Handbook of Set Theory

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Contents

I. Structural Consequences of AD

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1. Introduction

In this article we survey recent advances in descriptive set theory, starting (roughly) from where Moschovakis' book [32] ends. Our survey is not intended to be complete, but focuses mainly on the structural consequences of determinacy for the model $L(\mathbb{R})$, including the important case of the projective sets. By "structural" we are referring to the combinatorial theory of the pointclasses (for example, the scale property which in some sense describes the structure of the set) as well as the cardinal structure up to the natural ordinal associated with these pointclasses. This might include determining their cofinalities, partition properties. etc.

The axiom of determinacy, AD, is the assertion that every two-player integer game is determined; we review the basic concepts below. The axiom was introduced by Mycielski and Steinhaus in the 60's, and it soon became apparent that AD was a powerful tool for unlocking the combinatorial structure of sets of reals, and a program for doing this was begun. One of the central achievements during this period was the extension, assuming projective determinacy, to the general projective sets of the basic structural theory of the $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_1$ sets developed by the "classical" descriptive set theorists from the 20's through the 40's. [32] and [19] give a detailed account of the history of these developments. The results we discuss here can thus be viewed as extensions and refinements of the basic determinacy theory developed by the descriptive set-theorists of the 60's and 70's as described in [32], [19].

We note that more recent work of Martin, Steel, and Woodin [30, 41] (described in a little more detail below) has pinpointed the connection between large cardinal axioms and various levels of determinacy, including $\mathrm{AD}^{L(\mathbb{R})}$.

We work throughout this paper in the base theory $\mathrm{ZF} + \mathrm{DC}$ (except at a few points where we mention the use of DC). However, we will be dealing almost entirely in this paper with the consequences of AD. In particular we assume AD throughout §3–§5. In §2 we mention explicitly the hypotheses as needed. The reader should recall that AD contradicts choice as well as many of its consequences. In particular, successor cardinals need not be regular and measurable cardinals need not be limit cardinals.

There were two ways in which the earlier theory was inadequate. First, the theory of the projective sets was described largely in terms of the so-called projective ordinals, the $\boldsymbol{\delta}^1_n$. The first four of these were computed, and several general results were proved (a good reference here is [17]). Kunen initiated a program for computing the $\boldsymbol{\delta}^1_n$. The idea of a homogeneous tree, which plays a central part in the program, arose independently in the work of Kunen and Martin. Kechris and Martin independently (see [18]) then formulated the general notion of a homogeneous tree. Despite this important progress, however, the values of $\boldsymbol{\delta}^1_n$ for $n \geq 5$, and the combinatorial structure of the intervening cardinals were left open. Secondly, the projective sets represent only the sets of reals occurring in the first level of the $L(\mathbb{R})$ hierarchy. It is generally believed, however, that AD should suffice to develop the structural theory (in some reasonable sense) of the entire model $L(\mathbb{R})$. Thus, two central goals emerge: first, to refine the arguments from the projective sets to permit a detailed analysis of the cardinal structure within the projective ordinals, and secondly, to extend this analysis as far as possible.

In the early 80's, Martin and Steel [29] showed that $(\boldsymbol{\Sigma}^2_1)^{L(\mathbb{R})}$ was the largest pointclass in $L(\mathbb{R})$ having the scale property. Extending their methods, Steel [37] developed an analysis of the scale property in $L(\mathbb{R})$. This important work can be thought of as extending both the scale analysis of the projective sets and also the "fine-structure" theory of L (developed by Jensen), and uses methods of both. This fine-structure theory of $L(\mathbb{R})$, like the earlier theory of the projective sets, is not detailed enough to analyze the fine combinatorial structure of the cardinals, nor to answer many questions about the model $L(\mathbb{R})$ (though it suffices to answer many "scale type" questions, e.g., showing which pointclasses are the κ-Suslin sets for some κ, or showing every reliable cardinal is a Suslin cardinal; see [37]).

In the early 80's, Martin [27] obtained a result on the ultrapowers of $\boldsymbol{\delta}^1_3$ by the normal measures on $\boldsymbol{\delta}^1_3$. Building on this and some joint work with Martin, the author computed $\boldsymbol{\delta}^1_5$. In the mid–80's, this was extended to compute all the $\boldsymbol{\delta}^1_n$, and to develop the combinatorics of the cardinal structure of the cardinals up to that point. The analysis, naturally, proceeded by induction. The complete "first-step" of the induction appears in [11]. The analysis revealed a rich combinatorial structure to these cardinals. Indeed, even the answer $\boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$ hints at such a structure (in general $\boldsymbol{\delta}^1_{2n+1} = \aleph_{\omega(2n-1)+1}$, where $\omega(0) = 1$ and $\omega(n+1) = \omega^{\omega(n)}$). A goal, then, is to extend some version of this "very-fine" structure theory to the entire model $L(\mathbb{R})$. In the late 80's the author extended the analysis further, up to the least inaccessible cardinal in $L(\mathbb{R})$, although this lengthy analysis has never been written up. It was clear, however, that new, serious problems were being encountered shortly past the least inaccessible. In [10], for example, results were given that show the theory fell far short of $\kappa^{\mathbb{R}}$, the ordinal of the inductive sets (the Wadge ordinal of the least non-selfdual pointclass closed under real quantification).

More recently, attempts have been made to isolate some of the "global" obstructions to extending the detailed $L(\mathbb{R})$ analysis. Some combinatorial principles were thus formulated which seem to be necessary for extending the theory sufficiently high in $L(\mathbb{R})$ and which seem not to be provable by an inductive "from below" argument. The most important principle along these lines is called the weak square property, $\boxminus_{\kappa,\lambda}$, for κ a Suslin cardinal and λ an ordinal $< \Theta$ (the supremum of the lengths of prewellorderings of the reals). Recently, the author has established this choice-like principle (to be defined in §6). This has a number of consequences. For example, it follows that every regular κ which is either a Suslin cardinal or the successor of a Suslin cardinal is $\boldsymbol{\delta}^2_1$ supercompact in $L(\mathbb{R})$. This and other global choice-like principles will be discussed in §6.

At the time of the present writing, then, we may summarize the current situation as follows. A detailed structural analysis for an initial segment of $L(\mathbb{R})$ including the projective sets is known. Certain choice-like principles which seem to be important for further extensions have been established globally (that is below Θ). What remains is to identify the further global principles required, and integrate them with the inductive analysis.

In the present paper we will survey both approaches mentioned above. We will adopt a somewhat informal style at times in order to keep this paper reasonably self-contained, still give proofs, and keep the discussion to a reasonable length. In §2 we will collect together and review various results from descriptive set theory and determinacy theory we will need. We will present proofs for some of these results of particular significance for us, and reference the others. In §3 we develop the AD theory of the Suslin cardinals, culminating in a classification theorem. In §4 we will develop a theory of "trivial" descriptions. This is not actually necessary, and in fact was omitted from [11]. Descriptions are the combinatorial ingredients that "describe" how to generate equivalence classes of functions with respect to certain measures. At the level of ω_1 (here the measures are simply the n-fold products of the normal measure), descriptions are trivial enough objects that they need not be introduced explicitly, but rather absorbed into the notation. Nevertheless, by introducing them explicitly the reader can see the general flavor of the arguments while the combinatorics is still trivial. Using this approach, we will show in this section the strong partition relation on ω_1, the weak partition relation on $\boldsymbol{\delta}^1_3$, and give a new proof of the Kechris-Martin theorem for $\boldsymbol{\Pi}^1_3$. In all cases, our proofs will use only the theory of the trivial descriptions and techniques that will generalize to higher levels (in particular, no use is made of the theory of indiscernibles for L). In §5 we will introduce the (non-trivial) notion of a description of level 1. This corresponds to the analysis for computing $\boldsymbol{\delta}^1_5$ and proving the strong partition relation on $\boldsymbol{\delta}^1_3$, etc. Although we state complete definitions and theorems, we will illustrate proofs here frequently by considering examples which show the reader the ideas involved without getting lost in details. In §6 we introduce $\boxminus_{\kappa,\lambda}$ and other global choice-like principles. Some of the results presented here are of interest in their own right, and several are new.

We hope §§4,5 will provide a good introduction to, and an overview of, the modern theory of the projective sets (that is, the developments since [32]), and §6 will give some insight into the problems being faced in extending the theory and their possible solutions.

Although the focus of this paper is on the consequences of AD, we mention briefly some connections with other hypotheses. Martin (see [24] and [25]) showed Borel determinacy is a theorem of ZFC, although H. Friedman [4] showed that $\aleph_1$ iterations of the power set axiom applied to the reals is needed to prove it. Martin also showed $\boldsymbol{\Pi}^1_1$ determinacy followed from the existence of $x^\#$ for every $x \in \omega^\omega$, and Harrington proved the converse. Martin also showed the curious result that $\boldsymbol{\Delta}^1_{2n}$ determinacy implies $\boldsymbol{\Sigma}^1_{2n}$ determinacy (see [15] for another proof). More recently, Martin and Steel [30] showed that the existence of n Woodin cardinals (see [30] for the definition) plus a larger measurable implies $\boldsymbol{\Delta}^1_{n+1}$-determinacy. Woodin showed that the existence of ω many Woodin cardinals plus a larger measurable implies $\mathrm{AD}^{L(\mathbb{R})}$. In fact, $\boldsymbol{\Delta}^1_{n+1}$-determinacy is equiconsistent with the existence of n Woodin cardinals, and AD is equiconsistent with the existence of ω many Woodin cardinals. Thus, the AD hypothesis now has the added measure of respect that $\mathrm{AD}^{L(\mathbb{R})}$ follows from more "conventional" large cardinal hypotheses. It should be noted, however, that the program of using AD to explore the structural theory of the projective sets and beyond was begun in the 60's, well before this connection was known.

We review now some notation and terminology that we will use throughout the paper. We generally work in the Polish space (complete, separable, metric space) ω^ω, the space of functions from ω to ω topologized with the product of the discrete topologies on ω. As a topological space this is homeomorphic to the space of irrationals, but any two uncountable Polish spaces are Borel isomorphic (in fact, isomorphic by a $\boldsymbol{\Delta}^0_3$-measurable function). We follow the usual convention of referring to the elements of ω^ω as "reals." By a *perfect product space* we mean a space of the form $X = X_1 \times \cdots \times X_n$, where each $X_i = \omega$ or ω^ω (ω always with the discrete topology), and at least one factor is ω^ω. All perfect product spaces are recursively homeomorphic to ω^ω by recursive coding and decoding maps, whose notation we now standardize.

For each $n \geq 2$, fix a recursive bijection

$$(m_0, \dots, m_{n-1}) \to \langle m_0, \dots, m_{n-1} \rangle_n$$

from ω^n to ω which is increasing in each argument, and let

$$m \to ((m)_0, \dots, (m)_{n-1})$$

denote the recursive inverse map. Let $(x_0, x_1, \dots) \to \langle x_0, x_1, \dots \rangle$ also denote the induced recursive bijection from $(\omega^\omega)^\omega$ to ω^ω defined by

$$\langle x_0, x_1, \dots \rangle(m) = x_{m_0}(m_1),$$

where (m_0, m_1) refers to the inverse of the coding map $\langle\ \rangle_2$. Similarly, for any perfect product space $X = X_1 \times \cdots \times X_{n-1}$, there is a recursive bijection between X and ω^ω, we will use the same notation $(x_0, \dots, x_{n-1}) \to \langle x_0, \dots, x_{n-1} \rangle_n$, and $x \to ((x)_0, \dots, (x)_{n-1})$, for the coding and decoding maps. Since the precise meaning is generally clear from the context, we will usually drop the subscripts and extra parentheses from the notation.

For X a perfect product space and $A \subseteq X$, we write A^c for $X - A$ (it will always be clear which X we are referring to). If $R \subseteq X \times Y$, the domain of R is defined by $\mathrm{dom}(R) = \{x \colon \exists y\ (x, y) \in R\}$. For any $x \in X$, we let R_x denote the section of R at x, that is, $R_x = \{y \colon (x, y) \in R\}$.

A (boldface) *pointclass* $\mathbf{\Gamma}$ is a collection of subsets of perfect product spaces X which is closed under continuous inverse image. That is, if $f \colon X \to Y$ is continuous and $A \subseteq Y$ is in $\mathbf{\Gamma}$, then $B = f^{-1}(A)$ is in $\mathbf{\Gamma}$. We also say A is *Wadge reducible* to B, written $A \leq_w B$. As is customary in descriptive set theory, we frequently use logical notation in describing sets, and thus we write $A(x)$ for $x \in A$. Thus we may rewrite the above as $B(x) \leftrightarrow A(f(x))$, and for this reason pointclasses are referred to as being closed under continuous substitution (or Wadge reduction). Likewise $\neg A(x)$ means $x \notin A$. For $\mathbf{\Gamma}$ a pointclass, $\check{\mathbf{\Gamma}}$ denotes the *dual* pointclass, that is, $A \in \check{\mathbf{\Gamma}}$ iff $A^c \in \mathbf{\Gamma}$. We say $\mathbf{\Gamma}$ is *non-selfdual* if $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$, and otherwise say $\mathbf{\Gamma}$ is *selfdual*. We let $\exists^\omega$ and $\exists^{\omega^\omega}$ denote existential quantification over ω and ω^ω respectively, and likewise for $\forall^\omega$ and $\forall^{\omega^\omega}$. We apply this notation also to pointclasses. For example, $\exists^{\omega^\omega}\mathbf{\Gamma}$ denotes the $A \subseteq X$ for which there is a $B \subseteq X \times \omega^\omega$ such that $\forall x\ (A(x) \leftrightarrow \exists y \in \omega^\omega\ B(x, y))$. For $\mathbf{\Gamma}$ a (usually non-selfdual) pointclass we let $\mathbf{\Delta}(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$. When $\mathbf{\Gamma}$ is understood we frequently just write $\mathbf{\Delta}$.

Let X, Y be perfect product spaces, and $A \subseteq X$, $B \subseteq Y$. Assuming AD, Wadge's lemma asserts that either $A \leq_w B$ or $B \leq_w A^c$. In fact, the proof shows something stronger. For example, suppose $X = Y = \omega^\omega$, and let $A, B \subseteq \omega^\omega$. Consider the integer game where I plays integers $a(i)$, and II plays $b(i)$, thereby producing reals $a, b \in \omega^\omega$, and II wins the run iff $(a \in A \leftrightarrow b \in B)$. If II has a winning strategy τ, then τ defines a

Lipschitz continuous function (which we also call τ) from ω^ω to ω^ω. By this we mean $\tau(a)\restriction n$ depends only on $a\restriction n$. Also, $a \in A \leftrightarrow \tau(a) \in B$. If I has a winning strategy σ, we likewise get a Lipschitz continuous function σ such that $b \in B \leftrightarrow \sigma(b) \in A^c$. If we let $\leq_l$ denote reduction by a Lipschitz continuous function, we therefore have either $A \leq_l B$ or $B \leq_l A^c$.

For X a Polish space, we let $\boldsymbol{\Sigma}^0_1$ (respectively $\boldsymbol{\Pi}^0_1$) denote the collection of open (respectively closed) subsets of X. For $\alpha < \omega_1$, we recursively define $\boldsymbol{\Sigma}^0_\alpha$ to the sets $A \subseteq X$ which are countable unions of sets A_n, each of which lies in $\boldsymbol{\Pi}^0_\beta$ for some $\beta < \alpha$. Also, $\boldsymbol{\Pi}^0_\alpha = \check{\boldsymbol{\Sigma}}^0_\alpha$, and $\boldsymbol{\Delta}^0_\alpha = \boldsymbol{\Sigma}^0_\alpha \cap \boldsymbol{\Pi}^0_\alpha$. The sets which are $\boldsymbol{\Sigma}^0_\alpha$ for some $\alpha < \omega_1$ (equivalently $\boldsymbol{\Pi}^0_\alpha$ for some $\alpha < \omega_1$) are the *Borel* sets. The *projective hierarchy* is defined as follows. The $\boldsymbol{\Sigma}^1_1$ (or *analytic*) sets are the sets which are continuous images of closed sets in Polish spaces. An equivalent definition is $\boldsymbol{\Sigma}^1_1 = \exists^{\omega^\omega}\boldsymbol{\Pi}^0_1$. Also, we define $\boldsymbol{\Pi}^1_1 = \check{\boldsymbol{\Sigma}}^1_1$, and $\boldsymbol{\Delta}^1_1 = \boldsymbol{\Sigma}^1_1 \cap \boldsymbol{\Pi}^1_1$. Suslin's theorem says that the $\boldsymbol{\Delta}^1_1$ sets are exactly the Borel sets. In general, we define $\boldsymbol{\Sigma}^1_n = \exists^{\omega^\omega}\boldsymbol{\Pi}^1_{n-1}$, $\boldsymbol{\Pi}^1_n = \check{\boldsymbol{\Sigma}}^1_n$, and $\boldsymbol{\Delta}^1_n = \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n$. The *lightface* projective classes, Σ^1_n, Π^1_n, Δ^1_n are defined an an analogous manner, except at the bottom we take Σ^0_1 to be the collection of sets of the form $\bigcup_{n\in\omega} V_{f(n)}$, where $f : \omega \to \omega$ is recursive, and $\{V_n\}_{n\in\omega}$ is a recursive presentation for the space X. See [32] for further details (we will always have X a perfect product space, in which case any reasonable enumeration of the usual basis for X will be a recursive presentation).

For s a finite sequence (*i.e.*, a function with domain $n \in \omega$), we let $\mathrm{lh}(s) = n$ denote the length of s. Thus, $s\restriction m$, for $m \leq n$, is the initial subsequence of s of length m.

