and [7] the same definition is used for finite $\nu$ and $\Phi \in \mathcal{S}$, but the use of specifications is avoided by writing the conditional measures directly in terms of the exponentiated Hamiltonians in 3.1, and calling the set of such measures $\mathfrak{G}_{\Phi,\nu}$ rather than $\mathfrak{G}_{\eta^{\Phi,\nu}}$. Both Dobrushin and the team of Lanford and Ruelle are attributed with the discovery that this system of conditional measures is the appropriate "infinte volume" generalization of the classical Gibbs ensemble for finite cardinality configuration spaces (or purely atomic measures.) Consequently, the collective equations 3.2 are known as *the DLR equations.*

If $\Phi \in \mathcal{B}$ or if $\Phi$ has well defined Hamiltonians, then it has a well defined local energy function

$$\hat{A}_\Phi = \sum_{\Lambda \ni \circ 0} \Phi_\Lambda,$$

wherein the symbol

$$\Lambda \ni \circ\lambda$$

indicates that the lattice site $\lambda$ is the $\left\lceil \frac{|\Lambda|}{2} \right\rceil$th, or middle, element of $\Lambda$ in lexicographic order. Note the series $\hat{A}_\Phi$ converges in $\| \ \|_\infty$ if $\Phi \in \mathcal{B}$. The physical interpretation of $\hat{A}(\Phi)$ is specific internal energy, i.e. internal energy per lattice site. Although it is calculated at the origin, translation invariance implies it is the same throughout the lattice. In defining local energy it is perhaps a bit arbitrary to assign the interaction in a region $\Lambda$ to it's lexicographical middle. Indeed a more common definition of local energy is

$$A_\Phi = \sum_{0 \in \Lambda \in \mathcal{P}_F(\mathbb{Z}^d)} |\Lambda|^{-1} \Phi_\Lambda.$$

Both maps $\hat{A}$ and $A$ are bounded linear maps $(\mathcal{B}, \|\ \|_B) \to \left(BUC\left(\mathbb{N}^{\mathbb{Z}^d}\right), \|\ \|_\infty\right)$. However, $\hat{A}$ has the advantage of admitting a natural construction of an element $\Phi \in \hat{A}^{-1}(f)$ for every $f \in BUC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$. The construction is due to Ruelle ([13]), but we present the details below in Fact 3.3. Interestingly, the exact image of $\mathcal{B}$ under the averaging approach $A$ to local energy appears to be unknown. The two local energies are equivalent in the sense that for any shift invariant finite measure $\mu$ on $\mathbb{N}^{\mathbb{Z}^d}$ the integrals coincide: $\int A(\Phi)d\mu = \int \hat{A}_\Phi d\mu$. Thus they define the same variational equilibrium states. We turn next to a formulation of the DLR equations in terms of these two local energies.

**Proposition 3.1.** *Let $\Phi$ be an interaction admissible by a finite a priori measure $\nu$. If for each $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ and $t \in E^{\Lambda^C}$ there is a uniform rate of convergence for all the series*

$$H_\Lambda(\Phi)(\omega, t) : \ \omega \in E^\Lambda,$$

*then for each $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ and $t \in E^{\Lambda^C}$, and $B \in \mathcal{E}^{\mathbb{Z}^d}$*

$$\eta_\Lambda^{\Phi,\nu}(B|\cdot t) = \lim_{n\to\infty} \frac{\int_{E^\Lambda} \mathbb{1}_B(\omega,t) e^{-\left(\hat{A}_\Phi\right)_n(\omega,t)} d\nu^\Lambda(\omega)}{\int_{E^\Lambda} e^{-\left(\hat{A}_\Phi\right)_n(\omega',t)} d\nu^\Lambda(\omega')}. \tag{3.3}$$

