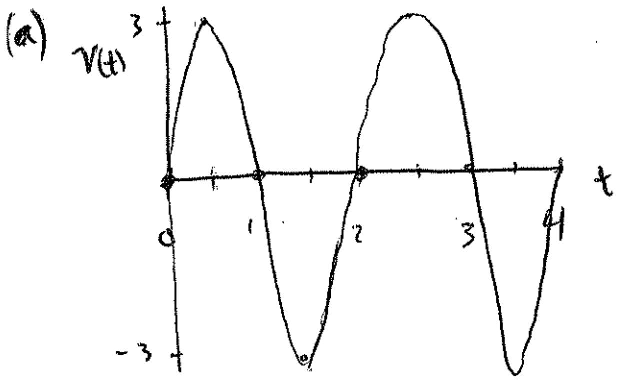
Answer Key | §6.1 # 12, 20, 22, 44
§6.2 # 11, 16, 28, 34, 52, 60

§6.1

#12. $v(t) = 3\sin(\pi t)$; $0 \le t \le 4$; $s(0) = 1$.

(a)



Moving in pos. direction for t in $(0,1)$, $(2,3)$;
moving in neg. direction for t in $(1,2)$, $(3,4)$.

(b) • Displacement on $[0,t]$ is $\int_0^t 3\sin(\pi\tau)\,d\tau = \int_0^t v(\tau)\,d\tau =$ $\left(\begin{array}{l}\text{let } u = \pi\tau \\ du = \pi\, d\tau\end{array}\right)$

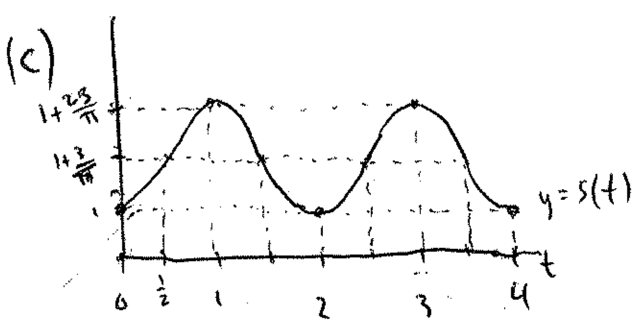
$= 3\int_0^t \sin(u)\cdot\frac{1}{\pi}du = \frac{3}{\pi}\Big[-\cos u\Big]_0^t = \frac{3}{\pi}\Big[-\cos(\pi t) + \cos(0)\Big]$;

therefore position at time t is $s(t) = s(0) + \text{displacement}$

(By F.T.C. $s(t) - s(0) = \int_0^t v(\tau)\,d\tau$.) $= 1 + \frac{3}{\pi}\big(1 - \cos(\pi t)\big)$.

• Antiderivative of velocity is $-\frac{3}{\pi}\cos(\pi t) + C$; solving initial value problem gives $-\frac{3}{\pi}\cdot 1 + C = 1$; so $C = 1 + \frac{3}{\pi}$ and $s(t) = 1 + \frac{3}{\pi} - \frac{3}{\pi}\cos(\pi t)$.

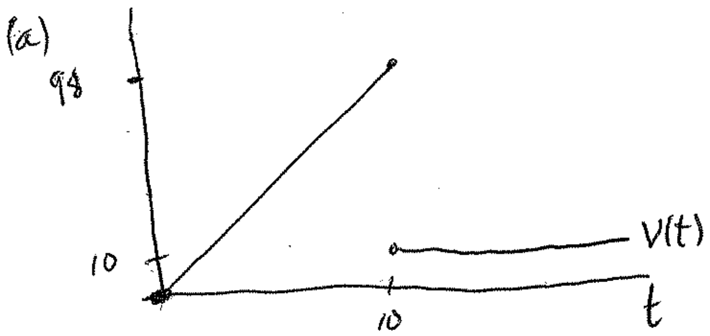
(c)



#20. $v(t) = 9.8t$ m/s for $0 \le t \le 10$ s

$v(t) = 10$ m/s for $t > 10$ s.

(a)



(b) Displacement in 1st 30s is $\displaystyle\int_0^{30} v(t)\,dt = \underbrace{\tfrac{1}{2}\cdot 10\cdot 98}_{\int_0^{10} v(t)\,dt} + \underbrace{20\cdot 10}_{\int_{10}^{30} v(t)\,dt}$

$= 10(49+20) = 690$ meters.