By a tree on a set X we mean a set $T \subseteq X^{<\omega}$ closed under subsequence. Given sets X and Y, we frequently identify a tree T on the set $X \times Y$ with a set $T \subseteq \{(s,t) \in (X \times Y)^{<\omega} : \mathrm{lh}(s) = \mathrm{lh}(t)\}$ closed under initial segment, that is, if $(s,t) \in T$, then $(s\restriction m, t\restriction m) \in T$ for any $m < \mathrm{lh}(s)$. Similarly, we identify trees on $X_1 \times \cdots \times X_n$ with subsets of $X_1^{<\omega} \times \cdots \times X_n^{<\omega}$. $[T]$ denotes the set of paths or branches through T. We say a tree is well-founded iff it is well-founded under the extension relation, and otherwise say it is ill-founded. Thus, T is ill-founded iff $[T] \neq \emptyset$. If T is ill-founded, and X is equipped with a well-order $<_X$, then T has a *left-most* branch $f \in [T]$. That is, for all $g \in [T]$, if $f \neq g$, then for the least n such that $f(n) \neq g(n)$ we have $f(n) <_X g(n)$. For T a tree on $X \times Y$, and $x \in X^\omega$, we let $T_x = \{s \in Y^{<\omega} \colon (x\restriction \mathrm{lh}(s), s) \in T\}$ denote the section of the tree T at x. For T a tree on $X \times Y$, we let $p[T]$ denote the projection of $[T]$, that is,

$$p[T] = \{x \in X^\omega : \exists y \in Y^\omega\ \forall n\ (x\restriction n, y\restriction n) \in T\}.$$

If T is a tree on $\lambda_1 \times \cdots \times \lambda_n$, and $\beta \in \mathrm{ON}$, then we let

$$T\restriction\beta = \{(s_1,\dots,s_n) \in T \colon \forall m < \mathrm{lh}(s_1)\ (s_1(m),\dots,s_n(m) < \beta)\}.$$

If $(X, <_X)$ is a well-ordered set, then for any $n \in \omega$ the set X^n is well-ordered by the induced lexicographic ordering defined by:

$$s <_{\text{lex}} t \leftrightarrow \exists k(s(k) <_X t(k) \wedge \forall l < k \ s(l) = t(l)).$$

We let $|s|_{\text{lex}}$ denote the rank of $s \in X^n$ in the lexicographic ordering. If T is a tree on X, then the well-order $<_X$ also induces a linear order on T, known as the Kleene-Brouwer order, defined by:

$$s <_{\text{KB}} t \leftrightarrow (s \text{ extends } t) \text{ or } \exists k < \min\{\text{lh}(s), \text{lh}(t)\} \quad (s(k) <_X t(k) \wedge \forall l < k \ s(l) = t(l)).$$

The Kleene-Brouwer order on T is a well-ordering iff T is well-founded, that is, $[T] = \emptyset$. If T is a tree on X and $s \in T$, we let $T(s)$ denote $\{t \in T : t \text{ extends } s\}$. If T is a well-founded tree, we let $|T|$ denote the rank of T. In this case, we let $|T|_{\text{KB}}$ denote the rank of T in the Kleene-Brouwer order (this also implicitly depends on the well-order of X). We always have $|T| \leq |T|_{\text{KB}}$. Note that $|T(s)|$ is the rank of s in T, and we also denote this by $|s|_T$. Also, $|(T{\restriction}\alpha)(s)|$ denotes the rank of s in the tree $T{\restriction}\alpha$, and likewise for $|(T{\restriction}\alpha)(s)|_{\text{KB}}$. More generally, if $\prec$ is any well-founded relation on $\delta \in \text{ON}$, we let $\prec{\restriction}\alpha$ denote $\prec \cap \{(\beta, \gamma) : \beta, \gamma < \alpha\}$. Likewise, $\prec(\alpha) = \prec \cap \{(\beta, \gamma) : \beta, \gamma \prec \alpha\}$. We let $|\prec|$ denote the rank of $\prec$. Similarly, $|\prec(\alpha)| = |\alpha|_\prec$ both denote the rank of α in the well-founded relation $\prec$.

By a game on a set X we mean a two player game where players I and II alternate playing $x_0, x_1, \ldots \in X$ building $\vec{x} \in X^\omega$, and I wins the run iff $\vec{x} \in A$, where $A \subseteq X^\omega$ is the *payoff* set. Although the game is officially identified with its payoff set $A \subseteq X^\omega$, we sometimes write G_A to denote this game for conceptual clarity. A strategy for I (II) in a game on X is a function from the sequences from X of even (odd) length into X. The strategy is winning for I (II) if every run of the game where I (II) follows the strategy is a win for I (II). A game A is *determined* if one of the players has a winning strategy. AD is the assertion that every game on ω is determined. A *quasi-strategy* for I (II) is a relation $R \subseteq X^{<\omega} \times X$ such that $\forall s \in X^{<\omega} \ \exists x \in X \ R(s, x)$, and $\forall s \in X^{<\omega}$ of odd (even) length, $\forall x \in X \ R(s, x)$. We think of a quasi-strategy as a multi-valued strategy. A quasi-strategy is winning for I (II) if every run $\vec{x} \in X^\omega$ of the game such that $\forall n \ R(\vec{x}{\restriction}n, x(n))$ is a win for I (II). A game G_A is *quasi-determined* if one of the two players has a winning quasi-strategy. If X can be well-ordered (*e.g.*, $X = \omega$), then from a winning quasi-strategy for one of the players we easily get a winning strategy for the same player. If X is a perfect product space, Y is a set, and $A \subseteq X \times Y^\omega$, then $\Game^Y A$ is the set $B \subseteq X$ defined by $x \in B$ iff I has a winning strategy in the game $G_A(x)$ where I and II alternate playing $y_0, y_1, y_2, \dots \in Y$ producing $\vec{y} \in Y^\omega$, and I wins the run if $(x, \vec{y}) \in A$. We abbreviate this by writing $x \in B \leftrightarrow \exists y_0 \ \forall y_1 \ \exists y_2 \ \forall y_3 \cdots (x, \vec{y}) \in A$.

By a *measure* on a set X we mean a countably additive ultrafilter on X. Recall that assuming AD, every ultrafilter on a set X is countably additive, that is, a measure. If ν is a measure on X, and f is a function with domain X, we let $[f]_\nu$ denote the equivalence class of f in the ultrapower, that is, $f \sim g \leftrightarrow \nu(\{x \in X : f(x) = g(x)\}) = 1$. When considering ultrapowers by measures, we also let $[f]_\nu$ denote the image of $[f]_\nu$ in the transitive collapse of the ultrapower. We often say "A has measure one" in place of $\nu(A) = 1$. We also write $\forall^*_\nu x \in X$ to abbreviate "for ν measure one many $x \in X$."

We introduce a useful notational convention. Let $\nu_1, \dots, \nu_n$ be measures. If $P \subseteq \mathrm{ON}$ and $\delta \in \mathrm{ON}$, we let $\forall^*_{\nu_1}\alpha_1\ \forall^*_{\nu_2}\alpha_2\ \cdots \forall^*_{\nu_n}\alpha_n\ P(\delta(\alpha_1, \dots, \alpha_n))$ abbreviate the statement: if $[f_1]_{\nu_1} = \delta$ then for ν_1 almost all α_1, if $[f_2]_{\nu_2} = f_1(\alpha_1)$, then for ν_2 almost all α_2, if $[f_3]_{\nu_3} = f_2(\alpha_2)$, ..., for ν_n almost all α_n, $P(f_n(\alpha_n))$. It is easily checked that this statement is well-defined. We also extend this convention to properties of pairs of ordinals, etc. For example, given measures $\nu_1, \dots, \nu_n$ and ordinals δ, ϵ, we might write $\forall^*_{\nu_1}\alpha_1 \cdots \forall^*_{\nu_n}\alpha_n$ $\mathrm{cof}(\delta(\alpha_1, \dots, \alpha_n)) \leq \epsilon(\alpha_1, \dots, \alpha_n)$. If the measures are understood, we simply write $\forall^*\alpha_1, \dots, \alpha_n\ P(\delta(\alpha_1, \dots, \alpha_n))$, etc.

If $f \colon X \to \mathrm{ON}$ is a function and $\alpha \in \mathrm{ON}$, we write $N_f(\alpha)$ for the least ordinal in the range of f greater than α. Likewise, if $A \subseteq \mathrm{ON}$, we write $N_A(\alpha)$ for the least ordinal greater than α in A.

2. Survey of Basic Notions

Throughout §2 we work in the base theory ZF+DC, stating other hypotheses explicitly as needed.

2.1. Prewellordering, Scales, and Periodicity

We begin with a review of the basic concepts of scale and prewellordering. The definition of a scale was introduced by Moschovakis, and represents a distillation of the key ideas in the Novikov-Kondo proof of uniformization for $\boldsymbol{\Pi}^1_1$ sets.

Recall the basic notions of pointclass, Wadge reduction, *etc.*, that were defined in the introduction. If $\boldsymbol{\Gamma}$ is a pointclass, we say $A \subseteq \omega^\omega \times X$ in $\boldsymbol{\Gamma}$ is *universal* for $\boldsymbol{\Gamma}\restriction X$ (where $X = X_0 \times \cdots \times X_n$ is a perfect product space) if for every $B \subseteq X$ in $\boldsymbol{\Gamma}$, there is a $y \in \omega^\omega$ such that $B = U_y = \{x \colon (y, x) \in U\}$. Assuming AD, Wadge's lemma implies that every non-selfdual pointclass $\boldsymbol{\Gamma}$ has universal sets. For suppose $A \in \boldsymbol{\Gamma} - \check{\boldsymbol{\Gamma}}$. For every product space $X = X_0 \times \cdots \times X_n$, define $U_X \subseteq \omega^\omega \times X$ by $U_X(y, x_0, \cdots, x_n) \leftrightarrow f_y(\langle x_0, \cdots, x_n\rangle) \in A$ where we view every $y \in \omega^\omega$ as coding a Lipschitz continuous function $f_y \colon \omega^\omega \to \omega^\omega$ (say by $f_y(a_0, \cdots, a_n) = (y(\langle a_0\rangle), \dots, y(\langle a_0, \dots, a_n\rangle))$. Clearly $U_X \in \boldsymbol{\Gamma}$. If $B \subseteq X$ is in $\boldsymbol{\Gamma}$, then by Wadge $B \leq_l A$, so for some y we have $B = (U_X)_y$.

The usual diagonal argument shows that a universal $\boldsymbol{\Gamma}$ set $A \subseteq \omega^\omega \times \omega^\omega$ cannot lie in $\check{\boldsymbol{\Gamma}}$, and thus if $\boldsymbol{\Gamma}$ has a universal set, it is non-selfdual. Also, the *s-m-n* and recursion theorems go through at this level of generality. Specifically, we have:

2.1 Theorem *Let* $\mathbf{\Gamma}$ *be a pointclass with a universal set. Then there are universal sets* $U_X \subseteq \omega^\omega \times X$ *for all product spaces* X *with the following properties:*

1. *(s-m-n theorem) For every* $X = X_1 \times \cdots \times X_n$*,* $Y = X_1 \times \cdots \times X_n \times \cdots \times X_m$ *where* $m > n$*, there is a continuous* $s_{Y,X} \colon \omega^\omega \times X \to \omega^\omega$ *such that*

   $$U_Y(y, x_1, \dots, x_n, \dots, x_m) \leftrightarrow U_{X'}(s_{Y,X}(y, x_1, \dots, x_n), x_{n+1}, \dots, x_m)$$

   where $X' = X_{n+1} \times \cdots \times X_m$*.*

2. *(recursion theorem) For every product space* $X = X_1 \times \cdots \times X_n$ *and* $\mathbf{\Gamma}$ *set* $A \subseteq \omega^\omega \times X$*, there is a* $y^* \in \omega^\omega$ *such that for all* $x \in X$*,* $U_X(y^*, x) \leftrightarrow A(y^*, x)$*.*

Proof. Let $U \subseteq \omega^\omega \times \omega^\omega$ in $\mathbf{\Gamma}$ be universal for $\mathbf{\Gamma}$ subsets of ω^ω. For $X = X_1 \times \cdots \times X_n$ define $U_X(y, (x_1, \dots, x_n)) \leftrightarrow U(y_0, \langle y_1, x_1, \dots, x_n \rangle)$, where here $y \to (y_0, y_1)$ denotes our recursive bijection from ω^ω to $\omega^\omega \times \omega^\omega$. Suppose $Y = X_1 \times \cdots \times X_n \times \cdots \times X_m$. Then

$$U_Y(y, (x_1, \dots, x_n, \dots, x_m)) \leftrightarrow U(y_0, \langle y_1, x_1, \dots, x_n, \dots, x_m \rangle)$$

and

$$U_{X'}(s, (x_{n+1}, \dots, x_m)) \leftrightarrow U(s_0, \langle s_1, x_{n+1}, \dots, x_m \rangle).$$

Thus, it suffices to take $s_{Y,X}(y, x_1, \dots, x_n) = \langle \epsilon, \langle y, x_1, \dots, x_n \rangle \rangle$ where ϵ is such that $U(\epsilon, \langle \langle y, x_1, \dots, x_n \rangle, x_{n+1}, \dots, x_m \rangle) \leftrightarrow U(y_0, \langle y_1, x_1, \dots, x_m \rangle)$ for all $y, x_1, \dots, x_m$. That is, choose ϵ so that for all z

$$U(\epsilon, z) \leftrightarrow U(z_{0,0,0}, \langle z_{0,0,1}, z_{0,1}, \dots, z_{0,n}, z_1, \dots, z_m \rangle)$$

which is possible as U is universal (here $z_{0,0,0}$ abbreviates $(((z)_0)_0)_0$, *etc.*, and these decoding functions refer to the obvious product spaces).

As for the recursion theorem, fix $X = X_1 \times \cdots \times X_m$, and let $A \subseteq \omega^\omega \times X$ be in $\mathbf{\Gamma}$. Let $\epsilon \in \omega^\omega$ be such that $U(\epsilon, y, x) \leftrightarrow A(s(y, y), x)$, where s is the *s-m-n* function from $(\omega^\omega)^2$ to ω^ω corresponding to the spaces ω^ω and $\omega^\omega \times X$. Thus, $U(s(\epsilon, y), x) \leftrightarrow U(\epsilon, y, x) \leftrightarrow A(s(y, y), x)$ for all y, x, where we have dropped the cumbersome subscripts on the U. Let then $y = \epsilon$, and thus $y^* = s(\epsilon, \epsilon)$. ⊣

Following Moschovakis, we call sets U_X satisfying theorem 2.1 *good* universal sets. We shall frequently implicitly assume (without loss of generality) that our universal sets are good. Note that the construction of the U_X is uniform in the universal sets A.

We review now some of the general theory of prewellorderings and scales.

2.2 Definition *A (regular) norm on a set $A \subseteq \omega^\omega$ is a map ϕ from A into (onto) some ordinal. A norm $\phi\colon A \to ON$ is said to be a $\boldsymbol{\Gamma}$-norm if there are $\boldsymbol{\Gamma}, \check{\boldsymbol{\Gamma}}$ binary relations $\leq_\phi^{\boldsymbol{\Gamma}}$, $\leq_\phi^{\check{\boldsymbol{\Gamma}}}$ on ω^ω such that for all $y \in A$, $\forall x\ [(x \in A \wedge \phi(x) \leq \phi(y)) \leftrightarrow x \leq_\phi^{\boldsymbol{\Gamma}} y \leftrightarrow x \leq_\phi^{\check{\boldsymbol{\Gamma}}} y$. A pointclass $\boldsymbol{\Gamma}$ has the prewellordering property, written* pwo($\boldsymbol{\Gamma}$)*, if every $A \in \boldsymbol{\Gamma}$ admits a $\boldsymbol{\Gamma}$-norm.*

Norms can be identified with *prewellorderings* of A (that is, transitive, reflexive, connected binary relations $\preceq$ on A). We let $\prec$ denote the strict part of a prewellordering $\preceq$ and vice versa (i.e., $x \prec y \leftrightarrow x \preceq y \wedge \neg y \preceq x$).

The above definition generalizes immediately to any perfect product space X as well. A standard and straightforward lemma (theorem 4B.1 of [32]) says that if $\boldsymbol{\Gamma}$ is closed under $\wedge, \vee$, then $\phi\colon A \to \mathrm{ON}$ is a $\boldsymbol{\Gamma}$-norm on $A \in \boldsymbol{\Gamma}$ iff the following relations are in $\boldsymbol{\Gamma}$:

$$x \leq_\phi^* y \leftrightarrow x \in A \wedge (y \notin A \vee \phi(x) \leq \phi(y))$$
$$x <_\phi^* y \leftrightarrow x \in A \wedge (y \notin A \vee \phi(x) < \phi(y))$$

In fact, to show that the existence of the starred relations implies the prewellordering property requires no closure assumptions on $\boldsymbol{\Gamma}$.

If $\boldsymbol{\Gamma}$ has the prewellordering property and is closed under $\vee$, $\wedge$, then any two $\boldsymbol{\Gamma}$ sets A, B have comparable $\boldsymbol{\Gamma}$-norms. That is, there are $\boldsymbol{\Gamma}$-norms ϕ, ψ on A, B respectively and $\boldsymbol{\Gamma}$ relations $\leq_{\psi,\phi}^{\boldsymbol{\Gamma}}$, $\leq_{\phi,\psi}^{\boldsymbol{\Gamma}}$ and $\check{\boldsymbol{\Gamma}}$ relations $\leq_{\psi,\phi}^{\check{\boldsymbol{\Gamma}}}$, $\leq_{\phi,\psi}^{\check{\boldsymbol{\Gamma}}}$ such that $\forall y \in A\ \forall x\ [(x \in B \wedge \psi(x) \leq \phi(y)) \leftrightarrow x \leq_{\psi,\phi}^{\boldsymbol{\Gamma}} y \leftrightarrow x \leq_{\psi,\phi}^{\check{\boldsymbol{\Gamma}}} y$, and likewise $\forall y \in B\ \forall x\ [(x \in A \wedge \phi(x) \leq \psi(y)) \leftrightarrow x \leq_{\phi,\psi}^{\boldsymbol{\Gamma}} y \leftrightarrow x \leq_{\phi,\psi}^{\check{\boldsymbol{\Gamma}}} y]$. To see this, let $E = \{\langle i, z\rangle \colon (i = 0 \wedge z \in A) \vee (i = 1 \wedge z \in B)\}$. Let ρ be a $\boldsymbol{\Gamma}$-norm on E, and let $\phi(x) = \rho(\langle 0, x\rangle)$ for $x \in A$, and $\psi(x) = \rho(\langle 1, x\rangle)$ for $x \in B$. We can take, for example $x \leq_{\phi,\psi}^{\boldsymbol{\Gamma}} y$ iff $\langle 0, x\rangle \leq_\rho^{\boldsymbol{\Gamma}} \langle 1, y\rangle$. Note, however, that these norms are not regular.

2.3 Definition *A set $A \subseteq \omega^\omega$ is κ-Suslin if there is a tree T on $\omega \times \kappa$ such that $A = p[T]$. Let S_κ denote the pointclass of κ-Suslin sets. A cardinal κ is a Suslin cardinal if $S_\kappa - \bigcup_{\kappa' < \kappa} S_{\kappa'} \neq \emptyset$.*

A closely related concept (see lemma 2.5) is that of a scale.

2.4 Definition *A semi-scale $\{\phi_n\}_{n\in\omega}$ on a set $A \subseteq X$ (X a perfect product space) is a collection of norms ϕ_n on A such that if $\{x_m\}_{m\in\omega} \subseteq A$ is a sequence of points in A converging to x, and for all n, $\phi_n(x_m)$ is eventually constant, then $x \in A$. We say $\{\phi_n\}$ is an α semi-scale if all norms map into α. We say $\{\phi_n\}$ is a* scale *if it in addition satisfies the lower semi-continuity property: $\forall n\ \phi_n(x) \leq \lambda_n \doteq \lim_{m\to\infty} \phi_n(x_m)$. Likewise we define α-scale.*

A (semi)-scale $\{\phi_n\}$ on A is a good *(semi)-scale if whenever $x_m \in A$ and for all n $\phi_n(x_m)$ is eventually constant, then $x = \lim_{m\to\infty} x_m$ exists (and thus $x \in A$).*

A (semi)-scale is called very good *if it is good and whenever $x, y \in A$ and $\phi_n(x) \leq \phi_n(y)$, then $\phi_i(x) \leq \phi_i(y)$ for all $i < n$.*

A (semi)-scale is called excellent *if it is very good and whenever $x, y \in A$ and $\phi_n(x) = \phi_n(y)$ then $x \restriction n = y \restriction n$ (assuming now $X = \omega^\omega$).*

The notions of good α-scale, etc.*, are defined in the same manner, requiring the norms to map into α.*

The next lemma shows the essential equivalence of these concepts.