*Proof.* Fix $\Lambda$, $t$, and $B$ as above and examine the nth term of the sequence, assuming W.L.o.G. that $\Lambda_n \supseteq \Lambda$. Cancelling all factors $e^{-\Phi_{\Lambda'}}$ for which $\Lambda' \cap \Lambda = \emptyset$ yields

$$\frac{\displaystyle\int_{E^\Lambda} \mathbb{1}_B(\omega,t) e^{-\left(\hat{A}_\Phi\right)_n(\omega,t)} d\nu^\Lambda(\omega)}{\displaystyle\int_{E^\Lambda} e^{-\left(\hat{A}_\Phi\right)_n(\omega',t)} d\nu^\Lambda(\omega')} = \frac{\displaystyle\int_{E^\Lambda} \mathbb{1}_B(\omega,t) \exp\left(-\sum_{\lambda\in\Lambda_n} \sum_{\substack{\Lambda'\ni\circ\lambda,\\ \Lambda'\cap\Lambda\neq\emptyset}} \Phi_{\Lambda'}(\omega,t) d\nu^\Lambda(\omega)\right)}{\displaystyle\int_{E^\Lambda} \exp\left(-\sum_{\lambda\in\Lambda_n} \sum_{\substack{\Lambda'\ni\circ\lambda,\\ \Lambda'\cap\Lambda\neq\emptyset}} \Phi_{\Lambda'}(\omega,t)\right) d\nu^\Lambda(\omega)}. \tag{3.4}$$

Now in the exponent of both integrands we recognize a partial sum of $H_\Lambda(\Phi)$. With the assumed strength of convergence of the Hamiltonians, for every $\epsilon > 0$ there is an index $n_\epsilon$ so large that for every $n \geq n_\epsilon$ and $\omega \in E^\Lambda$

$$\left| H_\Lambda(\Phi)(\omega,t) - \sum_{\lambda\in\Lambda_n} \sum_{\substack{\Lambda'\ni\circ\lambda,\\ \Lambda'\cap\Lambda\neq\emptyset}} \Phi_{\Lambda'}(\omega,t) \right| < \epsilon$$

and hence, applying the mean value theorem to $e^x$ to get

$$|e^a - e^b| < |a-b|e^{|a-b|}e^{\max(a,b)} < |a-b|e^{2|a-b|}e^a,$$

we can conclude

$$\left| e^{-H_\Lambda(\Phi)(\omega,t)} - \exp\left( - \sum_{\lambda\in\Lambda_n} \sum_{\substack{\Lambda'\ni\circ\lambda,\\ \Lambda'\cap\Lambda\neq\emptyset}} \Phi_{\Lambda'}(\omega,t) \right) \right| < \epsilon\, e^{2\epsilon}\, e^{-H_\Lambda(\Phi)(\omega,t)}.$$

Thus

$$\left| \int_{E^\Lambda} \mathbb{1}_B(\omega,t) e^{-H_\Lambda(\Phi)(\omega,t)} d\nu^\Lambda(\omega) - \int_{E^\Lambda} \mathbb{1}_B(\omega,t) \exp\left( - \sum_{\lambda\in\Lambda_n} \sum_{\substack{\Lambda'\ni\circ\lambda,\\ \Lambda'\cap\Lambda\neq\emptyset}} \Phi_{\Lambda'}(\omega,t) \right) d\nu^\Lambda(\omega) \right| <$$

$$< \epsilon\, e^{2\epsilon} \int_{E^\Lambda} \mathbb{1}_B(\omega,t) e^{-H_\Lambda(\omega,t)} d\nu^\Lambda(\omega)$$

$$\leq \epsilon\, e^{2\epsilon} Z^t_\Lambda(\Phi).$$

The denominator of the right hand side of equation 3.4 converges to $Z^t_\Lambda(\Phi)$ by the same analysis. Numerator and denominator converge, therefore the ratio converges to $\eta^{\Phi,\nu}_\Lambda(B|\cdot t)$, as was claimed. Note this includes the cases where $\eta^{\Phi,\nu}_\Lambda(B|\cdot t) = 0$. □

For the "averaged" local energy $A_\Phi$ we require slightly stronger convergence of the Hamiltonians, namely $\Phi \in \mathcal{S}$.