(c) Displacement after an arbitrary time $t_1 > 10$ is

$\displaystyle\int_0^{10} v(t)\,dt + \int_{10}^{t_1} v(t)\,dt = 490 + 10\cdot t_1$ meters.

Set equal to $3\text{km} = 3000\,\text{m}$, solve for t_1 : $3000 = 490 + 10t_1 \longrightarrow$

$\longrightarrow$ so $t_1 = \dfrac{2510}{10} = 251$ seconds.

#22. Solve for $v(t)$ and $s(t)$, given that $a(t) = 20 - 4t$, $v(0) = 60$, $s(0) = 40$.

First by antiderivative $v(t) = 20t - 2t^2 + C_1$, and by initial value

$60 = C_1$; thus $v(t) = 20t - 2t^2 + 60$. Then by antiderivative

$s(t) = 10t^2 - \frac{2}{3}t^3 + 60t + C_2$, and by initial value $40 = C_2$;

thus $s(t) = \frac{-2}{3}t^3 + 10t^2 + 60t + 40$.

§6.1

#44. Let $v(t) = 2\sin t$ for $0 \le t \le \pi$.

The distance traveled by the object is $\int_0^\pi |2\sin t|\,dt = \int_0^\pi |v(t)|\,dt =$

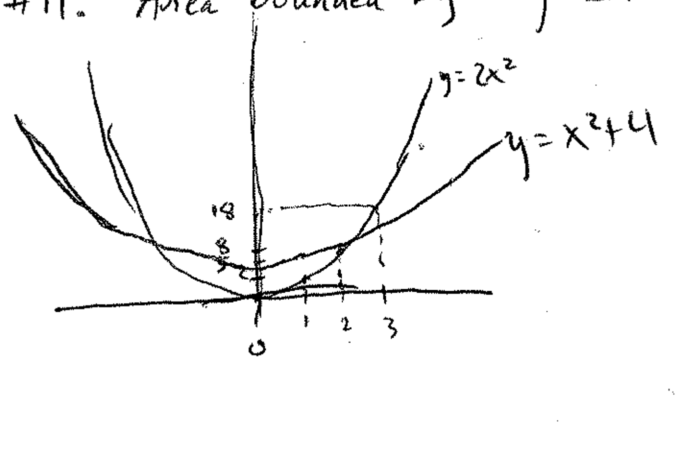
$= \int_0^\pi 2\sin t\,dt = 2\left[-\cos t\right]_0^\pi = 2\left[-(-1)-(-1)\right] = 4.$

(since $\sin t$ is positive on $[0,\pi]$, anyway)

The average speed is $\dfrac{\text{distance traveled}}{\text{time elapsed}} = \dfrac{4}{\pi}$. Thus the same distance could have been traveled over the same time interval at a constant velocity of $\frac{4}{\pi}$.

§6.2

#11. Area bounded by $y = 2x^2$ and $y = x^2 + 4$



$$\text{Area} = \int_{-2}^{2} x^2 + 4 - 2x^2\,dx$$

$$= 2\int_0^2 x^2 + 4 - 2x^2\,dx =$$

$$= 2\int_0^2 -x^2 + 4\,dx = 2\left[-\frac{x^3}{3} + 4x\right]_0^2 = 2\left[-\frac{8}{3} + 8 - 0\right] = \frac{32}{3}.$$

#16. Area $= \int_0^{\pi/3} \sin(2x) - \sin(x)\, dx + \int_{\pi/3}^{\pi} \sin(x) - \sin(2x)\, dx$

$= \left[-\frac{1}{2}\cos(2x) + \cos x\right]_0^{\pi/3} + \left[-\cos(x) + \frac{1}{2}\cos(2x)\right]_{\pi/3}^{\pi}$

$= \left(-\frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{2}\right) - \left(-\frac{1}{2} + 1\right) + \left(-(-1) + \frac{1}{2}\right) - \left(-\frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\right)$

$= \frac{3}{4} - \frac{1}{2} + \frac{3}{2} + \frac{3}{4} = 2\frac{1}{2}$. (i.e. 2.5).