2.5 Lemma *For every $A \subseteq \omega^\omega$ and every $\alpha \in ON$, A is α-Suslin iff A has an α-semiscale iff A has an α-scale iff A has an excellent α-scale.*

Proof. Clearly excellent scale $\to$ very good scale $\to$ good scale $\to$ scale $\to$ semi-scale, for any α. If $\{\phi_n\}$ is a semi-scale on A into α, define the tree of the semi-scale as follows:

$$((a_0, \ldots, a_{n-1}), (\beta_0, \ldots, \beta_{n-1})) \in T_\phi \leftrightarrow \exists x \in A\ [x \restriction n = (a_0, \ldots, a_{n-1}) \wedge \phi_0(x) = \beta_0, \ldots, \phi_{n-1}(x) = \beta_{n-1}].$$

Clearly $A \subseteq p[T]$. If $(x, f) \in [T]$, then $\exists x_m \in A$ such that $x_m \to x$ and $\phi_n(x_m) \to f(n)$ for all n. Thus, $x \in A$ by definition of a semi-scale. Hence, $A = p[T]$.

Thus, it suffices to show that A is α-Suslin implies A admits an excellent α-scale. First note that A is α-Suslin iff A is κ-Suslin, where $\kappa = |\alpha|$. Thus we may assume $\alpha = \kappa$ is a cardinal. Fix a tree T on $\omega \times \kappa$ such that $A = p[T]$. We consider two cases.

First assume $\mathrm{cof}(\kappa) > \omega$. Then $\forall x \in A\ \exists \beta < \kappa\ (x \in p[T \restriction \beta])$. Define a tree S by:

$$((a_0, \ldots, a_{n-1}), (\beta_0, \ldots, \beta_{n-1})) \in S \leftrightarrow \beta_0 > \beta_1, \ldots, \beta_{n-1} \wedge ((a_0, \ldots, a_{n-2}), (\beta_1, \ldots, \beta_{n-1})) \in T.$$

Thus, $A = p[S]$ as well. For $x \in A$, define

$$\phi_n(x) = |(f(0), x(0), \ldots, f(n-1), x(n-1)|^*_{lex},$$

where $f\colon \omega \to \kappa$ is the leftmost branch through S_x, and $|\vec{s}|^*_{lex}$ denotes the rank of $\vec{s}$ in the lexicographic ordering restricted to $\vec{t} \in \kappa^{2n}$ such that $t(0) > t(1), \ldots, t(2n-1)$. Thus, $\phi_n\colon A \to \kappa$. Suppose $\{x_m\} \subseteq A$ and $\phi_n(x_m) \to \lambda_n$ for all n. Let $\vec{s}_n$ be such that $|\vec{s}_n|^*_{\mathrm{lex}} = \lambda_n$. Then $\vec{s}_{n+1}$ extends $\vec{s}_n$ for all n, so $x_m \to x \in \omega^\omega$ and the $\vec{s}_n$ define an $f\colon \omega \to \kappa$

for which $(x, f) \in [T]$. This shows $\{\phi_n\}$ is a semi-scale, and the lower semi-continuity and excellence are easily verified.

Suppose next that $\mathrm{cof}(\kappa) = \omega$. Let $\kappa_i < \kappa$ with $\sup_{i \in \omega} \kappa_i = \kappa$. We may assume T is a tree on $\omega \times (\kappa - \omega)$. Define a tree S on $\omega \times \kappa$ by "padding" T as follows. An element of S will be of the form

$$((a_0, \ldots, a_{n-1}), (k_0, 0, \ldots, 0, \beta_0, \ldots, k_i, 0, \ldots, 0, \beta_i, \ldots))$$

such that:

1. Each $k_l \in \omega$, and after k_l occur k_l 0's. Also, $\beta_l < \kappa_{k_l}$.

2. $((a_0, \ldots, a_j), (\beta_0, \ldots, \beta_j)) \in T$, where j is maximal so that $(k_0 + 2) + \cdots + (k_j + 2) \leq n$.

Note that if $(\vec{a}, \vec{s}) \in S$, then $\forall i\ \vec{s}(i) < \kappa_i$. Easily, $A = p[S]$ as well. We now define ϕ_n as in the previous case (using lexicographic ordering on $(\kappa_n)^{2n}$). It is easily checked that $\{\phi_n\}$ is an excellent α-scale. ⊣

On standard consequence of scales is the Kunen-Martin theorem (c.f. theorem 2G.2 of [32]) which we now state.

2.6 Theorem (Kunen, Martin) *Every κ-Suslin wellfounded relation on ω^ω has length less than κ^+.*

We next recall the fundamental notion of a $\mathbf{\Gamma}$-scale, a notion introduced by Moschovakis.

2.7 Definition *A scale $\{\phi_n\}$ on a set A is a $\mathbf{\Gamma}$-scale if all of the norms ϕ_n are $\mathbf{\Gamma}$-norms. We say $\mathbf{\Gamma}$ has the scale property,* scale($\mathbf{\Gamma}$)*, if every $A \in \mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-scale.*

The prewellordering and scale properties are the basic structural ingredients in descriptive set theory, and have numerous applications there (this theory is developed in [32]). For example, if pwo($\mathbf{\Gamma}$) and $\mathbf{\Gamma}$ is closed under $\forall^\omega$ and $\wedge$, $\vee$, then $\mathbf{\Gamma}$ has the number uniformization property, that is, every $A \subseteq \omega^\omega \times \omega$ in $\mathbf{\Gamma}$ can be uniformized by a $\mathbf{\Gamma}$ relation $B \subseteq A$. Namely, set

$$B(x, n) \leftrightarrow (\forall m\ (x, n) \leq^* (x, m)) \wedge (\forall m < n\ (x, n) <^* (x, m)),$$

where $\leq^*, <^*$ correspond to a $\mathbf{\Gamma}$-norm on A. [The number uniformization property can also be shown directly for pointclasses of the form $\exists^\omega \mathbf{\Gamma}$ where $\mathbf{\Gamma}$ is closed under $\forall^\omega$ but not $\exists^\omega$]. Likewise, if $\mathbf{\Gamma}$ has the scale property and is closed under $\forall^{\omega^\omega}$ and $\wedge, \vee$, then every $\mathbf{\Gamma}$ relation $A \subseteq \omega^\omega \times \omega^\omega$ has a $\mathbf{\Gamma}$

uniformization. To see this, note that if $\{\phi_n\}$ is a $\mathbf{\Gamma}$ scale on $A \subseteq \omega^\omega \times \omega^\omega$, and we define

$$\psi_n(x,y) = |(\phi_0(x,y), x(0), y(0), \dots, \phi_{n-1}(x,y), x(n-1), y(n-1)|_{\text{lex}},$$

then $\{\psi_n\}$ is a very good $\mathbf{\Gamma}$-scale on A. For $x \in \text{dom}(A)$, $n \in \omega$, let $s_n = (\alpha_0, x(0), y(0), \dots, \alpha_{n-1}, x(n-1), y(n-1))$ be lexicographically least such that for some $(x, y_n) \in A$, $\psi_n(x, y_n) = |s_n|_{\text{lex}}$. Note that s_{n+1} extends s_n. By the scale property, there is a $(x,y) \in A$ with $\psi_n(x,y) = |s_n|_{\text{lex}}$ for all n, and by very goodness this y is unique. Thus if we define $B(x,y) \leftrightarrow \forall z \forall n (x,y) \leq^*_{\psi_n} (x,z)$, then B uniformizes A.

It is a relatively straightforward ZF result that the prewellordering and scale properties propagate from a pointclass $\mathbf{\Gamma}$ closed under $\forall^{\omega^\omega}$ to $\exists^{\omega^\omega}\mathbf{\Gamma}$. The important periodicity theorems assert that, granted sufficient determinacy, they also propagate from a pointclass $\mathbf{\Gamma}$ closed under $\exists^{\omega^\omega}$ to $\forall^{\omega^\omega}\mathbf{\Gamma}$. We state without proof the first two of the three periodicity theorems (proofs may be found in [32]). These theorems are due to Martin-Moschovakis, Moschovakis, and Moschovakis respectively. We note that DC is not required for the following two theorems.

2.8 Theorem (First Periodicity) *Let $\mathbf{\Gamma}$ be a pointclass closed under $\exists^{\omega^\omega}$ with* pwo($\mathbf{\Gamma}$)*. If $\mathbf{\Delta}$-determinacy holds, then* pwo($\forall^{\omega^\omega}\mathbf{\Gamma}$).

2.9 Theorem (Second Periodicity) *Let $\mathbf{\Gamma}$ be a pointclass closed under $\exists^{\omega^\omega}$ and $\wedge, \vee$ with the scale property. If $\mathbf{\Delta}$-determinacy holds, then $\forall^{\omega^\omega}\mathbf{\Gamma}$ has the scale property.*

2.10 Remark *The proof of the second periodicity theorem also shows that if $A \subseteq \lambda^\omega \times \omega^\omega$ admits a scale (that is, is Suslin), then so does $B \subseteq \lambda^\omega$, where $B(\vec{\alpha}) \leftrightarrow \forall x \in \omega^\omega \ A(\vec{\alpha}, x)$.*

Thus, assuming projective determinacy the pointclasses amongst the $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ having the scale property are $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_3$, $\mathbf{\Sigma}^1_4, \dots$, exhibiting a periodicity of order two.

We also recall a version of the third periodicity theorem, due also to Moschovakis. Because we will have specific need for this result later, we give the proof. For the version we state, we require DC.

Let X be a set, and $A \subseteq X^\omega$. Recall G_A is the game where I, II alternate playing $x_0, x_1, \ldots \in X$, and I wins iff $(x_0, x_1, \ldots) \in A$. Assume I has a winning quasi-strategy in the game G_A, and A admits a very good semi-scale $\{\phi_n\}$ (defined in an obvious way using X^ω in place of ω^ω, where X is given the discrete topology). We define (assuming sufficient determinacy) a canonical winning quasi-strategy τ for I in G_A as follows. Suppose $s, t \in X^{<\omega}$ are winning positions for I in G_A of the same odd length (i.e., II's turn to move). For $n \in \omega$, consider the game $G^n_{s,t}$ played as follows:

s F $a(0)$ S $a(1)$ F $a(2)$ S $a(3)$

t S $b(0)$ F $b(1)$ S $b(2)$ F $b(3)$

The game consists of two players F and S (first and second), making moves from X as shown. Let $a, b \in X^\omega$ be the sequences they produce. We say S wins the run of the game iff $(s^\frown a) \leq^*_{\phi_n} (t^\frown b)$. Let W_m, for odd m, be the set of winning positions for I in G_A of length m (i.e., I has a winning quasi-strategy starting from that position). For $s, t \in W_m$ set $s \leq^m_n t$ iff S has a winning quasi-strategy in $G^n_{s,t}$. We assume here that the games $G^n_{s,t}$ are quasi-determined. We claim that each $\leq^m_n$ is a prewellordering on each W_m. First note that there cannot be an infinite sequence $s_0, s_1, \ldots \in W_m$ such that $\forall i\ s_i \not\leq^m_n s_{i+1}$. For if so, fix winning quasi-strategies for F in all of the games $G^n_{s_i,s_{i+1}}$, and fill in the sequence of games as shown in figure I.1, using DC.

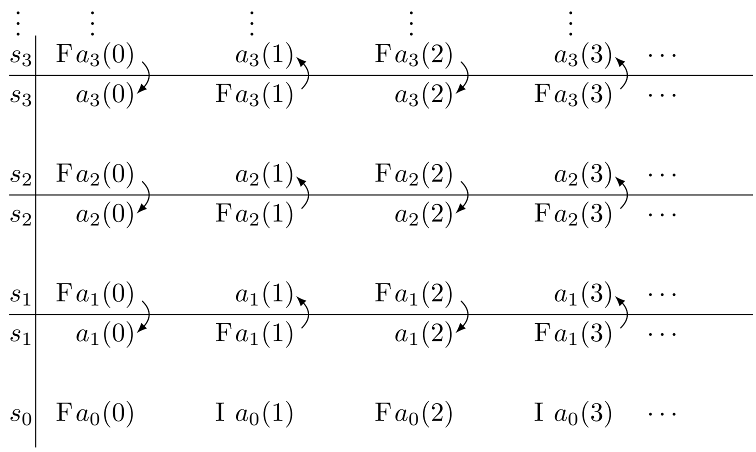

Figure I.1:

Here F follows the fixed winning quasi-strategies on all of the boards (moves made by following one of F's winning quasi-strategies are marked with an F), S's moves in the various boards are obtained by copying as shown, except in the bottom run where S follows a fixed winning quasi-strategy for the game G_A starting from s_0 (these moves are marked with a I). Let a_0, a_1, a_1, a_2, $a_2 \ldots$, be the sequences they produce. Thus $s_0^\frown a_0 \in A$

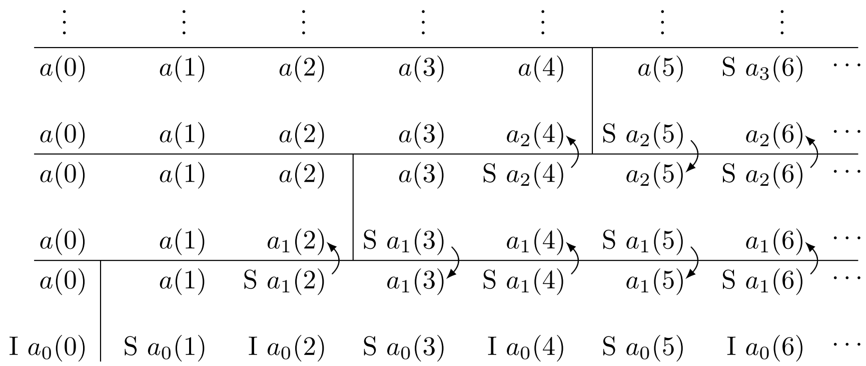

Figure I.2:

and $\phi_n(s_0^\frown a_0) > \phi_n(s_1^\frown a_1) > \ldots$, a contradiction. It follows that $\leq_n^m$ is well-founded, reflexive, and connected on each W_m. Transitivity of $\leq_n^m$ also easily follows, since if $s \leq_n^m t$, $t \leq_n^m u$, but $\neg s \leq_n^m u$, we could play quasi-strategies for S in the first two games against one for F in the third game to get a contradiction. Thus, each $\leq_n^m$ is a prewellordering on each W_m.

Define the quasi-strategy τ as follows. If $s = (s(0), \ldots, s(2n-1))$ is a winning position for I in G_A of even length, $s^\frown a \in \tau$ iff $s^\frown a$ is a winning position for I and $s^\frown a \leq_n^{2n+1} s^\frown b$ for all $b \in X$ such that $s^\frown b$ is a winning position for I.

To see this is a winning quasi-strategy, suppose $a = (a(0), a(1), \ldots)$ is a run following τ. Consider then the play of the games as shown in figure I.2.

Here I is following a fixed winning quasi-strategy for G_A on the bottom run (these moves are marked with a I), and S is following winning quasi-strategies for the $G_{s,t}^n$, $s = (a(0),\ldots,a(2n))$, $t = (a(0),\ldots,a(2n-1), a_n(2n))$, on all the boards (these moves are marked with an S). If we let $a_n = (a(0), \ldots, a(2n-1), a_n(2n), a_n(2n+1), \ldots)$, then $a_n \in A$ and $\phi_n(a_n) \leq \phi_n(a_{n-1})$. Since $\{\phi_n\}$ is a very good scale, it follows that all the $\phi_n(a_m)$ are eventually constant and thus $a \in A$.

We state now our version of the third periodicity theorem.

2.11 Theorem (Third Periodicity) *Let X be a set, $A \subseteq X^\omega$, and $\{\phi_n\}$ a very good semi-scale on A. Assume I has a winning quasi-strategy in G_A, and all of the games $G_{s,t}^n$ defined above are quasi-determined. Then the canonical quasi-strategy τ defined above is winning for I. If each of the* *games $G^n_{s,t}$ is determined, that is one of the players has a winning strategy, then each of the relations $\leq^{m*}_n, <^{m*}_n$ corresponding to the prewellordering $\leq^m_n$ on W_m is in $\Game^X \phi_n$. Specifically,*

$$s \leq^{m*}_n t \leftrightarrow \forall a(0) \exists b(0) \forall b(1) \exists a(1) \ldots (s^\frown a) \leq^*_{\phi_n} (t^\frown b)$$
$$s <^{m*}_n t \leftrightarrow \exists b(0) \forall a(0) \exists a(1) \forall b(1) \ldots (s^\frown a) <^*_{\phi_n} (t^\frown b).$$

Proof. Assuming all the games $G^n_{s,t}$ are quasi-determined, we have defined the quasi-strategy τ for I, and shown that it is winning for I. Assume now that all of the games $G^n_{s,t}$ are actually determined. For odd m, let $\leq^{m*}_n$, $<^{m*}_n$ be the starred relations corresponding to the prewellordering $\leq^m_n$ defined above; we must verify the equivalences stated in the theorem. Let $s, t \in X^m$, and suppose first $s \leq^{m*}_n t$. In particular, $s \in W_m$. If $t \in W_m$ as well, so $s \leq^m_n t$, then the right hand side of the first equivalence holds, since it just asserts II has a winning strategy in the game $G^n_{s,t}$, which is the definition of $s \leq^m_n t$ in this case. If $t \notin W_m$, II easily wins $G^n_{s,t}$ by playing so the a, b produced in the run of $G^n_{s,t}$ satisfy $(s^\frown a) \in A$ and $(t^\frown b) \notin A$. Assume now the right-hand side of the first equivalence, that is, II wins $G^n_{s,t}$. We must have $s \in W_m$, as otherwise I could easily win this game by playing to ensure $(s^\frown a) \notin A$. If $t \notin W_m$, $s \leq^{m*}_n t$ holds by definition, and if $t \in W_m$ then again by definition $s \leq^m_n t$ and so $s \leq^{m*}_n t$.

For the second equivalence, note first that the right-hand side is asserting that F has a winning strategy in the game $H^n_{s,t}$:

s		S $a(0)$	F $a(1)$			S $a(2)$	F $a(3)$
t	F $b(0)$			S $b(1)$	F $b(2)$		

where F wins the run iff $(s^\frown a) <^*_{\phi_n} (t^\frown b)$.