**Proposition 3.2.** *If $\Phi \in \mathcal{S}$ then for every finite a priori measure $\nu$ and every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$, $t \in E^{\Lambda^C}$, and $B \in \mathcal{E}^{\mathbb{Z}^d}$*

$$\eta^{\Phi,\nu}_\Lambda(B|\cdot t) \;=\; \lim_{n\to\infty} \frac{\int_{E^\Lambda} \mathbb{1}_B(\omega,t) e^{-(A_\Phi)_n(\omega,t)} d\nu^\Lambda(\omega)}{\int_{E^\Lambda} e^{-(A_\Phi)_n(\omega',t)} d\nu^\Lambda(\omega')}. \tag{3.5}$$

*Proof.* For any $n \geq 1$

$$\begin{aligned} (A_\Phi)_{\Lambda_n} &\equiv \sum_{\lambda\in\Lambda_n} A_\Phi \circ T^\lambda \\ \text{(by invariance of } \Phi) \quad &= \sum_{\lambda\in\Lambda_n} \sum_{\Lambda'\ni 0} |\Lambda'|^{-1} \Phi_{\Lambda'+\lambda} \\ &= \sum_{\lambda\in\Lambda'} \sum_{\Lambda'\ni\lambda} |\Lambda'|^{-1} \Phi_{\Lambda'} \\ &= \sum_{\Lambda'\cap\Lambda_n\neq\emptyset} \frac{|\Lambda'\cap\Lambda_n|}{|\Lambda'|} \Phi_{\Lambda'}. \qquad (3.6) \end{aligned}$$

For every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$, $t \in E^{\Lambda^C}$, $B \in \mathcal{E}^{\mathbb{Z}^d}$, and $n$ so large that $\Lambda_n \supseteq \Lambda$,
(3.7)

$$\frac{\int_{E^\Lambda} 1\!\!1_B(\omega,t)\exp\left(-A_\Phi\right)_n(\omega,t)\; d\nu^\Lambda(\omega)}{\int \exp\left(-A_\Phi\right)_m(\omega',t)\; d\nu^\Lambda(\omega')} = \frac{\int_{E^\Lambda} 1\!\!1_B(\omega,t)\exp\left(\sum_{\Lambda'\cap\Lambda_n\neq\emptyset} \frac{|\Lambda'\cap\Lambda_n|}{|\Lambda'|}\Phi_{\Lambda'}(\omega,t)\right) d\nu^\Lambda(\omega)}{\int \exp\left(-\sum_{\Lambda'\cap\Lambda_n\neq\emptyset} \frac{|\Lambda'\cap\Lambda_n|}{|\Lambda'|}\Phi_{\Lambda'}(\omega',t)\right) d\nu^\Lambda(\omega')}.$$

Again the cancellation from the first line to the second of those factors which intersect $\Lambda_n$ but not $\Lambda$ reflects the fact that if $\Lambda'\cap\Lambda = \emptyset$ then $\Phi_{\Lambda'}(\omega,x) = \Phi_{\Lambda'}(\omega',x)$ holds for every $\omega' \in E^\Lambda$. For $\Phi \in \mathcal{S}$ all the Hamiltonians converge in $\|\ \|_\infty$ and so

$$\left\| H_\Lambda(\Phi) - \sum_{\Lambda'\cap\Lambda_n\neq\emptyset} \frac{|\Lambda'\cap\Lambda_n|}{|\Lambda'|}\Phi_{\Lambda'} \right\|_\infty \leq \sum_{\substack{\Lambda'\cap\Lambda_n\neq\emptyset,\\ \Lambda'\cap\Lambda_n^C\neq\emptyset}} \frac{|\Lambda'\cap\Lambda_n|}{|\Lambda'|}\|\Phi_{\Lambda'}\|_\infty \to 0$$

as $n \to \infty$. So for any $\epsilon > 0$ choose $n$ so large that the preceding norm is less than $\epsilon$. Applying the mean value theorem to $e^x$ in the same way as before,

$$\left| \int_{E^\Lambda} 1\!\!1_B(\omega,t)e^{-H_\Lambda(\omega,t)}d\nu^\Lambda(\omega) - \int_{E^\Lambda} 1\!\!1_B(\omega,t)\exp\left(\sum_{\Lambda'\cap\Lambda_n\neq\emptyset} \frac{|\Lambda'\cap\Lambda_n|}{|\Lambda'|}\Phi_{\Lambda'}(\omega,t)\right) d\nu^\Lambda(\omega) \right|$$
$$\leq \epsilon\, e^\epsilon\, e^{\|H_\Lambda(\Phi)\|_\infty}\nu(E).$$

The same argument shows that the denominator in the right hand of equation 3.7 converges to $Z^t_\Lambda(\Phi)$, and so the ratio converges to $\eta^{\Phi,\nu}_\Lambda(B|\cdot t)$.