(Here it was necessary to first solve for the intersection point

x: $\sin(x) = \sin(2x) = 2\sin(x)\cos(x)$

i.e.: $1 = 2\cos x$

$x = \frac{\pi}{3}$ (only solution in $[0, \pi]$).)

#28. The intersection point x: $2x - 8 = x^2 - 4x$

$x^2 - 6x + 8 = 0$

$(x-4)(x-2) = 0$

$x = 4$ or $x = 2$.

(a) $\int_{x=0}^{2} 0 - (x^2 - 4x)\, dx + \int_{x=2}^{4} 0 - (2x - 8)\, dx$.

(b) Invert: $y = 2x - 8 \iff x = \frac{y+8}{2}$.

#28. (b) continued:

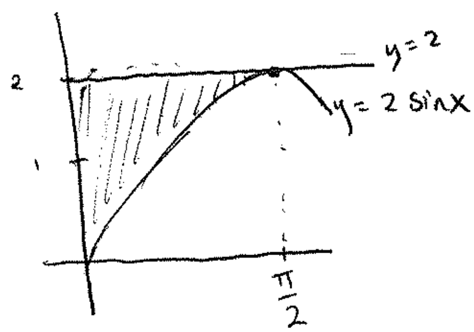
Invert: $y = x^2 - 4x \iff x^2 - 4x - y = 0$

$$x = \frac{4 \pm \sqrt{16 + 4y}}{2} = 2 \pm \sqrt{4+y}.$$

$$\text{Area} = \int_{y=-4}^{0} \frac{y+8}{2} - \left(2 - \sqrt{4+y}\right)\, dy.$$

$y=-4$: y-coordinate of intersection point

#34.



$$\text{Area} = \int_0^{\pi/2} 2 - 2\sin x \, dx =$$

$$= \left[2x + 2\cos x\right]_0^{\pi/2}$$

$$= (\pi + 0) - (0 + 2) = \pi - 2.$$

#52. Intersection of $y = 2x^2$ and $y = -4x + 6$ is

x: $2x^2 = -4x + 6$ i.e. $2x^2 + 4x - 6 = 0$

$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$
$$x = 1.$$

Intersection of $y = 4\sqrt{2x}$ and $y = -4x + 6$ is

x: $4\sqrt{2x} = -4x + 6$ i.e. $2\sqrt{2x} = -2x + 3$

$$8x = 4x^2 - 12x + 9$$
$$4x^2 - 20x + 9 = 0 \qquad x = \tfrac{1}{2} \text{ or } \cancel{\tfrac{9}{2}}.$$
$$(2x-9)(2x-1) = 0$$

Intersection of $y = 4\sqrt{2x}$ and $y = 2x^2$ is

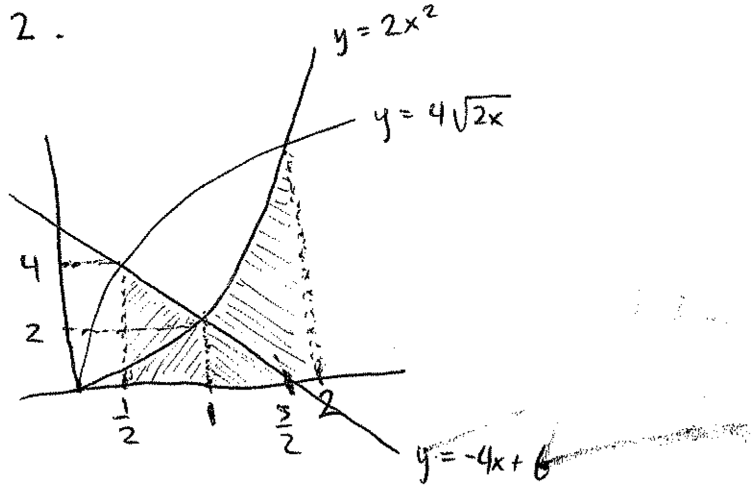
x: $4\sqrt{2x} = 2x^2$ i.e. $2\sqrt{2x} = x^2$

$$8x = x^4$$

(since $x \neq 0$) $8 = x^3$

$$x = 2.$$

Thus the picture is this



(by geometry, area of trapezoid, $= \int_{1/2}^{1} (-4x+6)\,dx$.)