Assume first now that $s, t \in X^m$ and $s <^{m*}_n t$. In particular $s \in W_m$. If $t \notin W_m$, then easily F has a winning strategy in $H^n_{s,t}$ by playing to ensure that $(s^\frown a) \in A$ and $(s^\frown b) \notin A$. So, assume $t \in W_m$. Suppose, toward a contradiction, that S has a winning strategy ρ in $H^n_{s,t}$. Since $\neg(t \leq^m_n s)$, we may fix also a winning strategy σ for F in $G^n_{t,s}$. Using DC, fill in the runs of the games as in figure I.3.

In the bottom run, the moves marked with a I are those following a winning quasi-strategy to produce a_0 with $t^\frown a_0 \in A$. In the boards with moves marked by F, those moves are made in accordance with σ. In the boards with moves marked by S, those moves are made in accordance with ρ. The other moves are made by copying as shown. Thus $t^\frown a_0 \in A$, and from the

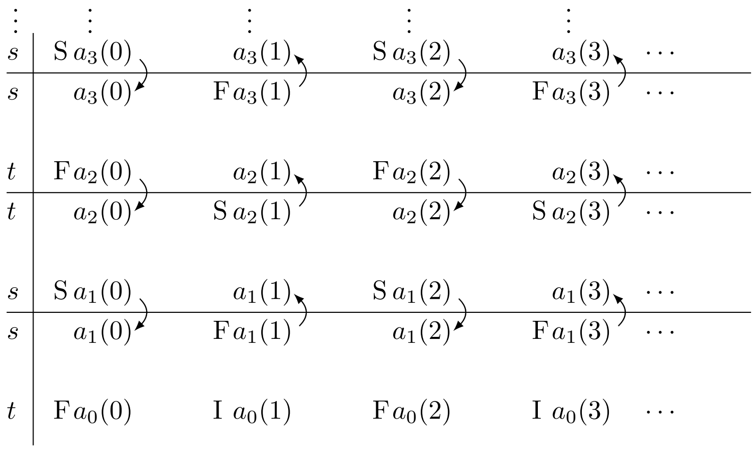

Figure I.3:

definitions of ρ, σ we have:

$$\phi_n(t^\frown a_0) > \phi_n(s^\frown a_1) \geq \phi_n(t^\frown a_2) > \phi_n(s^\frown a_3) \ldots,$$

a contradiction. Assume finally the right-hand side of the second equivalence, that is, F has a winning strategy in $H^n_{s,t}$. Easily this implies $s \in W_m$. If $t \notin W_m$, then the left-hand side is true by definition, so assume $t \in W_m$ as well. If $\neg(s <^{m*}_n t)$, then we would have $t \leq^m_n s$, and so S would have a winning strategy in $G^n_{t,s}$. These two strategies may be directly played against each other to get a contradiction. $\dashv$

A case of particular importance is when $X = \omega$, and $A \subseteq \omega^\omega$ is $\boldsymbol{\Sigma}^1_{2n}$. Assuming $\boldsymbol{\Delta}^1_{2n}$ determinacy ($\leftrightarrow \boldsymbol{\Sigma}^1_{2n}$ determinacy), theorem 2.11 shows that if I wins G_A, then I has a $\boldsymbol{\Delta}^1_{2n+1}$ winning strategy. For we may define a canonical winning strategy τ for I in G_A by:

$$\tau(s) = k \leftrightarrow \forall m \ (s^\frown k) \leq^{2l+1*}_l (s^\frown m) \wedge \forall m < k \ (s^\frown k) <^{2l+1*}_l (s^\frown m),$$

where $\mathrm{lh}(s) = 2l$. Thus the relation $\tau(s) = k$ is $\boldsymbol{\Pi}^1_{2n+1}$ from theorem 2.11 since $\boldsymbol{\Sigma}^1_{2n}$ has the scale property and $\boldsymbol{\Pi}^1_{2n+1} = \Game^{\omega^\omega} \boldsymbol{\Sigma}^1_{2n}$. Since this relation does define a strategy, it follows that the relation is $\boldsymbol{\Delta}^1_{2n+1}$. Similarly, if I wins a $\boldsymbol{\Pi}^1_{2n+1}$ game, then, assuming $\boldsymbol{\Pi}^1_{2n+1}$ determinacy, I has a $\boldsymbol{\Delta}^1_{2n+2}$ winning strategy.

2.2. Projective Ordinals, Sets, and the Coding Lemma

The Moschovakis coding lemma is a basic tool in determinacy theory. We present the result in a general form.

2.12 Theorem (AD + DC) (Coding Lemma) *Let* $\boldsymbol{\Gamma}$ *be a non-selfdual pointclass closed under* $\exists^{\omega^\omega}$ *and* $\wedge$*, and* $\prec$ *a* $\boldsymbol{\Gamma}$ *well-founded relation on* ω^ω *of rank* $\theta \in ON$*. Let* $R \subseteq \mathrm{dom}(\prec) \times \omega^\omega$ *be such that* $\forall x \in \mathrm{dom}(\prec)\ \exists y\ R(x,y)$*. Then there is a* $\boldsymbol{\Gamma}$ *set* $A \subseteq \mathrm{dom}(\prec) \times \omega^\omega$ *which is a choice set for* R*, that is:*

1. $\forall \alpha < \theta\ \exists x \in \mathrm{dom}(\prec)\ \exists y\ [|x|_\prec = \alpha \wedge A(x,y)]$.
2. $\forall x \forall y\ A(x,y) \to [x \in \mathrm{dom}(\prec) \wedge R(x,y)]$.

Proof. We may assume θ is minimal so that the theorem fails, and fix $\prec$, R, and a good universal set $U \subseteq (\omega^\omega)^3$ for the $\boldsymbol{\Gamma}$ subsets of $(\omega^\omega)^2$. Easily, θ is a limit ordinal. For $\delta < \theta$, say $u \in \omega^\omega$ codes a δ-choice set provided (1) holds for $\alpha \leq \delta$ using $A = U_u$, and (2) holds for $A = U_u$ where we replace $x \in \mathrm{dom}(\prec)$ with $x \in \mathrm{dom}(\prec) \wedge |x|_\prec \leq \delta$. By minimality of θ, for all $\delta < \theta$ there are δ-choice sets. Play the game where I, II play out $u, v \in \omega^\omega$, and II wins provided that if u codes a δ_1-choice set for some $\delta_1 < \theta$, then v codes a δ_2-choice set for some $\delta_2 > \delta_1$. If I has a winning strategy, we get a $\boldsymbol{\Sigma}^1_1$ set B of reals coding δ-choice sets for arbitrarily large $\delta < \theta$. Define then $A(x,y) \leftrightarrow \exists w \in B\ U(w,x,y)$, which easily works.

Suppose now τ is a winning strategy for II. From the *s-m-n* theorem, let $s\colon (\omega^\omega)^2 \to \omega^\omega$ be continuous such that for all ϵ, x, t, w, $U(s(\epsilon,x),t,w) \leftrightarrow \exists y\ \exists z\ [y \prec x \wedge U(\epsilon,y,z) \wedge U(z,t,w)]$. By the recursion theorem, let ϵ_0 be such that $U(\epsilon_0,x,z) \leftrightarrow z = \tau(s(\epsilon_0,x))$. A straightforward induction on $|x|_\prec$ for $x \in \mathrm{dom}(\prec)$ shows that $\forall x \in \mathrm{dom}(\prec)\ \exists! z\ U(\epsilon_0,x,z)$, and $\forall x \in \mathrm{dom}(\prec)$ $\forall z\ [U(\epsilon_0,x,z) \to z$ codes a $\geq |x|_\prec$-choice set$]$. Let $A(x,y) \leftrightarrow \exists z \in \mathrm{dom}(\prec)$ $\exists w\ [U(\epsilon_0,z,w) \wedge U(w,x,y)]$. ⊣

The coding lemma is frequently used where $\prec$ is the strict part of a prewellordering $\preceq$ which is also in $\boldsymbol{\Gamma}$ ($\boldsymbol{\Gamma}$ as in theorem 2.12), and where the set R is invariant, that is, there is an $R' \subseteq \theta \times \omega^\omega$ such that $R(x,y) \leftrightarrow R'(|x|_\prec, y)$. In this case, the relation A may be taken to have domain $\mathrm{dom}(\preceq)$. For we may define $A(x,y) \leftrightarrow \exists x'\ [x' \preceq x \wedge x \preceq x' \wedge A'(x',y)]$, where A' is the $\boldsymbol{\Gamma}$ choice set from theorem 2.12. A useful consequence of this is that if there is a $\boldsymbol{\Gamma}$ prewellordering $\preceq$ of length α whose strict part $\prec$ is also in $\boldsymbol{\Gamma}$, then every $S \subseteq \alpha$ is $\boldsymbol{\Delta}$ in the codes provided by $\prec$. That is, there are $\boldsymbol{\Gamma}, \check{\boldsymbol{\Gamma}}$ sets C, D such that for all $x \in \mathrm{dom}(\preceq)$, $S(|x|_\prec) \leftrightarrow C(x) \leftrightarrow D(x)$. To see this, apply the coding lemma to the relation $R(x,a) \leftrightarrow (|x|_\prec \in S \wedge a = 1) \vee (|x|_\prec \notin S \wedge a = 0)$ (identifying $0,1$ with two reals). Let A be an invariant choice set for R in $\boldsymbol{\Gamma}$, and set $C(x) \leftrightarrow (x,1) \in A$, $D(x) \leftrightarrow (x,0) \notin A$.

Finally, if $\boldsymbol{\Gamma}$ is a non-selfdual pointclass closed under $\forall^{\omega^\omega}$, $\vee$, and $\mathrm{pwo}(\boldsymbol{\Gamma})$, then we may improve the definability estimate. Namely, suppose $P \in \boldsymbol{\Gamma} - \check{\boldsymbol{\Gamma}}$

and ϕ is a $\mathbf{\Gamma}$ norm on P mapping onto α. Then every $S \subseteq \alpha$ is $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ in the codes provided by ϕ (rather than $\mathbf{\Delta}(\exists^{\omega^\omega}\mathbf{\Gamma})$). To see this, let U be universal for $\check{\mathbf{\Gamma}}\restriction\omega^\omega \times \omega$. For $\beta < \alpha$ (we may assume α is a limit ordinal), say y codes $S\restriction\beta$ if for all (x, a),

$$U_y(x, a) \leftrightarrow (x \in P \wedge \phi(x) < \beta \wedge x \in S \wedge a = 1) \vee \\ (x \in P \wedge \phi(x) < \beta \wedge x \notin S \wedge a = 0).$$

From the coding lemma, for all $\beta < \alpha$ there is a y coding $S\restriction\beta$. Play the integer game where I plays x, II plays y, and II wins iff $[x \in P \rightarrow \exists\beta > \phi(x)\ (y \text{ codes } S\restriction\beta)]$. II wins by boundedness (a winning strategy for I would give a $\mathbf{\Sigma}^1_1$ set $S \subseteq P$ coding cofinally in α many ordinals, from which we would compute $P \in \check{\mathbf{\Gamma}}$ by $x \in P \leftrightarrow \exists y \in S\ x \leq_{\check{\mathbf{\Gamma}}} y$), and if τ is winning for II, define $D(x) \leftrightarrow U_{\tau(x)}(x, 1)$ and $C(x) \leftrightarrow \neg U_{\tau(x)}(x, 0)$.

There is also a "uniform" version of the coding lemma (see [20]). Roughly speaking, this asserts that A may be chosen so that $A \cap \{(x, w) \colon |x|_{\preceq} \leq \delta\}$ is $\mathbf{\Sigma}_1(\preceq_\delta)$, where $\preceq_\delta$ denotes the prewellordering restricted to reals of rank $\leq \delta$. We refer the reader to [20] for a precise statement. This is particularly useful for long prewellorderings, where the initial segments may be much simpler than the overall prewellordering (this also provides another proof of the result of the previous paragraph).

Recall from the introduction the definitions of the projective pointclasses $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$. Also, assuming projective determinacy, $\mathbf{\Pi}^1_{2n+1}$ and $\mathbf{\Sigma}^1_{2n+2}$ have the scale property for all n. We now define the projective ordinals and establish their basic properties.

2.13 Definition $\boldsymbol{\delta}^1_n$ = *the supremum of the lengths of the* $\mathbf{\Delta}^1_n$ *prewellorderings of the reals.*

2.14 Theorem (AD+DC) *For all n, $\boldsymbol{\delta}^1_n$ is the supremum of the lengths of the $\mathbf{\Sigma}^1_n$ well-founded relations. Each $\boldsymbol{\delta}^1_n$ is a regular cardinal, $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$, and $\boldsymbol{\delta}^1_{2n+1} = \lambda^+_{2n+1}$ for some Suslin cardinal λ_{2n+1} of cofinality ω. $\mathbf{\Sigma}^1_{2n+2} = \boldsymbol{\delta}^1_{2n+1}$-Suslin and $\mathbf{\Sigma}^1_{2n+1} = \lambda_{2n+1}$-Suslin. The Suslin cardinals within the projective ordinals are exactly the λ_{2n+1} and the $\boldsymbol{\delta}^1_{2n+1}$.*

It is worthwhile to first isolate a result of Martin that we need for the proof of theorem 2.14, and we will also make use of later.

2.15 Theorem (AD + DC) (Martin) *The pointclass $\mathbf{\Delta}^1_{2n+1}$ is closed under $< \boldsymbol{\delta}^1_{2n+1}$ length unions and intersections.*

Proof. It is enough to show closure under $< \boldsymbol{\delta}^1_{2n+1}$ length unions. Towards a contradiction, let $\lambda < \boldsymbol{\delta}^1_{2n+1}$ and A_α, for $\alpha < \lambda$, be a sequence of $\mathbf{\Delta}^1_{2n+1}$ sets whose union $A = \bigcup_{\alpha<\lambda} A_\alpha$ is not in $\mathbf{\Delta}^1_{2n+1}$. We may assume λ is least so that the theorem fails, and thus we may assume the A_α form an increasing sequence. Since there is a $\mathbf{\Delta}^1_{2n+1}$ prewellordering of length λ, the coding lemma implies $A \in \mathbf{\Sigma}^1_{2n+1}$, and we are assuming $A \notin \mathbf{\Delta}^1_{2n+1}$. For $x \in A$, let $\phi(x) =$ the least α such that $x \in A_\alpha$, and let $<^*_\phi$, $\leq^*_\phi$ be the starred relations associated to ϕ. Note that $x <^*_\phi y \leftrightarrow \exists \alpha < \lambda\ (x \in A_\alpha \wedge y \notin A_\alpha)$. This shows $<^*_\phi$ is a λ union of $\mathbf{\Delta}^1_{2n+1}$ sets, and thus by the coding lemma $<^*_\phi$ is $\mathbf{\Sigma}^1_{2n+1}$. Likewise $\leq^*_\phi$ is $\mathbf{\Sigma}^1_{2n+1}$. This shows A admits a $\mathbf{\Sigma}^1_{2n+1}$ prewellordering, and since every $\mathbf{\Sigma}^1_{2n+1}$ set is a continuous preimage of A (by Wadge's lemma), we have pwo($\mathbf{\Sigma}^1_{2n+1}$). It is well known, however, that $\mathbf{\Sigma}^1_{2n+1}$, $\mathbf{\Pi}^1_{2n+1}$ cannot both have the prewellordering property, thus we have a contradiction. [This last fact can be argued by contradiction as follows: let $A \subseteq \omega^\omega \times \omega^\omega$ be a universal $\mathbf{\Pi}^1_{2n+1}$ set, and ϕ a $\mathbf{\Pi}^1_{2n+1}$ norm on A. Define $B(x, y) \leftrightarrow (x_0, y) <^*_\phi (x_1, y)$, and $C(x, y) \leftrightarrow (x_1, y) <^*_\phi (x_0, y)$, so $B, C \in \mathbf{\Pi}^1_{2n+1}$, and $B \cap C = \emptyset$. Let ψ_1, ψ_2 be comparable $\mathbf{\Sigma}^1_{2n+1}$-norms on B^c, C^c (see the discussion after definition 2.2). Define $E(x, z) \leftrightarrow (x, z) <^*_{\psi_1,\psi_2} (x, z)$. Note also $E(x, z) \leftrightarrow \neg(x, z) \leq^*_{\psi_2,\psi_1} (x, z)$, since $B^c \cup C^c = \omega^\omega \times \omega^\omega$. So $E \in \mathbf{\Delta}^1_{2n+1}$. However, E is also universal for $\mathbf{\Delta}^1_{2n+1}$. For if $D \subseteq \omega^\omega$ is $\mathbf{\Delta}^1_{2n+1}$, let x be such that $D^c = A_{x_0}$, $D = A_{x_1}$, and hence $D^c = B_x$, $D = C_x$. Then $E_x = D$. Being selfdual, however, the pointclass $\mathbf{\Delta}^1_{2n+1}$ cannot have a universal set by the usual diagonal argument (the set $S(x) \leftrightarrow \neg E(x, x)$ cannot be in $\mathbf{\Delta}^1_{2n+1}$)]. ⊣

proof of theorem 2.14. If $\phi \colon A \to \delta$ is a regular $\mathbf{\Pi}^1_{2n+1}$ norm on a $\mathbf{\Pi}^1_{2n+1}$ set A, then by definition $\delta \leq \boldsymbol{\delta}^1_{2n+1}$ (as all initial segments of the prewellordering are in $\mathbf{\Delta}^1_{2n+1}$). Thus, from the scale property for $\mathbf{\Pi}^1_{2n+1}$, every $\mathbf{\Pi}^1_{2n+1}$, and hence also every $\mathbf{\Sigma}^1_{2n+2}$ set is $\boldsymbol{\delta}^1_{2n+1}$-Suslin. From the coding lemma it also follows that every $\boldsymbol{\delta}^1_{2n+1}$-Suslin set is $\mathbf{\Sigma}^1_{2n+2}$, so $\boldsymbol{\delta}^1_{2n+1}$-Suslin $= \mathbf{\Sigma}^1_{2n+2}$. If A is universal for $\mathbf{\Pi}^1_{2n+1}$, then in fact $\delta = \boldsymbol{\delta}^1_{2n+1}$. The proof is a typical application of the recursion theorem. Fix a $\mathbf{\Sigma}^1_{2n+1}$ well-founded relation $\prec$. Fix a universal $\mathbf{\Pi}^1_{2n+1}$ set $U \subseteq \omega^\omega \times \omega^\omega$ and $\mathbf{\Pi}^1_{2n+1}$ norm ϕ on U. By the recursion theorem, let $\alpha_0 \in \omega^\omega$ be such that $U(\alpha_0, y) \leftrightarrow \forall x\ (x \prec y \to (\alpha_0, x) <^*_\phi (\alpha_0, y))$. A straightforward induction on $|x|_\prec$ for $x \in \mathrm{dom}(\prec)$ shows that $U(\alpha_0, x)$ and $\phi(\alpha_0, x) \geq |x|_\prec$. Thus, $\boldsymbol{\delta}^1_{2n+1} =$ the supremum of the lengths of the $\mathbf{\Sigma}^1_{2n+1}$ well-founded relations (this also follows from the Kunen-Martin theorem mentioned below, but the argument above does not need scales). The coding lemma easily shows that the supremum of the lengths of the $\mathbf{\Sigma}^1_n$ well-founded relations must have cofinality $\geq \boldsymbol{\delta}^1_n$. Thus, $\boldsymbol{\delta}^1_{2n+1}$ is a regular cardinal.