□

The question remains: how can an arbitrary regular, exp-summable (and therefore unbounded) function $f$ be related to a local energy $A_\Phi$ or $\hat{A}_\Phi$ for some interaction $\Phi$? The answer was suggested to the author by H.O. Georgii. First of all let

$$\tilde{f}(x) = f(x) - \sup f|_{[x_0]}; \tag{3.8}$$

a bounded version of $f$. Then define the finite a priori measure $\nu$ on $\mathbb{N}$ by

$$\nu(i) = e^{\sup f|_{[i]}}. \tag{3.9}$$

For the sake of clarity we'll flesh out the details (in $\mathbb{N}^{\mathbb{Z}^d}$) of Ruelle's aforementioned construction of $\Phi \in \hat{A}^{-1}(\tilde{f})$.

**Fact 3.3.** *For every $g \in BUC(\mathbb{N}^{\mathbb{Z}^d})$ there is an interaction $\Phi \in \mathcal{B}$ with $\hat{A}_\Phi = g$. If $g$ additionally satisfies the strong regularity property*

$$\sum_{n=1}^{\infty} n^d\,\delta_n(g) < \infty, \tag{3.10}$$

*then in fact $\Phi \in \mathcal{S}$.*

*Proof.* First we need to express $g$ as an absolutely, uniformly convergent series whose terms are all local functions. A standard choice could be

$$\begin{aligned} g(x) &= \inf g|_{[x_0]} + \Big(\inf g|_{[x|_{\Lambda_2}]} - \inf g|_{[x_0]}\Big) + \Big(\inf g|_{[x|_{\Lambda_3}]} - \inf g|_{[x|_{\Lambda_2}]}\Big) + \dots \\ &\equiv \underline{g}_1 + \underline{g}_2 + \underline{g}_3 + \dots \end{aligned}$$

Notice $\|\underline{g}_n\|_\infty < \delta_{n-1}(g)$, so regularity of $g$ implies convergence of $\sum_{n=1}^{\infty} \|\underline{g}_n\|_\infty$. Now for each $n \geq 1$ define the local interaction $\Phi^n$ by

$$\Phi^n_{\Lambda_n} = \underline{g}_n; \quad \Phi^n_{\Lambda_n+\lambda} = \underline{g}_n \circ T^\lambda \ \forall\ \lambda \in \Lambda_n,$$

and

$$\Phi^n_{\Lambda'} = 0$$

for all other $\Lambda' \in \mathcal{P}_F(\mathbb{Z}^d)$. Now $\Phi^n$ has the following properties

$$\hat{A}_{\Phi^n} = \Phi^n_{\Lambda_n} = \underline{g}_n,$$

$$\|\Phi^n\|_B = \|\underline{g}_n\|_\infty,$$

and

$$\|\Phi^n\|_S = |\Lambda_n| \|\underline{g}_n\|_\infty.$$

Thus, by continuity and linearity of the map $\hat{A}$, we conclude

$$\mathcal{B} \ni \Phi \equiv \sum_{n=1}^{\infty} \Phi^n,$$

$$\hat{A}_\Phi = g,$$

and

$$\|\Phi\|_S \leq \sum_{n=1}^{\infty} \|\Phi^n\|_S \leq \sum_{n=1}^{\infty} |\Lambda_n| \delta_{n-1}(g).$$

Given the strong regularity expressed in Equation 3.10 and the fact $|\Lambda_n| = O(n^d)$, this last sum clearly converges. □

Let us refer to the strong regularity defined in Equation 3.10 as *d-regularity.* Now we can prove the promised compatibility of our notion of a Gibbs measure for a regular,exp-summable function of $\mathbb{N}^{\mathbb{Z}^d}$ with the standard (physical) interaction-Hamiltonian notion of Gibbs measure as defined by the DLR equations 3.2.