$$\text{Area} = \int_{1/2}^{2} 4\sqrt{2x}\,dx - \frac{3}{2} - \int_{1}^{2} 2x^2\,dx$$

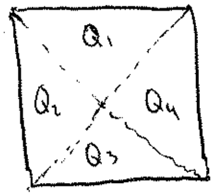
$$= 4\sqrt{2}\left[\frac{2}{3}x^{3/2}\right]_{1/2}^{2} - 2\left[\frac{1}{3}x^3\right]_{1}^{2} - \frac{3}{2}$$

$$= \frac{8\sqrt{2}}{3}\left(2\sqrt{2} - \frac{1}{2\sqrt{2}}\right) - \frac{2}{3}(8-1) - \frac{3}{2}$$

$$= \frac{8}{3}\left(4 - \frac{1}{2}\right) - \frac{14}{3} - \frac{3}{2} = \frac{8}{3}\cdot\frac{7}{2} - \frac{14}{3} - \frac{3}{2} = \frac{28-14}{3} - \frac{3}{2} =$$

$$= \frac{14}{3} - \frac{3}{2} = \frac{28-9}{6} = \frac{19}{6}.$$

#60. (a)

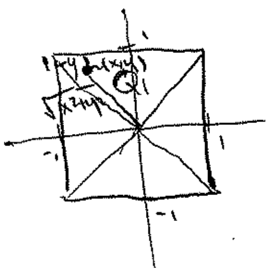


In each of the 4 quadrants, the areas closer to the center and closer to the edge, respectively, are the same.

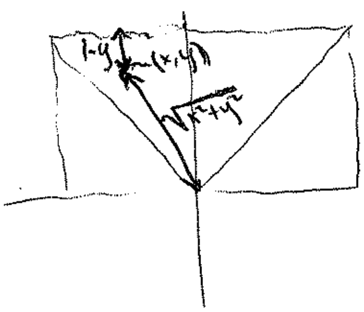
If a point in Q_1 is closer to the edge than the center then it must be closer to the top edge (not the left or right edge).

Thus the area closer to the edge than the center throughout the whole board is 4 · the area closer to the top edge than the center in Q_1.

(b) In Q_1

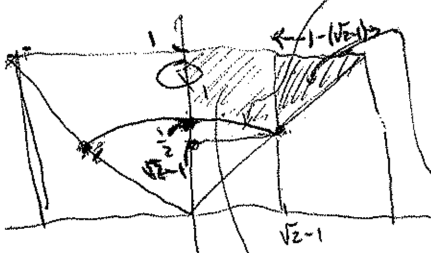


the points (x,y) that are equidistant from the top edge and the center satisfy $1-y = \sqrt{x^2+y^2}$



$$1-2y+\cancel{y^2} = x^2+\cancel{y^2}$$

$$y = \frac{x^2-1}{-2} = \frac{1-x^2}{2}. \quad \left(\text{This is the curve } C.\right)$$



Intersection of $x = \frac{1-x^2}{2}$ is at $x = \sqrt{2}-1$.

Area of triangle $= \frac{1}{2}\left(1-(\sqrt{2}-1)\right)^2 = \frac{1}{2}(2-\sqrt{2})^2$.

(c)

Area of  $= \int_{x=0}^{1/2} 1-\left(\frac{1-x^2}{2}\right)\,dx =$

$$= \int_0^{1/2} 1-\frac{1}{2}+\frac{1}{2}x^2\,dx = \frac{1}{2}\int_0^{1/2} 1+x^2\,dx = \frac{1}{2}\left[x+\frac{x^3}{3}\right]_0^{1/2} = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{24}\right) = \frac{13}{48}.$$

Total area above curve C in Q_1 is

$$2\cdot\left(\frac{1}{2}(2-\sqrt{2})^2 + \frac{13}{48}\right) = \frac{13}{24} + 6 - 4\sqrt{2} \approx .8848$$

Since the area of Q_1 is 1, this is also the ratio $\dfrac{\text{Area closer to edge than center in } Q_1}{\text{Total area in } Q_1}$,

and the ratio for the whole board is the same.

Thus the "probability" of hitting closer to the edge than the center is ≈ 0.8848.