From the Kunen-Martin theorem 2.6, $\boldsymbol{\delta}^1_{2n+2} \leq$ the supremum of the lengths of the $\mathbf{\Sigma}^1_{2n+2}$ well-founded relations $\leq (\boldsymbol{\delta}^1_{2n+1})^+$. Conversely, let $\prec$ be a well-ordering of $\boldsymbol{\delta}^1_{2n+1}$. The coding lemma implies that $\prec$ is $\mathbf{\Delta}^1_{2n+2}$

in the codes relative to a norm ϕ on a $\mathbf{\Pi}^1_{2n+1}$ universal set A, that is, the relation $(x, y \in A \wedge \phi(x) \prec \phi(y))$ is $\mathbf{\Delta}^1_{2n+2}$. Thus, $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$, and $\boldsymbol{\delta}^1_{2n+2} =$ the supremum of the $\mathbf{\Sigma}^1_{2n+2}$ well-founded relations. By our remarks above, this shows $\boldsymbol{\delta}^1_{2n+2}$ is also regular.

Note that $\boldsymbol{\delta}^1_n < \boldsymbol{\delta}^1_{n+1}$ as the $\mathbf{\Sigma}^1_n$ well-founded relations can be "put together" into a single $\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n$ well-founded relation via a universal $\mathbf{\Sigma}^1_n$ set.

Suppose A is a universal $\mathbf{\Sigma}^1_{2n+1}$ set, and write $A(x) \leftrightarrow \exists y B(x, y)$ where $B \in \mathbf{\Pi}^1_{2n}$. Clearly if B is κ-Suslin, then so in A. Now B admits a $\mathbf{\Delta}^1_{2n+1}$ scale (using scale($\mathbf{\Pi}^1_{2n+1}$)), and thus A is λ-Suslin for some $\lambda < \boldsymbol{\delta}^1_{2n+1}$. Let λ_{2n+1} be the least such λ. It follows that every $\mathbf{\Sigma}^1_{2n+1}$ set is λ_{2n+1}-Suslin. From the Kunen-Martin theorem and the definition of $\boldsymbol{\delta}^1_{2n+1}$, it follows that $\boldsymbol{\delta}^1_{2n+1} = \lambda^+_{2n+1}$. We claim that $\mathrm{cof}(\lambda_{2n+1}) = \omega$. For if not, then A could be written as a λ_{2n+1} union of sets A_α, each of which is $< \lambda_{2n+1}$-Suslin. Each A_α must be $\mathbf{\Sigma}^1_{2n}$, as otherwise by Wadge, some $\mathbf{\Pi}^1_{2n}$-complete set, and hence every $\mathbf{\Sigma}^1_{2n+1}$ set would be $< \lambda_{2n+1}$-Suslin. By theorem 2.15, $A \in \mathbf{\Delta}^1_{2n+1}$, a contradiction. So, $\mathbf{\Sigma}^1_{2n+1} = \lambda_{2n+1}$-Suslin.

For any κ the pointclass of κ-Suslin sets is closed under $\exists^{\omega^\omega}$ (as well as $\wedge, \vee$). From Wadge's lemma, the only pointclasses within the projective hierarchy that are closed under $\exists^{\omega^\omega}$ (and contain the closed sets) are the $\mathbf{\Sigma}^1_n$, and thus we have determined all the Suslin classes and cardinals within the projective hierarchy. ⊣

It follows also from our discussion above that $\boldsymbol{\delta}^1_1 = \omega_1$ and $\boldsymbol{\delta}^1_2 = \omega_2$. Martin and Solovay also computed $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, and $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$ (see also the next section). In section 2.6 we will show (assuming AD) that each $\boldsymbol{\delta}^1_n$ is measurable, and in fact has the countable exponent partition property $\boldsymbol{\delta}^1_n \to (\boldsymbol{\delta}^1_n)^\lambda$ for all $\lambda < \omega_1$ (defined in §2.6).

2.3. Wadge Degrees and Abstract Pointclasses

We now recall some of the abstract theory of pointclasses. Additional background may be found in [35], [36], and [39]. We assume AD+DC throughout this section, though the determinacy required is "local," *e.g.*, only projective determinacy is required within the projective sets, *etc.*

If $\mathbf{\Gamma}$ is a pointclass and κ a cardinal, let $\bigcup_\kappa \mathbf{\Gamma}$ denote those $A \subseteq \omega^\omega$ which can be written as $A = \bigcup_{\alpha<\kappa} A_a$ where each $A_\alpha \in \mathbf{\Gamma}$. We similarly define $\bigcap_\kappa \mathbf{\Gamma}$.

We note the simple observation that if $\mathbf{\Gamma}$ is a non-selfdual pointclass, then the closure of $\mathbf{\Gamma}$ under $\exists^\omega$ implies the closure of $\mathbf{\Gamma}$ under countable unions, and likewise the closure under $\forall^\omega$ implies the closure under countable intersections. For suppose $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$, and $A_n \in \mathbf{\Gamma}$ for $n \in \omega$. Thus, $A_n \leq_w A$ for all n, and it follows easily that $B \leq_w A$ as well, where $B(x) \leftrightarrow \bar{x} \in A_{x(0)}$ and $\bar{x}(i) = x(i+1)$. Then $x \in \bigcup_n A_n \leftrightarrow \exists i\ B(i^\frown x)$. Another simple but useful observation is that if $\boldsymbol{\Gamma}$ is non-selfdual and closed under countable intersections (respectively unions), then $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ (respectively $\forall^{\omega^\omega}\boldsymbol{\Gamma}$) is closed under countable unions and intersections. As we already noted, $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ is closed under countable unions (since $\boldsymbol{\Gamma}$ has a universal set, so does $\exists^{\omega^\omega}\boldsymbol{\Gamma}$, and hence it also is non-selfdual). To check countable intersections, let $A_n \in \exists^{\omega^\omega}\boldsymbol{\Gamma}$, say $A_n(x) \leftrightarrow \exists y\ B_n(x,y)$ with $B_n \in \boldsymbol{\Gamma}$. Then $x \in \bigcap_n A_n \leftrightarrow \exists y\ B(x,y)$, where $B = \bigcap_n B_n$ and $B_n(x,y) \leftrightarrow A_n(x,(y)_n)$. Thus, $\bigcap_n A_n \in \exists^{\omega^\omega}\boldsymbol{\Gamma}$.

Recall the definitions of Lipschitz reduction, $\leq_l$ and Wadge reduction $\leq_w$ from the introduction. We say a set $A \subseteq \omega^\omega$ is selfdual if $A \leq_l A^c$. A theorem of Steel [39] says that $A \leq_l A^c$ iff $A \leq_w A^c$. Thus, A is selfdual iff the pointclass generated by A, namely $\boldsymbol{\Gamma}_A = \{B : B \leq_w A\}$, is selfdual (*i.e.*, closed under complements).

Consider now pairs of the form (A, A^c) (if A is selfdual, we may equivalently take just A in what follows). We extend $\leq_l$ to such pairs by setting $(A, A^c) \leq_l (B, B^c)$ iff one of A, A^c is $\leq_l$ to one of B, B^c. This is easily seen to be transitive, reflexive, and by Wadge's lemma, connected. A Lipschitz degree, or l-degree, denotes an equivalence class of a pair under the relation $(A, A^c) \equiv_l (B, B^c)$ iff $(A, A^c) \leq_l (B, B^c)$ and $(B, B^c) \leq_l (A, A^c)$. An important basic result of Martin (*c.f.*[39]) asserts that the strict part of $\leq_l$ is well-founded.

The Wadge degrees, or w-degrees, are defined analogously, using $\leq_w$ in place of $\leq_l$. Of course, a Wadge degree is an amalgamation of l-degrees, and it is immediate that the Wadge degrees are also well-ordered. From the result of Steel mentioned above, it follows that only selfdual l-degrees are amalgamated in forming a w-degree. It is shown in [39] that for α a limit ordinal of cofinality ω, an l-degree of rank α must be selfdual, and for $\mathrm{cof}(\alpha) > \omega$ the pair is non-selfdual. Furthermore, following any non-selfdual l-degree (A, A^c), the next ω_1 l-degrees are all selfdual and of the same w-degree (that of the join of A and A^c, that is, $\{n^\frown x\colon (n \text{ is even } \wedge x \in A) \vee (n \text{ is odd } \wedge x \notin A)\}$). This gives a general picture of the w-degrees: the selfdual and non-selfdual w degrees alternate, and at limit ordinals α, a pair of Wadge degree α is selfdual iff $\mathrm{cof}(\alpha) = \omega$. For $A \subseteq \omega^\omega$, we let $o(A)$ denote the rank of (A, A^c) in $\leq_w$.

In [36], [35] some additional structural results for general pointclasses were obtained. Recall $\boldsymbol{\Gamma}$ has the reduction property, $\mathrm{red}(\boldsymbol{\Gamma})$, if for all $A, B \in \boldsymbol{\Gamma}$ $\exists A', B' \in \boldsymbol{\Gamma}$ such that $A' \subseteq A$, $B' \subseteq B$, $A' \cap B' = \emptyset$ and $A' \cup B' = A \cup B$. $\boldsymbol{\Gamma}$ has the separation property, $\mathrm{sep}(\boldsymbol{\Gamma})$, if for all $A, B \in \boldsymbol{\Gamma}$ with $A \cap B = \emptyset$, $\exists C \in \boldsymbol{\Delta}$ $(A \subseteq C \subseteq \neg B)$. A standard result in descriptive theory (see [32]) is that $\mathrm{pwo}(\boldsymbol{\Gamma}) \to \mathrm{red}(\boldsymbol{\Gamma})$ for $\boldsymbol{\Gamma}$ closed under $\wedge$, $\vee$, and $\mathrm{red}(\boldsymbol{\Gamma}) \to \mathrm{sep}(\check{\boldsymbol{\Gamma}})$. [36] shows that for any non-selfdual pointclass $\boldsymbol{\Gamma}$, either $\mathrm{sep}(\boldsymbol{\Gamma})$ or $\mathrm{sep}(\check{\boldsymbol{\Gamma}})$, and from [39], both sides cannot have the separation property. Also, if $\boldsymbol{\Gamma}$ is closed under $\wedge, \vee$ then $\mathrm{red}(\boldsymbol{\Gamma})$ or $\mathrm{red}(\check{\boldsymbol{\Gamma}})$. More generally, Steel [35] shows that if $\boldsymbol{\Gamma}$ is closed under $\wedge$ and $\neg\,\mathrm{sep}(\boldsymbol{\Gamma})$, then $\mathrm{red}(\boldsymbol{\Gamma})$. Finally, [35] shows that if $\boldsymbol{\Gamma}$ is a *Levy* class, that is closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$, and if we make the technical assumption that $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ is not closed under well-ordered unions (this is true in $L(\mathbb{R})$, for example, for all selfdual $\boldsymbol{\Delta} \neq \mathcal{P}(\omega^\omega)$) then either pwo($\boldsymbol{\Gamma}$) or pwo($\check{\boldsymbol{\Gamma}}$). It is also shown there that if $\boldsymbol{\Delta}$ is closed under real quantifiers, and sep($\boldsymbol{\Gamma}$), then $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$ (and thus $\check{\boldsymbol{\Gamma}}$ is closed under $\forall^{\omega^\omega}$).

2.16 Definition *Let $\boldsymbol{\Gamma}$ be a (possibly selfdual) pointclass. We let $o(\boldsymbol{\Gamma}) = \sup\{o(A) \colon A \in \boldsymbol{\Gamma}\}$. We let $\delta(\boldsymbol{\Gamma})$ = the supremum of the lengths of the $\boldsymbol{\Delta}$ prewellorderings of ω^ω (where $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$).*

In [21] it is shown that for $\boldsymbol{\Delta}$ closed under real quantification, $\wedge$ and $\vee$, $o(\boldsymbol{\Delta}) = \delta(\boldsymbol{\Delta})$ = the supremum of the $\boldsymbol{\Delta}$ wellfounded relations on ω^ω. We note that for $\boldsymbol{\Delta}$ closed under real quantification, closure under $\wedge$ and $\vee$ is almost automatic; it is needed only to rule out the case of a largest Wadge degree in $\boldsymbol{\Delta}$, which occurs only when $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cup \check{\boldsymbol{\Gamma}}$ for some non-selfdual closed under real quantification (by the hierarchy analysis below).

If $\boldsymbol{\Delta}$ is selfdual, closed under real quantifiers, $o(\boldsymbol{\Delta})$ has uncountable cofinality, and we again make the technical assumption that $\boldsymbol{\Delta}$ is not closed under well-ordered unions, then Steel [35] shows there is a non-selfdual pointclass $\boldsymbol{\Gamma}$ closed under $\forall^{\omega^\omega}$ with pwo($\boldsymbol{\Gamma}$) such that $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$. Steel establishes this by getting a useful representation for the $\boldsymbol{\Gamma}$ sets. Namely, if δ is the least ordinal such that $\boldsymbol{\Delta}$ is not closed under δ unions, then $\boldsymbol{\Gamma}$ is the collection of $\boldsymbol{\Sigma}^1_1$-bounded δ unions of $\boldsymbol{\Delta}$ sets. A union $A = \bigcup_{\alpha<\delta} A_\alpha$ is $\boldsymbol{\Sigma}^1_1$-bounded if for every $\boldsymbol{\Sigma}^1_1$ $B \subseteq A$, $\exists \delta' < \delta$ $B \subseteq \bigcup_{\alpha<\delta'} A_\alpha$.

Steel shows in [35] (generalizing results of [21]) that these results suffice to place the prewellordering property within the Levy classes, as well as to classify the Levy classes within projective-like hierarchies. We summarize the conclusions. Suppose $\boldsymbol{\Gamma}$ is non-selfdual and closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$. Let α be the supremum of the limit ordinals β such that $\boldsymbol{\Delta}_\beta \doteq \{A \colon o(A) < \beta\}$ is closed under real quantifiers and $\boldsymbol{\Delta}_\beta \subseteq \boldsymbol{\Gamma}$. We have the following cases:

Type I Hierarchy $\mathrm{cof}(\alpha) = \omega$. The pointclass $\boldsymbol{\Lambda}$ of Wadge degree α is selfdual, consisting of ω-joins of sets of smaller degree. Let $\boldsymbol{\Gamma}_0$ be the class of countable unions of sets, each of degree $< \alpha$. Then $\boldsymbol{\Gamma}_0$ is the smallest class closed under $\exists^\omega$ containing $\boldsymbol{\Lambda}$, and we have pwo($\boldsymbol{\Gamma}_0$). If we let $\boldsymbol{\Gamma}_1 = \forall^{\omega^\omega}\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_2 = \exists^{\omega^\omega}\boldsymbol{\Gamma}_1$, etc., then pwo($\boldsymbol{\Gamma}_n$) for all n by first periodicity. $\boldsymbol{\Gamma}_0$ is closed under countable unions and finite intersections, and $\boldsymbol{\Gamma}_n$ for $n \geq 1$ is closed under countable unions and intersections. Also, $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_i$ or $\check{\boldsymbol{\Gamma}}_i$ for some i.

Type II, III Hierarchies $\mathrm{cof}(\alpha) > \omega$, so there is a non-selfdual pointclass $\boldsymbol{\Gamma}_0$ of degree α closed under $\forall^{\omega^\omega}$ and with pwo($\boldsymbol{\Gamma}_0$). We assume in these cases that $\boldsymbol{\Gamma}_0$ is not closed under $\exists^{\omega^\omega}$. If we let $\boldsymbol{\Gamma}_1 = \exists^{\omega^\omega}\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_2 = \forall^{\omega^\omega}\boldsymbol{\Gamma}_1$, etc., then pwo($\boldsymbol{\Gamma}_n$) for all n. For $n \geq 1$, $\boldsymbol{\Gamma}_n$ is closed under countable unions and intersections. If $\boldsymbol{\Gamma}_0$ is as well (by a result of [35] this is equivalent to $\boldsymbol{\Gamma}_0$ being closed under finite unions), this is referred to as a type II hierarchy, otherwise a type III hierarchy. Clearly, $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_i$ or $= \check{\boldsymbol{\Gamma}}_i$ for some i.

Type IV Hierarchy $\mathrm{cof}(\alpha) > \omega$, and for $\boldsymbol{\Gamma}_0$ as in the previous case, $\boldsymbol{\Gamma}_0$ is closed under real quantifiers. Let $\boldsymbol{\Gamma}_1 = \{A \cap B \colon A \in \boldsymbol{\Gamma}_0, B \in \check{\boldsymbol{\Gamma}}_0\}$. Let $\boldsymbol{\Gamma}_2 = \exists^{\omega^\omega}\boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_3 = \forall^{\omega^\omega}\boldsymbol{\Gamma}_2$, etc. Then $\mathrm{pwo}(\boldsymbol{\Gamma}_n)$ for all n, and for $n \geq 2$ (or $n = 0$) $\boldsymbol{\Gamma}_n$ is closed under countable unions and intersections. Clearly, $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_i$ or $= \check{\boldsymbol{\Gamma}}_i$ for some i.

2.17 Remark *We refer to the pointclasses* $\boldsymbol{\Gamma}_0$ *as in the type II, III hierarchies above as Steel pointclasses.*

We present one more result in the abstract theory of pointclasses which we will need later, and which illustrates the usefulness of the hierarchy classification.