**Theorem 3.4.** *If $f$ is d-regular and exp-summable, $\tilde{f}$ and $\nu$ are as defined in equations 3.8 and 3.9, respectively, $\Phi$ is construction of Fact 3.3 applied to $-\tilde{f}$, (so that $\hat{A}_\Phi = -\tilde{f}$,) then a Borel probability measure $\mu$ satisfies the interaction DLR equations* 3.2 *if and only if it satisfies the local energy DLR equations 2.1. Hence $\mu \in \mathfrak{G}_{\Phi,\nu}$ if and only if $\mu \in$ G.S.$(f)$.*

*Proof.* Theorem 2.10 says $\mu \in$ G.S.$(f)$ if and only if it satisfies the local energy DLR equations 2.1. Proposition 3.1 says that $\mu$ is a Gibbs measure (in the sense of [5]) for the interaction $\Phi$ if and only if $\mu$ obeys the equations, $\forall\, \Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ $\forall\, B \in \text{Borel}\left(\mathbb{N}^{\mathbb{Z}^d}\right)$, and $\mu$-a.e. $x \in \mathbb{N}^{\mathbb{Z}^d}$

$$\mu(B|\mathcal{A}_{\Lambda^C})(x) = \lim_{m\to\infty} \frac{\displaystyle\sum_{\omega\in\mathbb{N}^\Lambda} \mathbb{1}_B\left(\omega, x|_{\Lambda^C}\right) \exp\left(-(\hat{A}_\Phi)_m(\omega, x|_{\Lambda^C})\right)\, \nu([\omega])}{\displaystyle\sum_{\xi\in\mathbb{N}^\Lambda} \exp\left(-(\hat{A}_\Phi)_m(\xi, x|_{\Lambda^C})\right)\, \nu([\xi])}.$$

But

$$\frac{\displaystyle\sum_{\omega\in\mathbb{N}^\Lambda} \mathbb{1}_B(\omega, x|_{\Lambda^C}) \exp\left(-(A_\Phi)_m(\omega, x|_{\Lambda^C})\right)\, \nu([\omega])}{\displaystyle\sum_{\xi\in\mathbb{N}^\Lambda} \exp\left(-(A_\Phi)_m(\xi, x|_{\Lambda^C})\right)\, \nu([\xi])} = \frac{\displaystyle\sum_{\omega\in\mathbb{N}^\Lambda} \mathbb{1}_B(\omega, x|_{\Lambda^C}) \exp\left(\tilde{f}_m(\omega, x|_{\Lambda^C})\right)\, \nu([\omega])}{\displaystyle\sum_{\xi\in\mathbb{N}^\Lambda} \exp\left(\tilde{f}_m(\xi, x|_{\Lambda^C})\right)\, \nu([\xi])}$$

$$= \frac{\displaystyle\sum_{\omega\in\mathbb{N}^\Lambda} \mathbb{1}_B(\omega, x|_{\Lambda^C}) \exp\left(f_m(\omega, x|_{\Lambda^C}) - \sum_{\lambda\in\Lambda_m} \sup f|_{[(\omega, x|_{\Lambda^C})_\lambda]}\right) \exp\left(\sum_{\lambda\in\Lambda} \sup f|_{[\omega_\lambda]}\right)}{\displaystyle\sum_{\xi\in\mathbb{N}^\Lambda} \exp\left(f_m(\xi, x|_{\Lambda^C}) - \sum_{\lambda\in\Lambda_m} \sup f|_{[(\xi, x|_{\Lambda^C})_\lambda]}\right) \exp\left(\sum_{\lambda\in\Lambda} \sup f|_{[\xi_\lambda}\right)}.$$

For $m$ so large that $\Lambda \subseteq \Lambda_m$, for sites $\lambda \in \Lambda_m \setminus \Lambda$, the factors $\exp\left(\sup f|_{[(\omega, x|_{\Lambda^C})_\lambda]}\right)$ in the numerator cancel with the factors $\exp\left(\sup f|_{[(\xi, x|_{\Lambda^C})_\lambda]}\right)$ in the denominator. For sites $\lambda \in \Lambda$, the factors $\exp\left(-\sup f|_{[(\omega, x|_{\Lambda^C})_\lambda]}\right)$ and $\exp\left(\sup f|_{[\omega_\lambda]}\right)$ cancel in the numerator and the factors $\exp\left(-\sup f|_{[(\xi, x|_{\Lambda^C})_\lambda]}\right)$ and $\exp\left(\sup f|_{[\xi_\lambda]}\right)$ cancel in the denominator. Thus we conclude $\mu$ satisfies the DLR equations 3.2 for $\Phi$ if and only if it obeys the equations, $\forall\, \Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ $\forall\, B \in \text{Borel}\left(\mathbb{N}^{\mathbb{Z}^d}\right)$, and $\mu$-a.e. $x \in \mathbb{N}^{\mathbb{Z}^d}$

$$\mu(B|\mathcal{A}_{\Lambda^C})(x) = \lim_{m\to\infty} \frac{\displaystyle\sum_{\omega\in\mathbb{N}^\Lambda} \mathbb{1}_B(\omega, x|_{\Lambda^C}) \exp\left(f_m(\omega, x|_{\Lambda^C})\right)}{\displaystyle\sum_{\xi\in\mathbb{N}^\Lambda} \exp\left(f_m(\xi, x|_{\Lambda^C})\right)}. \tag{3.11}$$