2.18 Lemma *If* $\boldsymbol{\Gamma}$ *is a non-selfdual pointclass closed under* $\exists^{\omega^\omega}$ *and* $\mathrm{pwo}(\boldsymbol{\Gamma})$*, then* $\boldsymbol{\Gamma}$ *is closed under well-ordered unions.*

Proof. If $\boldsymbol{\Gamma}$ is closed under $\forall^{\omega^\omega}$ as well, this is theorem 1.1 of [14], so assume $\forall^{\omega^\omega}\boldsymbol{\Gamma} \neq \boldsymbol{\Gamma}$. If $\boldsymbol{\Gamma}$ is closed under countable unions and intersections, the result follows from lemma 2.4.1 of [21]. Suppose now $\boldsymbol{\Gamma}$ is not closed under countable intersections. The hierarchy analysis above shows that $\boldsymbol{\Gamma}$ is the base of a type I hierarchy, that is, $\boldsymbol{\Gamma} = \bigcup_\omega \boldsymbol{\Delta}$, and $\boldsymbol{\Delta}$ is closed under real quantifiers. Note that $\boldsymbol{\Gamma}$ is closed under $\wedge$. Towards a contradiction, let κ be the least cardinal so that $\bigcup_\kappa \boldsymbol{\Gamma} \nsubseteq \boldsymbol{\Gamma}$. Thus, κ is regular. Let $\boldsymbol{\Gamma}_1 = \exists^{\omega^\omega}\check{\boldsymbol{\Gamma}}$. By Wadge's lemma, $\check{\boldsymbol{\Gamma}} \subseteq \bigcup_\kappa \boldsymbol{\Gamma}$, and thus $\boldsymbol{\Gamma}_1 \subseteq \bigcup_\kappa \boldsymbol{\Gamma}$. Using the regularity of κ, let $\langle A_\alpha | \alpha < \kappa\rangle$ be a strictly increasing κ sequence of sets in $\boldsymbol{\Gamma}$ whose union A is in $\check{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma}$. Let $B = \{x \colon S_x \subseteq A\}$, where $S \subseteq (\omega^\omega)^2$ is universal $\boldsymbol{\Sigma}^1_1$. $B \in \check{\boldsymbol{\Gamma}}$ as $\check{\boldsymbol{\Gamma}}$ is closed under $\forall^{\omega^\omega}$ and $\vee$. Let $B = \bigcup_{\alpha<\kappa} B_\alpha$, where $B_\alpha \in \boldsymbol{\Gamma}$. If we replace A_α by $\{y \colon \exists x \in B_\alpha\ (y \in S_x)\}$, then the A_α form a $\boldsymbol{\Sigma}^1_1$-bounded sequence of $\boldsymbol{\Gamma}$ sets with union A. Let $U \subseteq (\omega^\omega)^2$ be a universal $\boldsymbol{\Gamma}$ set. Play the game where I plays x, II plays y, z, and II wins iff $x \in A \to \exists \alpha > |x| (U_y = A_\alpha \wedge z \in A_\alpha - \bigcup_{\beta<\alpha} A_\beta)$, where $|x|$ is the least $\alpha < \kappa$ such that $x \in A_\alpha$. By $\boldsymbol{\Sigma}^1_1$-boundedness, II wins, say by τ. Define $x \prec y$ iff $x, y \in A \wedge \tau(y)_1 \notin U_{\tau(x)_0}$. Thus, $\prec$ is a $\check{\boldsymbol{\Gamma}}$ prewellordering of length κ. By the coding lemma, then, $\bigcup_\kappa \boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}_1$, and hence $\bigcup_\kappa \boldsymbol{\Gamma} = \boldsymbol{\Gamma}_1$. Now, $\boldsymbol{\Delta}_1 = \boldsymbol{\Gamma}_1 \cap \check{\boldsymbol{\Gamma}}_1$ is clearly also not closed under κ unions, and any κ union of sets in $\boldsymbol{\Delta}_1$ is in $\boldsymbol{\Gamma}_1$. Thus, $\boldsymbol{\Gamma}_1 = \bigcup_{\kappa'} \boldsymbol{\Delta}_1$, where $\kappa' \leq \kappa$ is least such that $\boldsymbol{\Delta}_1$ is not closed under κ' unions. This, however, shows $\mathrm{pwo}(\boldsymbol{\Gamma}_1)$, a contradiction (this last part is Martin's argument from theorem 2.15 again). $\dashv$

In the case of a type II, III, or IV hierarchy, the following observations are occasionally useful.

2.19 Lemma *Let $\mathbf{\Gamma}$ be non-selfdual, closed under $\forall^{\omega^\omega}$, with* pwo($\mathbf{\Gamma}$)*, and assume $\mathbf{\Delta}$ is closed under real quantification. Let $\kappa = o(\mathbf{\Delta})$. Then* $\mathrm{cof}(\kappa)$ *is the least ordinal ρ such that $\mathbf{\Delta}$ is not closed under ρ-length unions. Furthermore, there is a κ strictly increasing sequence of sets in $\mathbf{\Delta}$.*

Proof. From pwo($\mathbf{\Gamma}$) we have that $\mathbf{\Delta}$ is not closed under $\delta(\mathbf{\Delta}) = \delta(\mathbf{\Delta}) = \kappa$ length unions. Let ρ be least so that $\mathbf{\Delta}$ is not closed under ρ-length unions. Clearly $\rho \leq \kappa$ is a regular cardinal. Suppose $\rho > \mathrm{cof}(\kappa)$. Let $\{A_\alpha\}_{\alpha<\rho}$ be an increasing sequence of $\mathbf{\Delta}$ sets whose union A is not in $\mathbf{\Delta}$. Since $\rho > \mathrm{cof}(\kappa)$ is regular, there is a $\beta < \kappa$ such that for cofinally in ρ many α we have $o(A_\alpha) \leq \beta$. By lemma 2.18 we may find a non-selfdual $\mathbf{\Gamma}_0 \subseteq \mathbf{\Delta}$ which is closed under wellordered unions and with $o(\mathbf{\Gamma}_0) > \beta$. Then $A \in \mathbf{\Gamma}_0$, a contradiction. Suppose $\rho < \mathrm{cof}(\kappa)$ and again consider a sequence $\{A_\alpha\}_{\alpha<\rho}$ as above. Since $\rho < \mathrm{cof}(\kappa)$, there is a $\beta < \kappa$ such that for all $\alpha < \rho$, $o(A_\alpha) \leq \beta$, and we reach the same contradiction as before. So, $\rho = \mathrm{cof}(\kappa)$.

Fix now a sequence $\{A_\alpha\}_{\alpha<\mathrm{cof}(\kappa)}$ of sets in $\mathbf{\Delta}$ whose union A is not in $\mathbf{\Delta}$. Let $h(\alpha) = o(A_\alpha)$, so h is cofinal in κ. Without loss of generality we may assume h is strictly increasing. Let $\preceq$ be a $\mathbf{\Delta}$ prewellordering of length $\mathrm{cof}(\kappa)$ (we may assume $\mathrm{cof}(\kappa) < \kappa$ as otherwise there is nothing to show in the second claim). View every real y as coding a Lipschitz continuous function $f_y\colon \omega^\omega \to \omega^\omega$. By the coding lemma there is $\mathbf{\Delta}$ relation $R \subseteq (\omega^\omega)^2$ with $\mathrm{dom}(R) = \mathrm{dom}(\preceq)$ and such that for all $(x, y) \in R$, $f_y^{-1}(A_{|x|+1})$ is a prewellordering of length $h(|x|)$, where $|x|$ denotes the rank of x in $\preceq$. For $\beta < \kappa$ define E_β by:

$$\begin{aligned}(x, y, z) \in E_\beta \leftrightarrow x \in \mathrm{dom}(\preceq) \wedge (\forall \gamma < |x|\ h(\gamma) \leq \beta) \wedge R(x, y)\\ \wedge |z|_{f_y^{-1}(A_{|x|+1})} \leq \beta.\end{aligned}$$

Clearly the E_β form a κ-length strictly increasing sequence. To see that $E_\beta \in \mathbf{\Delta}$, let $\alpha_0 < \mathrm{cof}(\kappa)$ be least such that $h(\alpha_0) > \beta$. Then $E_\beta = \bigcup_{\alpha\leq\alpha_0} E_{\alpha,\beta}$ where:

$$(x, y, z) \in E_{\alpha,\beta} \leftrightarrow x \in \mathrm{dom}(\preceq) \wedge (|x| = \alpha) \wedge R(x, y) \wedge |z|_{f_y^{-1}(A_{\alpha+1})} \leq \beta.$$

Since $\mathbf{\Delta}$ is closed under $< \mathrm{cof}(\kappa)$ unions, it is enough to show each $E_{\alpha,\beta} \in \mathbf{\Delta}$. The first three conjuncts are clearly in $\mathbf{\Delta}$. For the last, note that $P_y \doteq f_y^{-1}(A_{\alpha+1})$ is a $\mathbf{\Delta}$ prewellordering computed uniformly from y, that is, $(u, v) \in P_y \leftrightarrow f_y(\langle u, v\rangle) \in A_{\alpha+1}$. From the coding lemma it is straightforward to compute that $\{(y, z) \colon |z|_{P_y} \leq \beta\}$ is projective in any pointclass of Wadge degree at least $o(A_{\alpha+1})$. ⊣

2.4. The Scale Theory of $L(\mathbb{R})$

The pointclass results of §2.3 can be considered to be a generalization of the "Spector" theory of the projective sets, that is, the theory which uses only the prewellordering property for the $\mathbf{\Pi}^1_{2n+1}$, $\mathbf{\Sigma}^1_{2n+2}$ sets. There is likewise a generalization of the scale theory of the projective sets to the sets of reals in $L(\mathbb{R})$. This theory is developed in [37]. We survey without proof the main results of this theory. We assume $\mathrm{AD} + V = L(\mathbb{R})$. Recall Θ is the supremum of the lengths of the prewellorderings of the reals.

Recall that the $J_\alpha(\mathbb{R})$ hierarchy building up $L(\mathbb{R})$ is defined similarly to the J_α hierarchy for L, except we start with $J_1(\mathbb{R}) = V_{\omega+1}$. Thus, for limit α, $J_\alpha(\mathbb{R}) = \bigcup_{\alpha'<\alpha} J_{\alpha'}(\mathbb{R})$, and $J_{\alpha+1}(\mathbb{R})$ is the closure of $J_\alpha(\mathbb{R}) \cup \{J_\alpha(\mathbb{R})\}$ under suitable rudimentary functions. $\mathbf{\Sigma}_n(J_\alpha(\mathbb{R}))$ denotes the subsets of $J_\alpha(\mathbb{R})$ which are $\mathbf{\Sigma}_n$-definable over $J_\alpha(\mathbb{R})$ using parameters from $J_\alpha(\mathbb{R})$. We also let $\mathbf{\Sigma}_n(J_\alpha(\mathbb{R}))$ denote the pointclass $\mathbf{\Sigma}_n(J_\alpha(\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$. Note that $\mathbf{\Sigma}^1_n = \mathbf{\Sigma}_n(J_1(\mathbb{R}))$, so the $\mathbf{\Sigma}_n(J_\alpha(\mathbb{R}))$ hierarchy provides an extension of the projective hierarchy to all the pointclasses in $L(\mathbb{R})$. Recall that for $X = \omega^\omega$ or $X = \mathcal{P}(\omega^\omega)$, a relation $R \subseteq X$ is $\mathbf{\Sigma}^2_1$ if it can be written in the form $R(x) \leftrightarrow \exists B \subseteq \omega^\omega\ P(x, B)$, where P is projective. That is, $P(x, B) \leftrightarrow \exists z_1 \in \omega^\omega\ \forall z_2 \in \omega^\omega \cdots \exists(\forall) z_n \in \omega^\omega\ Q(x, B, z_1, \dots, z_n)$, where Q is in the smallest collection containing any Borel relation on the real coordinates, the relations $z_i \in B$, $z_i \in x$ (if $X = \mathcal{P}(\omega^\omega)$), and closed under countable unions, intersections, and complements. Let ${\boldsymbol{\delta}^2_1}^{L(\mathbb{R})}$ be the supremum of the lengths of the $(\mathbf{\Delta}^2_1)^{L(\mathbb{R})}$ prewellorderings. We will henceforth just write $\boldsymbol{\delta}^2_1$ in place of ${\boldsymbol{\delta}^2_1}^{L(\mathbb{R})}$ (we will never consider $\boldsymbol{\delta}^2_1$ in a context outside of $L(\mathbb{R})$). $\boldsymbol{\delta}^2_1$ is the least ordinal δ such that $J_\delta(\mathbb{R}) \prec^{\mathbb{R}}_1 L(\mathbb{R})$, that is elementary for $\mathbf{\Sigma}_1$ formulas with real parameters. Also, $(\mathbf{\Sigma}^2_1)^{L(\mathbb{R})} = \mathbf{\Sigma}_1(J_{\boldsymbol{\delta}^2_1}(\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$ and $(\mathbf{\Delta}^2_1)^{L(\mathbb{R})} = J_{\boldsymbol{\delta}^2_1}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$.

Martin and Steel [29] (using an idea of Moschovakis) show that $(\mathbf{\Sigma}^2_1)^{L(\mathbb{R})}$ has the scale property, and is the largest scaled pointclass in $L(\mathbb{R})$. Steel [37] refines this analysis as follows. Following Steel, call $[\alpha, \beta]$, where $\alpha \leq \beta$, a $\mathbf{\Sigma}_1$-gap if $J_\alpha(\mathbb{R}) \prec^{\mathbb{R}}_1 J_\beta(\mathbb{R})$ and the interval $[\alpha, \beta]$ is maximal with this property. The gaps thus partition the ordinals in $[1, \Theta]$. $[\boldsymbol{\delta}^2_1, \Theta]$ is the last gap, and for the first non-trivial gap, that is where $\beta > \alpha$, $\mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ already contains all the inductive sets (the smallest non-selfdual pointclass closed under real quantification). For α beginning a gap $[\alpha, \beta]$, $\mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ has the scale property, and $\mathbf{\Sigma}_1(J_\alpha(\mathbb{R})) = \mathbf{\Sigma}_1(J_\alpha(\mathbb{R}); \mathbb{R})$, that is, every $\mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ set is $\mathbf{\Sigma}_1$ definable over $J_\alpha(\mathbb{R})$ using only parameters from $\mathbb{R}$. If $\mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ is not closed under real quantifiers, then periodicity propagates the scale property to $\mathbf{\Pi}_{2n}(J_\alpha(\mathbb{R}))$, $\mathbf{\Sigma}_{2n+1}(J_\alpha(\mathbb{R}))$. Otherwise (by a result of Martin), none of the $\mathbf{\Sigma}_n(J_\alpha(\mathbb{R}))$, $\mathbf{\Pi}_n(J_\alpha(\mathbb{R}))$ have the scale property for $n \geq 2$. If $\beta > \alpha$, then none of the classes $\mathbf{\Sigma}_n(J_\gamma(\mathbb{R}))$, $\mathbf{\Pi}_n(J_\gamma(\mathbb{R}))$ for $\alpha < \gamma < \beta$ have the scale property. The existence of scales at the end of a gap hinges on whether the gap satisfies a certain reflection property (a "strong" gap in the terminology of [37]). If so, there are no new scaled classes at the end of the gap. If not (a "weak" gap), then $\mathbf{\Sigma}_n(J_\beta(\mathbb{R}))$ has the scale property, where n is least such that $\mathbf{\Sigma}_n(J_\beta(\mathbb{R})) \cap \mathcal{P}(\mathbb{R}) \not\subseteq J_\beta(\mathbb{R})$. A periodicity argument then propagates the scale property to the $\mathbf{\Sigma}_{n+2k}(J_\beta(\mathbb{R}))$, $\mathbf{\Pi}_{n+2k+1}(J_\beta(\mathbb{R}))$. These results place exactly the scale property among the classes $\mathbf{\Sigma}_n(J_\alpha(\mathbb{R}))$, $\mathbf{\Pi}_n(J_\alpha(\mathbb{R}))$. With a little extra argument, this suffices to place exactly the scale property among the Levy classes in $L(\mathbb{R})$ (the only additional classes with the scale property are the Steel classes $\mathbf{\Gamma}_0$ such that $\exists^{\omega^\omega}\mathbf{\Gamma}_0 = \mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ for some α beginning a gap).

The analysis also shows that for α beginning a gap, a universal $\mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ set U_α and a $\mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ scale $\{\phi_n^\alpha\}$ on U_α can be constructed uniformly in α. This uniformity, however, fails for the Steel pointclasses having the scale property; this presents an obstacle in some arguments.

2.5. Determinacy and Coding Results

We begin by recalling a useful ordinal determinacy result. If $\lambda_1, \ldots, \lambda_n \in$ ON and $A \subseteq \lambda_1^\omega \times \cdots \times \lambda_n^\omega$, we say A is Suslin if there is a tree T on $\lambda_1 \times \cdots \times \lambda_n \times \lambda_{n+1}$ for some $\lambda_{n+1} \in$ ON such that $(\vec{\alpha}_1, \ldots, \vec{\alpha}_n) \in A$ iff $\exists \vec{\alpha}_{n+1} \in \lambda_{n+1}^\omega\ (\vec{\alpha}_1, \ldots, \vec{\alpha}_{n+1}) \in [T]$. The collection of Suslin A contains the open and closed sets (${\lambda_i}^\omega$ endowed with the product of the discrete topology on λ_i), is closed under $\exists^{\omega^\omega}$, countable unions and intersections, continuous preimages, and assuming AD, $\forall^{\omega^\omega}$ (the only non-trivial part is closure under $\forall^{\omega^\omega}$ which follows from the proof of the second periodicity theorem). We say A is *co-Suslin* if $(\lambda_1^\omega \times \cdots \times \lambda_n^\omega) - A$ is Suslin.

The following theorem is the basis for many ordinal determinacy results.

2.20 Theorem (AD) *Let $\lambda < \Theta$, and $A \subseteq \lambda^\omega$ be Suslin and co-Suslin. Then the ordinal game G_A is determined.*

The theorem is due originally to Moschovakis, and appears as theorem 2.2 of [31] (though in a weaker form). The proof there is similar to that of the third periodicity theorem, using also the Harrington-Kechris theorem to ensure the determinacy of certain real games. A second proof appears in [20] (c.f. theorem 2.5), and is a more direct combinatorial proof.

An important tool in the theory of $L(\mathbb{R})$ is the Solovay basis theorem. This, along with theorem 2.20, will provide the determinacy of some of the games we will consider later.