In particular, letting $B = [\omega]$ for $\omega \in \mathbb{N}^\Lambda$ in equations 3.11 and referencing Theorem 2.10 shows that if $\mu \in \mathfrak{G}_{\Phi,\nu}$, then $\mu \in \text{G.S.}(f)$. At the same time it shows that if $\mu \in \text{G.S.}(f)$ then for every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ the class of sets

$$\left\{S \in \text{Borel}\left(\mathbb{N}^{\mathbb{Z}^d}\right) \,:\, \mu(S|\mathcal{A}_{\Lambda^C})(x) = \eta_\Lambda(S|x) \text{ for } \mu-\text{a.e.}x \in \mathbb{N}^{\mathbb{Z}^d}\right\}$$

includes all cylinders $[\omega]$ where $\omega \in \mathbb{N}^\Lambda$. Just as we argued in Observation 2.12, the basic properties of conditional expectation combined with the Lebesgue dominated convergence theorem imply that this class actually includes all the Borel sets of $\mathbb{N}^{\mathbb{Z}^d}$. Thus $\text{G.S.}(f) \subseteq \mathfrak{G}_{\Phi,\nu}$, and the theorem is proved. □

## REFERENCES

[1] Bowen, R.: Equilibrium states and the ergodic theory of Anosov Diffeomorphisms. Springer-Verlag, Berlin;New York (1975)
[2] Bowen, R.: Some systems with unique equilibrium states. Math. Systems Theory 8, 193-202 (1974)
[3] Capocaccia, D.: A Definition of Gibbs state for a compact set with $\mathbb{Z}^n$ action. Commun. Math. Phys. 48, 85-88 (1976)
[4] Cohn, D.: Measure Theory. Birkhauser, Boston (1980)
[5] Georgii, H-O.: Gibbs Measures and Phase Transitions. DeGruyter, (1981)
[6] Haydn, N.T.A., Ruelle, D.: Equivalence of Gibbs and equilibrium states for homeomorphisms satisfying expansiveness and specification. Commun. Math. Phys. 148, 155-167 (1992)
[7] Israel, Robert B.: Convexity in the Theory of Lattice Gases. Princeton Series in Physics, Princeton Univ. Press, Princeton, N. J. (1979)
[8] Keller, G.: Equilibrium States in Ergodic Theory. London Mathematical Society Student Texts 42. Cambridge University Press, Cambridge (1998)
[9] Mauldin, R.D., Urbański, M.: Gibbs states on the symbolic space over an infinite alphabet, Israel. J. of Math., 125 (2001), 93-130.
[10] Mauldin, R.D., Urbański, M.: Graph directed Markov systems: geometry and dynamics of limit sets.Cambridge University Press, Cambridge (2003)
[11] Parthasarathy, K.R. Introduction to probability and measure. Hindustan Book Agency (January 2005)
[12] Ruelle, D.: Statistical mechanics: rigorous results. W.A. Benjamin, New York (1969)
[13] Ruelle, D.: Thermodynamical formalism: the mathematical structures of classical equilibrium statistical mechanics. Addison-Wesley Pub. Co., Advanced Book Program, Reading, Mass. (1978)
[14] Sarig, O.: Thermodynamic Formalism for Countable Markov shifts. Erg. Th. Dyn. Sys. 19, 1565-1593 (1999)
[15] Simon, B.: The Statistical Mechanics of Lattice Gases, Vol. I. Princeton University Press, Princeton (1993)
[16] Fernandez, R., Sokal, A., VanEnter, A.C.D: J. Stat. Phys. **72**: 891-959 (1993)

STEPHEN MUIR
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF NORTH TEXAS,
P.O. BOX 311430,
DENTON, TX 76203-1430,
USA
*E-mail address*: `srm0070@unt.edu Web: www.math.unt.edu/∼srm0070`