2.21 Theorem (ZF) (Solovay Basis Theorem) *Let $P(A)$ be a $\mathbf{\Sigma}_1^2$ relation on sets $A \subseteq \omega^\omega$. If $L(\mathbb{R}) \models \exists A\ P(A)$ then $L(\mathbb{R}) \models \exists A \in \mathbf{\Delta}_1^2\ P(A)$.*

Proof. Write $P(A) \leftrightarrow \exists B \subseteq \omega^\omega\ Q(A, B)$ where Q is projective. Work inside $L(\mathbb{R})$. Since every set in $L(\mathbb{R})$ is ordinal definable from a real, we may fix reals x_0, y_0 and formulas ϕ_1, ϕ_2 such that for some $\alpha, \beta \in$ ON we have

$$L(\mathbb{R}) \models \exists ! A\ \exists ! B\ [\phi_1(x_0, \alpha, A) \wedge \phi_2(y_0, \beta, B) \wedge Q(A, B)].$$

Let $\psi(x_0, \alpha, y_0, \beta)$ abbreviate the right-hand side, and $\psi'(x_0, \alpha, y_0, \beta, A, B)$ denote the part inside the square brackets. Let $N \in \omega$ be large enough so that a transitive set model of $\mathrm{ZF}_N + V = L(\mathbb{R})$ containing the reals must be of the form $J_\delta(\mathbb{R})$. Let $(\delta_0, \alpha_0, \beta_0)$, where $\delta_0 > \alpha_0, \beta_0$, be the lexicographically least triple such that

$$J_{\delta_0}(\mathbb{R}) \models \mathrm{ZF}_N + \exists! A\ \exists! B[\phi_1(x_0, \alpha_0, A) \wedge \phi_2(y_0, \beta_0, B) \wedge Q(A, B)].$$

A hull argument shows that δ_0 exists, and there is a map from the reals onto $J_{\delta_0}(\mathbb{R})$, and thus $J_{\delta_0}(\mathbb{R})$ may be coded by a set of reals (that is, there is a structure $(\mathbb{R}, E)$ isomorphic to $J_{\delta_0}(\mathbb{R})$). Let A, B be the unique sets of reals in $J_{\delta_0}(\mathbb{R})$ such that $\phi_1(x_0, \alpha_0, A)$, $\phi_2(y_0, \beta_0, B)$, and $Q(A, B)$ hold in $J_{\delta_0}(\mathbb{R})$. Since Q is projective, $Q(A, B)$ holds in $L(\mathbb{R})$, and thus $P(A)$. We have:

$x \in A \leftrightarrow \exists E \subseteq \omega^\omega \times \omega^\omega\ \exists x', x'_0, y'_0 \in \mathbb{R}$ satisfying the following:

1.) $(\mathbb{R}, E)$ is well-founded, $(\mathbb{R}, E) \models \mathrm{ZF}_N + V = L(\mathbb{R})$, and $\mathbb{R} \subseteq \pi(\mathbb{R}, E)$, where π is the transitive collapse map.

2.) $\pi(x') = x$, $\pi(x'_0) = x_0, \pi(y'_0) = y_0$

3.) $(\mathbb{R}, E) \models \exists \alpha', \beta'\ [\psi(x'_0, \alpha', y'_0, \beta') \wedge$
$\forall (\alpha'', \beta'') <_{\text{lex}} (\alpha', \beta')\ \neg\psi(x'_0, \alpha'', y'_0, \beta'') \wedge$
$\exists A'\ \exists B'\ \psi'(x'_0, \alpha', y'_0, \beta', A', B') \wedge x' \in A']$.

4.) $(\mathbb{R}, E) \models \forall \delta' \in \mathrm{ON}\ J_{\delta'}(\mathbb{R}) \not\models \exists \alpha', \beta'\ \psi(x'_0, \alpha', y'_0, \beta')$.

Since (1)-(4) are projective statements about E, this shows that $A \in \mathbf{\Sigma}^2_1$, and a similar computation shows $\neg A \in \mathbf{\Sigma}^2_1$. ⊣

From this, we get a useful determinacy result.

2.22 Theorem $(\mathrm{AD} + V = L(\mathbb{R}))$ *Let $\lambda < \Theta$ and $F \colon \lambda^\omega \times (\omega^\omega)^n \to \omega^\omega$ be continuous, and $A \subseteq \omega^\omega$. Consider the game $G_{\lambda,F,A}$ on λ where I, II play $\alpha_0, \alpha_1, \ldots$ producing $\vec{\alpha} \in \lambda^\omega$, and I wins iff*

$$\exists x_1 \forall x_2 \cdots \exists(\forall) x_n F(\vec{\alpha}, x_1, x_2, \ldots, x_n) \in A.$$

Then $G_{\lambda,F,A}$ is determined.

Proof. Suppose the theorem fails. By a hull argument, there is an $E \subseteq \omega^\omega \times \omega^\omega$ such that $(\mathbb{R}, E)$ is well-founded, $\mathbb{R} \subseteq \pi(M, E)$, where π denotes the transitive collapse map, and there are $\lambda', F', A' \in \mathbb{R}$ such that $(\mathbb{R}, E) \models \mathrm{ZF}_N +$ "(λ', F', A') witnesses the theorem fails". From theorem 2.21, we may fix such an E which is $\mathbf{\Delta}^2_1$. Let $J_\delta(\mathbb{R}) = \pi(\mathbb{R}, E)$. Let $\lambda'' = \pi(\lambda')$, $A'' = \pi(A')$, $F'' = \pi(F')$. So, $J_\delta(\mathbb{R}) \models$ "(λ'', A'', F'') witness the theorem fails". Since $E \in \mathbf{\Delta}^2_1$, easily $A'' \in \mathbf{\Delta}^2_1$. Hence A'' is Suslin, co-Suslin in $L(\mathbb{R})$, and thus so is

$$\{(\vec{\alpha}, x_1, \ldots, x_n) \in \lambda''^\omega \times (\omega^\omega)^n \colon F''(\vec{\alpha}, x_1, \ldots, x_n,) \in A''\}$$

By periodicity, $G_{\lambda'',A'',F''}$ is Suslin, co-Suslin in $L(\mathbb{R})$ and therefore determined. However, a winning strategy for this game can be identified with a subset of $\lambda'' < \Theta^{J_\delta(\mathbb{R})}$. By the coding lemma, the strategy must then lie in $J_\delta(\mathbb{R})$, a contradiction. ⊣

2.23 Corollary $(\mathrm{AD} + V = L(\mathbb{R}))$ *Let $\lambda < \Theta$, $F\colon \lambda^\omega \to \omega^\omega$ be continuous, $A \subseteq \omega^\omega$, and G the game on λ with payoff $F^{-1}(A)$. Then G is determined.*

If $\boldsymbol{\Gamma}$ is a pointclass closed under $\forall^{\omega^\omega}, \vee$, and $\mathrm{pwo}(\boldsymbol{\Gamma})$, and if $\phi\colon A \xrightarrow{onto} \kappa$ is a $\boldsymbol{\Gamma}$ norm on the $\boldsymbol{\Gamma}$-complete set A, then the usual boundedness principle applies: every $B \subseteq A$ in $\check{\boldsymbol{\Gamma}}$ is bounded below κ with respect to ϕ. In this case, κ is the supremum of the lengths of the $\boldsymbol{\Delta}$ prewellorderings, and κ is regular. There is a useful generalization of this principle, due to Steel, which applies to all ordinals $\alpha < \Theta$. First, we recall one of the main results of [35]:

2.24 Theorem (AD) (Steel) *Let $\boldsymbol{\Gamma}$ be non-selfdual, and $\exists^{\omega^\omega}\boldsymbol{\Delta} \subseteq \boldsymbol{\Delta}$. Then $\boldsymbol{\Gamma}$ is closed under intersections with κ-Suslin sets for $\kappa < \mathrm{cof}(o(\boldsymbol{\Delta}))$.*

The non-trivial case of theorem 2.24 is when $\mathrm{sep}(\boldsymbol{\Gamma})$ holds, for if $\mathrm{sep}(\check{\boldsymbol{\Gamma}})$ then $\boldsymbol{\Gamma}$ is closed under $\wedge$ by [35].

Using theorem 2.24 we now have the following general boundedness principle. We follow the proof in [12]. We say a norm $\phi\colon A \xrightarrow{onto} \alpha$ is κ-Suslin bounded if for every $B \subseteq A$ which is κ-Suslin we have $\sup\{\phi(x)\colon x \in B\} < \alpha$.

2.25 Theorem (AD) (Steel) *Let $\alpha < \Theta$ be a limit ordinal. Then there is an $A \subseteq \omega^\omega$ and a norm $\phi\colon A \xrightarrow{onto} \alpha$ which is κ-Suslin bounded for all $\kappa < \mathrm{cof}(\alpha)$.*

Proof. First note that we may assume α is regular, since a norm of length $\mathrm{cof}(\alpha)$ which is κ-Suslin bounded for all $\kappa < \mathrm{cof}(\alpha)$ produces one of length α. For example, let $\delta = \mathrm{cof}(\alpha)$, $h\colon \delta \to \alpha$ be cofinal and increasing, and $\psi\colon B \to \delta$ a norm which is κ-Suslin bounded for all $\kappa < \delta$. Let $\rho\colon C \xrightarrow{onto} \alpha$ be a norm. Define for $\beta < \alpha$,

$$A_\beta(x) \leftrightarrow [x_0 \in C \ \wedge\ \rho(x_0) = \beta\ \wedge x_1 \in B\ \wedge\ \beta < h(\psi(x_1))]$$

Let $A = \bigcup_{\beta<\alpha} A_\beta$, and $\phi(x) = \rho(x_0)$ for $x \in A$. Suppose $S \subseteq A$ is κ-Suslin for some $\kappa < \delta$. Let $S_1 = \{x_1\colon x \in S\}$. Then S_1 is κ-Suslin and $S_1 \subseteq B$, and so $\eta \doteq \sup\{\psi(x_1)\colon x \in S\} < \delta$. But clearly then $\sup\{\phi(x)\colon x \in S\} \leq h(\eta)$. So assume α is regular.

Similarly, it suffices to produce a norm $\psi\colon A \xrightarrow{onto} \rho$ which is κ-Suslin bounded for all $\kappa < \alpha$, for some ρ of cofinality α. For suppose $\psi\colon A \xrightarrow{onto} \rho$ is such a norm, and $\text{cof}(\rho) = \alpha$. Let $h\colon \alpha \to \rho$ be cofinal. Define $\phi\colon A \overset{onto}{\longrightarrow} \alpha$ by $\phi(x) =$ the least $\beta < \alpha$ such that $h(\beta) > \psi(x)$. Then easily ϕ is κ-Suslin bounded for all $\kappa < \alpha$.

Let $\rho > \alpha$ be a limit cardinal of cofinality α such that the collection $\mathbf{\Delta}$ of sets of Wadge degree $< \rho$ is closed under $\exists^{\omega^\omega}$. We produce an A and a norm $\phi\colon A \overset{onto}{\longrightarrow} \rho$ which is κ-Suslin bounded for all $\kappa < \alpha$. Let $\mathbf{\Gamma}$ be the non-selfdual pointclass closed under $\forall^{\omega^\omega}$ with $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ (see [35], we assume $\alpha > \omega$ as otherwise the result is trivial). Let B be a $\mathbf{\Gamma}$ universal set. Define A by: $x \in A$ iff x_0, x_1 code continuous functions f_{x_0}, f_{x_1} with $f_{x_0}^{-1}(B) = \omega^\omega - f_{x_1}^{-1}(B)$. For $x \in A$, let $\phi(x)$ be the Wadge degree of the $\mathbf{\Delta}$ set $f_{x_0}^{-1}$. Clearly $\phi\colon A \overset{onto}{\longrightarrow} \rho$. Suppose $S \subseteq A$ is κ-Suslin for some $\kappa < \alpha$, and assume towards a contradiction that $\sup\{\phi(x)\colon x \in S\} = \rho$. Define

$$C(x,y) \overset{\text{def}}{\leftrightarrow} (x \in S \wedge f_{x_0}(y) \in B) \leftrightarrow (x \in S \wedge f_{x_1}(y) \notin B)$$

Thus, $C \in \mathbf{\Delta}$ by theorem 2.24. This is a contradiction, since any $\mathbf{\Delta}$ set is Wadge reducible to C. For let $D \in \mathbf{\Delta}$, and take $x \in S$ so that $f_{x_0}^{-1}(B) = \omega^\omega - f_{x_1}^{-1}(B) = D' \geq_w D$. Then $y \in D' \leftrightarrow (x,y) \in C$.

$\dashv$

2.6. Partition Relations

We recall some facts and terminology associated with partition relations that we will be using frequently. We give the definitions working in our base theory ZF + DC, although to obtain non-trivial results we will have to assume AD.

If $f\colon \alpha \to \text{ON}$, we say f has uniform cofinality ω if there is a $f'\colon \alpha \times \omega \to \text{ON}$ such that $\forall \beta < \alpha\ f(\beta) = \sup_{n\in\omega} f'(\beta,n)$ and f' is increasing in the second argument, that is, $\forall \beta\ \forall n,m\ [n < m \to f'(\beta,n) < f'(\beta,m)]$.

2.26 Definition *We say $f\colon \alpha \to ON$ is of the correct type if f is increasing, everywhere discontinuous (i.e., for all $\beta < \alpha$, $f(\beta) > \sup_{\beta'<\beta} f(\beta')$), and of uniform cofinality ω.*

Generalizing this, we define:

2.27 Definition *Let $f, S\colon \alpha \to ON$. We say f has uniform cofinality S if there is a function $l\colon \{(\beta,\gamma)\colon \beta < \alpha \wedge \gamma < S(\beta)\} \to ON$ which is increasing in the second argument and $\forall \beta < \alpha\ f(\beta) = \sup_{\gamma<S(\beta)} l(\beta,\gamma)$. We frequently just say $f(\beta)$ has uniform cofinality $S(\beta)$.*

If μ is a measure (i.e., a countably additive ultrafilter) on α, we say f has uniform cofinality S almost everywhere, for S as above, if $\forall^*_\mu \beta < \alpha\ f(\beta) =$

$\sup_{\gamma<S(\beta)} l(\beta,\gamma)$. *We usually just say $f(\beta)$ has uniform cofinality $S(\beta)$ almost everywhere with respect to μ.*

Note that the statement "f has uniform cofinality S almost everywhere with respect to μ" depends only on $[f]_\mu$, $[S]_\mu$.

For κ a cardinal and $\lambda \leq \kappa$, we let $(\kappa)^\lambda$ denote the set of increasing functions from λ to κ. We write $\kappa \to (\kappa)^\lambda$ to mean: for every partition $\mathcal{P}\colon (\kappa)^\lambda \to \{0,1\}$ of the increasing functions from λ to κ into two pieces, there is a homogeneous $H \subseteq \kappa$ of size κ. That is, there is an $i \in \{0,1\}$ such that for all $f \in (H)^\lambda$ we have $\mathcal{P}(f) = i$. We define a variation on this as follows. We say $\kappa \overset{\text{c.u.b.}}{\longrightarrow} (\kappa)^\lambda$ if for all partitions $\mathcal{P}\colon (\kappa)^\lambda \to \{0,1\}$ of the increasing functions from λ to κ into two pieces, there is a c.u.b. $C \subseteq \kappa$ such that for some $i \in \{0,1\}$ and all $f\colon \lambda \to C$ of the correct type, $\mathcal{P}(f) = i$.

The following well-known fact connects these two variations. The proof is straightforward, and left to the reader.

2.28 Fact *For all cardinals κ and ordinals $\lambda \leq \kappa$:*

1. $\kappa \overset{c.u.b.}{\longrightarrow} (\kappa)^\lambda \Rightarrow \kappa \to (\kappa)^\lambda$
2. $\kappa \to (\kappa)^{\omega\cdot\lambda} \Rightarrow \kappa \overset{c.u.b.}{\longrightarrow} (\kappa)^\lambda$

The instances of the partition property of particular importance to us are expressed in the following definition.

2.29 Definition *We say a cardinal κ has the strong partition property if $\kappa \to (\kappa)^\kappa$. We say κ has the* weak *partition property if $\kappa \to (\kappa)^\lambda$ for all $\lambda < \kappa$.*

From fact 2.28 it follows that the notions of strong and weak partition property of κ do not depend on which of the two variations of the definition are used. In all of the determinacy arguments, it is the "c.u.b." version of the partition relation which is relevant. Since we will never need the other variation, we therefore adopt the convention that henceforth, $\kappa \to (\kappa)^\lambda$ means $\kappa \overset{\text{c.u.b.}}{\longrightarrow} (\kappa)^\lambda$.

There are two slight generalizations of the strong partition property of κ which we will employ frequently. First, if $\prec$ is a well-ordering of some set $\text{dom}(\prec)$ of order-type κ, we have the strong partition property for partitions of functions $f\colon \text{dom}(\prec) \to \kappa$ of the correct type (defined in the obvious manner). Second, instead of considering functions $f\colon \kappa \to \kappa$ or $f\colon \text{dom}(\prec) \to \kappa$ of the correct type, we may consider f which are increasing, everywhere discontinuous, and of uniform cofinality S, for any fixed $S\colon \kappa \to \kappa$. Alternatively, we may consider partitions of functions f which are increasing, continuous at limit ordinals (or points of limit rank in $\prec$), and such that

$f(\alpha)$ has uniform cofinality $S(\alpha)$ at points of successor rank. In either case, the generalized version of the strong partition property follows easily from the usual strong partition relation.

We present now an abstract form of Martin's proof of the strong partition relation on ω_1. We state it in the most general form for which we are able to prove it.

2.30 Definition *Let κ be a regular cardinal, $\lambda \in ON$, $\lambda \leq \kappa$. We say κ is λ-reasonable there is a non-selfdual pointclass $\mathbf{\Gamma}$ closed under $\exists^{\omega^\omega}$, and a map ϕ with domain ω^ω satisfying (where $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$):*

1. *$\forall x\ \phi(x) \subseteq \lambda \times \kappa$.*
2. *$\forall F \colon \lambda \to \kappa\ \exists x\ (\phi(x) = F)$.*
3. *$\forall \beta < \lambda\ \forall \gamma < \kappa\ R_{\beta,\gamma} \in \mathbf{\Delta}$, where*

$$x \in R_{\beta,\gamma} \leftrightarrow \phi(x)(\beta,\gamma) \wedge \forall \gamma' < \kappa\ (\phi(x)(\beta,\gamma') \to \gamma' = \gamma).$$

4. *Suppose $\beta < \lambda$, $A \in \exists^{\omega^\omega}\mathbf{\Delta}$, and $A \subseteq R_\beta \doteq \{x \colon \exists \gamma < \kappa\ R_{\beta,\gamma}(x)\}$. Then $\exists \gamma_0 < \kappa\ \forall x \in A\ \exists \gamma < \gamma_0\ R_{\beta,\gamma}(x)$.*

We say κ is *reasonable* if it is κ-reasonable. If $\exists! \gamma\ \phi(x)(\beta,\gamma)$, then we write $\phi(x)(\beta)$ for this unique γ. Note that the pointclass hypotheses of the theorem are really just that $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ for some Levy class $\mathbf{\Gamma}$ (i.e., $\mathbf{\Gamma}$ is non-selfdual and closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$) as the hypotheses are symmetric between $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$. Recall that from AD we have either pwo($\mathbf{\Gamma}$) or pwo($\check{\mathbf{\Gamma}}$).

2.31 Theorem (AD) (Martin) *If κ is $\omega \cdot \lambda$-reasonable, then $\kappa \to (\kappa)^\lambda$.*

Proof. We will show below that $\mathbf{\Delta}$ is in fact closed under $< \kappa$ unions and intersections; we assume this for now. We refer below to the sets R_β, $R_{\beta,\gamma}$ of definition 2.30.

Fix a partition $\mathcal{P} \colon (\kappa)^\lambda \to \{0,1\}$. Play the integer game where I plays out $x \in \omega^\omega$, II plays out $y \in \omega^\omega$. If there is a least ordinal $\beta < \omega \cdot \lambda$ such that $x \notin R_\beta$ or $y \notin R_\beta$, then II wins provided $x \notin R_\beta$. Otherwise, let f_x, $f_y \colon \omega \cdot \lambda \to \kappa$ be the functions they determine (*e.g.*, $f_x(\beta) = \phi(x)(\beta)$). Define in this case $f_{x,y} \colon \lambda \to \kappa$ by $f_{x,y}(\beta) = \sup_{\beta' < \omega\cdot(\beta+1)} \max(f_x(\beta'), f_y(\beta'))$. II then wins iff $\mathcal{P}(f_{x,y}) = 1$.

Assume without loss of generality that II has a winning strategy τ. For $\beta < \omega \cdot \lambda$ and $\gamma < \kappa$, define $x \in S_{\beta,\gamma} \leftrightarrow \forall \beta' \leq \beta\ \exists \gamma' \leq \gamma\ x \in R_{\beta',\gamma'}$. Thus, $S_{\beta,\gamma} \in \mathbf{\Delta}$. Hence, for all $\beta < \omega \cdot \lambda$ and $\gamma < \kappa$, $\tau[S_{\beta,\gamma}] \in \exists^{\omega^\omega}\mathbf{\Delta}$ (note that for any Levy class $\mathbf{\Gamma}$ that $\exists^{\omega^\omega}\mathbf{\Delta}$ is closed under $\wedge$, $\vee$; an easy consequence of the hierarchy analysis of §2.3). Now, $\tau[S_{\beta,\gamma}] \subseteq R_\beta$. Thus,

$\theta(\beta,\gamma) \doteq \sup\{\phi(x)(\beta)\colon x \in \tau[S_{\beta,\gamma}]\} < \kappa$, from (4) of 2.30. Let $C \subseteq \kappa$ be the set of points closed under θ, and $C' \subseteq C$ the set of limit points of C.

Suppose $F\colon \lambda \to C'$ is of the correct type; we show that $\mathcal{P}(F) = 1$. Let x be such that $\phi(x)$ determines a function $f_x\colon \omega\cdot\lambda \to C$ such that $F(\beta) = \sup_{\beta'<\omega\cdot(\beta+1)} f_x(\beta')$. We may assume $f_x(\beta) \geq \beta$ for all β. Let $y = \tau(x)$. From the definition of C it follows that $\phi(y)$ determines a function $f_y\colon \omega\cdot\lambda \to \kappa$ such that $f_y(\beta) \leq f_x(\beta+1)$ for all β. Thus, $F = f_{x,y}$, so $\mathcal{P}(F) = 1$.

We show now that $\boldsymbol{\Delta}$ is closed under $< \kappa$ unions. Suppose not, and let $\delta < \kappa$ be least such that some union $A = \bigcup_{\alpha<\delta} A_\alpha$ is not in $\boldsymbol{\Delta}$. Note that $R_0 = \bigcup_{\gamma<\kappa} R_{0,\gamma}$ is a κ union of $\boldsymbol{\Delta}$ sets, and $R_0 \notin \exists^{\omega^\omega}\boldsymbol{\Delta}$. Suppose first pwo($\boldsymbol{\Gamma}$). Then $\boldsymbol{\Gamma}$ is closed under well-ordered unions by lemma 2.18. Thus $A \in \boldsymbol{\Gamma}$, and by Wadge's lemma, $R_0 = \bigcup_{\alpha<\delta} S_\alpha$ for some $S_\alpha \in \boldsymbol{\Delta}$. Since κ is regular, one of the $S_\alpha \subseteq R_0$ must be "unbounded" in κ, a contradiction to $\omega\cdot\lambda$-reasonableness. So assume pwo($\check{\boldsymbol{\Gamma}}$), and thus pwo($\boldsymbol{\Gamma}_1$), where $\boldsymbol{\Gamma}_1 = \exists^{\omega^\omega}\check{\boldsymbol{\Gamma}}$. Thus, $\boldsymbol{\Gamma}_1$ is closed under well-ordered unions, and so $R_0 \in \boldsymbol{\Gamma}_1$. We cannot have $\bigcup_\delta \boldsymbol{\Delta} = \boldsymbol{\Gamma}$, as otherwise Martin's argument (theorem 2.15) shows pwo($\boldsymbol{\Gamma}$). It follows that $\bigcup_\delta \boldsymbol{\Delta} \supseteq \check{\boldsymbol{\Gamma}}$, and so $\bigcup_\delta \exists^{\omega^\omega}\boldsymbol{\Delta} \supseteq \boldsymbol{\Gamma}_1$ (and hence actually $\boldsymbol{\Gamma}_1 = \bigcup_\delta \exists^{\omega^\omega}\boldsymbol{\Delta}$). Thus, $R_0 = \bigcup_{\alpha<\delta} S_\alpha$, with each $S_\alpha \in \exists^{\omega^\omega}\boldsymbol{\Delta}$. As before, this contradicts reasonableness. ⊣

2.32 Remark *The proof shows that if* $\boldsymbol{\Gamma}$,ϕ *witness the* λ*-reasonableness of* κ*, then* $\boldsymbol{\Delta}$ *is closed under* $< \kappa$ *unions. With a little extra work one can show* $\boldsymbol{\Gamma}$ *is closed under countable unions, intersections, and* pwo($\check{\boldsymbol{\Gamma}}$).

The next lemma shows that all the $\boldsymbol{\delta}^1_{2n+1}$, and in particular ω_1, have the countable exponent partition relation. We will take this as the start of our analysis in the next section.

2.33 Theorem (AD) *Let* $\boldsymbol{\Gamma}$ *be a non-selfdual pointclass closed under* $\forall^{\omega^\omega}$, $\wedge, \vee$ *and assume* pwo($\boldsymbol{\Gamma}$). *Let* $\delta = \delta(\boldsymbol{\Gamma}) =$ *the supremum of the lengths of the* $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ *prewellorderings. Then* $\delta \to (\delta)^\lambda$ *for all* $\lambda < \omega_1$.

Proof. Fix λ, and a bijection $\pi\colon \omega \to \lambda$. Fix also a $\boldsymbol{\Gamma}$ universal set P and a $\boldsymbol{\Gamma}$ norm ψ on P. We may assume ψ is onto an ordinal, in which case that ordinal is δ. We define the map ϕ so that $\check{\boldsymbol{\Gamma}}, \phi$ witness the λ- reasonableness of δ. Define $\phi(x)(\beta,\gamma)$ iff $x_n \in P \wedge \psi(x_n) = \gamma$, where $\pi(n) = \beta$. (1)-(3) are immediate, and (4) follows since a $\check{\boldsymbol{\Gamma}}$ subset of P is bounded. ⊣

Note that if we know directly that $\boldsymbol{\Gamma}$ is closed under countable unions and intersections, then the pointclass arguments in theorem 2.31 are not necessary for the application to theorem 2.33, as the sets $S(\alpha,\gamma)$ as defined there are in $\boldsymbol{\Delta}$ directly.

3. Suslin Cardinals

In this section we develop the basic theory of Suslin cardinals and scales assuming AD. Steel's analysis of scales in $L(\mathbb{R})$ (which we overviewed in §2.4) provides a more detailed description if one assumes in addition $V = L(\mathbb{R})$, but just assuming AD suffices to develop the main facts. The main result is theorem 3.19. We assume AD throughout this section. The arguments of this section are mainly due to Martin and appear in [23], which we follow.

We assume in this section a basic knowledge of homogeneous and weakly homogeneous trees, though we only need here (aside from theorem 3.2 below) the basic definitions and general properties of the homogeneous tree construction. The reader could skip ahead to definition 5.1 and the following paragraphs for a discussion.

Recall that $S(\kappa)$ denotes the pointclass of κ-Suslin sets. Recall also the definitions of $o(\mathbf{\Gamma})$, $\delta(\mathbf{\Gamma})$ from definition 2.16. We will use frequently the fact mentioned previously (from [21]) that for $\mathbf{\Delta}$ closed under real quantification, $\wedge$ and $\vee$, we have $o(\mathbf{\Delta}) = \delta(\mathbf{\Delta})$.

We state two theorems we will need for this analysis. The first, due to Steel and Woodin is the following.

3.1 Theorem (Steel, Woodin) *The set of Suslin cardinals is closed below their supremum.*

Thus, assuming AD, the set of Suslin cardinals is closed below Θ except that the supremum of the Suslin cardinals, if less than Θ, may perhaps not be a Suslin cardinal. Woodin [42] has isolated a strengthening of AD called AD^+ which implies that the Suslin cardinals are closed below Θ. It is apparently unknown whether AD implies AD^+. We refer the reader to [42] for further discussion of AD^+.

We will also need the following theorem of Martin and Woodin on weak homogeneity. We refer the reader to [26] for a proof.

3.2 Theorem (Martin, Woodin) *Let κ be less than the supremum of the Suslin cardinals. Then every tree on $\omega \times \kappa$ is weakly homogeneous.*

3.1. Pointclass Arguments

We recall the following fact about the homogeneously Suslin sets.

3.3 Lemma *Let κ be a cardinal. Let $\mathbf{\Gamma}$ be the collection of $A \subseteq \omega^\omega$ which can be written in the form $A = p[T]$ where T is a homogeneous tree on $\omega \times \kappa$. Then $\mathbf{\Gamma}$ is a pointclass and is closed under $\forall^{\omega^\omega}$.*

Proof. It is straightforward to check that $\mathbf{\Gamma}$ is a pointclass. We first show that $\forall^{\omega^\omega} \mathbf{\Gamma} \subseteq S(\kappa)$. Suppose $A(x) \leftrightarrow \forall y\, B(x, y)$ where $B \in \mathbf{\Gamma}$, say $B = p[T]$

where T is a homogeneous tree on $\omega \times \omega \times \kappa$. Let $\{s_i\}_{i\in\omega}$ enumerate $\omega^{<\omega}$ in a reasonable manner. Define the tree U on $\omega \times \kappa$ by:

$$(t, \vec{\alpha}) \in U \leftrightarrow \forall i < \mathrm{lh}(t)\ (t\restriction \mathrm{lh}(s_i), s_i, \vec{\beta}) \in T,$$

where $\vec{\beta} = (\alpha_{j(0)}, \alpha_{j(1)}, \ldots, \alpha_{j(\mathrm{lh}(s_i)-1)})$, and $j(a)$ is the integer such that $s_{j(a)} = s_i\restriction a$. Clearly $p[U] \subseteq A$. The inclusion $A \subseteq p[U]$ follows also if we have that for every $x \in A$ there is a Lipschitz continuous $f\colon \omega^\omega \to \kappa^\omega$ such that for all $y \in \omega^\omega$, $(x, y, f(y)) \in [T]$ (for this f will produce a branch through U_x). The existence of f follows from the homogeneity of T: play the (closed for II) game where I plays integers $y(i)$, II plays ordinals $\alpha(i) < \kappa$, and II wins the run iff for all n, $(x\restriction n, y\restriction n, \vec{\alpha}\restriction n) \in T$. Since T is homogeneous, II has a winning strategy in this game, and this gives the desired function f. So, $\forall^{\omega^\omega}\boldsymbol{\Gamma} \subseteq S(\kappa)$.

If $\boldsymbol{\Gamma} = S(\kappa)$, then this shows $\forall^{\omega^\omega}\boldsymbol{\Gamma} = \boldsymbol{\Gamma}$ (and $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$ as well). Suppose $\boldsymbol{\Gamma} \subsetneq S(\kappa)$. If κ is not the largest Suslin cardinal, then from lemma 3.2 we have $S(\kappa) = \exists^{\omega^\omega}\boldsymbol{\Gamma}$. If $\forall^{\omega^\omega}\boldsymbol{\Gamma} \neq \boldsymbol{\Gamma}$, then by Wadge's lemma $\forall^{\omega^\omega}\boldsymbol{\Gamma} \supseteq \forall^{\omega^\omega}\check{\boldsymbol{\Gamma}} = \check{S}(\kappa)$, a contradiction. If κ is the largest Suslin cardinal, then $S(\kappa)$ is closed under $\forall^{\omega^\omega}$ (as well as $\exists^{\omega^\omega}$) as otherwise periodicity would give a larger Suslin class. So, $\boldsymbol{\Delta}(S(\kappa))$ is closed under real quantification. Also, any $A \in S(\kappa) - \check{S}(\kappa)$ cannot be in $\boldsymbol{\Gamma}$ (or even be the projection of a weakly homogeneous tree), as otherwise $\omega^\omega - A$ would be Suslin from the homogeneous tree construction. Thus, in this case $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Delta}(S(\kappa))$. From theorem 3.2 it follows that $\boldsymbol{\Gamma} = \boldsymbol{\Delta}(S(\kappa))$, and so $\boldsymbol{\Gamma}$ is closed under $\forall^{\omega^\omega}$. ⊣

We need the following simple lemma.

3.4 Lemma *Let κ be a Suslin cardinal. Then there is a κ-length strictly increasing sequence of sets in $S(\kappa)$. If $\mathrm{cof}(\kappa) > \omega$, then there is a κ-length strictly increasing sequence of sets each of which is $< \kappa$-Suslin.*

Proof. The proof that every Suslin cardinal is reliable (c.f. lemma 4.6 of [37]) shows that for any Suslin cardinal κ there is an $A \in S(\kappa) - \check{S}(\kappa)$ and a scale $\{\phi_i\}$ on A with norms into κ and with ϕ_0 onto κ. We recall the argument. Let $B \in S(\kappa) - \check{S}(\kappa)$, and $\{\psi_i\}$ a regular scale on B with norms into κ. Let $A = \{x\colon x' \in B\}$, where $x'(n) = x(n+1)$. Define for $x \in A$, $\phi_0(x) = \psi_{x(0)}(x')$, and $\phi_{i+1}(x) = \psi_i(x')$. Then ϕ_0 is onto κ as otherwise $A \in S(\lambda)$ for some $\lambda < \kappa$. For $\alpha < \kappa$ let $A_\alpha = \{x \in A\colon \phi_0(x) \leq \alpha\}$. Each A_α is in $S(\kappa)$. Moreover, the A_α form a strictly increasing sequence of $S(\kappa)$ sets of length κ.

Suppose now that $\mathrm{cof}(\kappa) > \omega$. We now use the argument of lemma 2.1 of [14]. Recall ϕ_0 maps onto κ. For $\alpha < \beta < \kappa$ define

$$A_{\alpha,\beta} = \{x\colon \forall i\ \phi_i(x) < \beta\} \cup \{x\colon \forall i\ \phi_i(x) \leq \beta \wedge \phi_0(x) \leq \alpha\}.$$

Each $A_{\alpha,\beta}$ is β-Suslin, and hence λ-Suslin (as κ is the next Suslin cardinal after λ). If we view the increasing pairs (α, β) as ordered first by the second coordinate and then the first, clearly the $A_{\alpha,\beta}$ form a (not necessarily strictly) increasing sequence of order type κ. For each $\alpha < \kappa$, there is an x with $\phi_0(x) = \alpha$ and for this x there is a least $\beta > \alpha$ such that $\forall i\ \phi_i(x) \leq \beta$. It follows that for these α, β that $A_{\alpha,\beta} - \bigcup_{\alpha',\beta'} A_{\alpha',\beta'} \neq \emptyset$ where the union ranges over (α', β') less than (α, β) in the ordering described. Thus there is a κ length subsequence of the $A_{\alpha,\beta}$ which is strictly increasing. ⊣

We will need the following result, due to Chuang, in the theory of pointclasses. The methods used in the proof are similar to those of [14].

3.5 Theorem (Chuang) *Let* $\mathbf{\Gamma}$ *be non-selfdual, closed under* $\forall^{\omega^\omega}$, $\vee$, *and assume* pwo($\mathbf{\Gamma}$). *Then there is no strictly increasing or decreasing sequence of* $\mathbf{\Gamma}$ *sets of length* $(\delta(\mathbf{\Gamma}))^+$.

Proof. Let $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ as usual. Note that $\mathbf{\Gamma}$ is closed under countable intersections and $\exists^{\omega^\omega}\mathbf{\Gamma}$ (which may be $\mathbf{\Gamma}$) is closed under countable unions and intersections. We fix a universal $\mathbf{\Gamma}$ set $U \subseteq \omega^\omega \times \omega^\omega$, so every real x codes a $\mathbf{\Gamma}$ set $U_x \subseteq \omega^\omega$. Let δ_0 be the supremum of those limit β such that $\{A\colon o(A) < \beta\}$ is closed under real quantification and is contained within $\mathbf{\Delta}$. Let $\mathbf{\Delta}_0 = \{A\colon o(A) < \delta_0\}$. Let $\delta = \delta(\mathbf{\Gamma}) =$ the supremum of the lengths of the $\mathbf{\Delta}$ prewellorderings. Suppose $\{A_\alpha\}_{\alpha<\delta^+}$ is a strictly increasing sequence of $\mathbf{\Gamma}$ sets. By thinning the sequence we may assume that for all $\alpha < \delta^+$ that $\bigcup_{\beta<\alpha} A_\beta \subsetneq A_\alpha$. Let $A = \bigcup_\alpha A_\alpha$. For $x \in A$ let $\phi(x) < \delta^+$ be the least ordinal α such that $x \in A_\alpha$. Let $\prec$ be the strict prewellordering defined by

$$x \prec y \leftrightarrow x, y \in A \wedge \phi(x) < \phi(y).$$

Thus, $\prec$ has length δ^+.

We consider two cases, though the argument in each case is similar.

Case I. $\mathbf{\Gamma}$ is not closed under $\exists^{\omega^\omega}$.

By periodicity pwo($\exists^{\omega^\omega}\mathbf{\Gamma}$), and so by lemma 2.18 $\exists^{\omega^\omega}\mathbf{\Gamma}$ is closed under well-ordered unions. It follows that $\prec\ \in \exists^{\omega^\omega}\mathbf{\Gamma}$ since $x \prec y$ iff $\exists \alpha < \beta < \delta^+$ $(x \in A_\alpha \wedge y \in A_\beta \wedge y \notin A_\alpha)$. We use here the fact that $\check{\mathbf{\Gamma}} \subseteq \exists^{\omega^\omega}\mathbf{\Gamma}$ as $\mathbf{\Gamma}$ is not closed under $\exists^{\omega^\omega}$. Let $C \subseteq (\omega^\omega)^3$ be defined by:

$$(x, y, z) \in C \leftrightarrow \exists \alpha < \delta^+\ (U_x = A_\alpha \wedge y, z \in A \wedge \phi(y) = \alpha \wedge \phi(z) = \alpha + 1).$$

Applying the coding lemma to the $\exists^{\omega^\omega}\mathbf{\Gamma}$ relation $\prec$ gives an $\exists^{\omega^\omega}\mathbf{\Gamma}$ set $S \subseteq C$ such that for all $\alpha < \delta^+$ there is an $(x, y, z) \in S$ with $\phi(y) = \alpha$. From pwo($\mathbf{\Gamma}$) and the closure of $\mathbf{\Gamma}$ under $\vee$ the usual boundedness argument shows that if ψ is a regular $\mathbf{\Gamma}$ norm on a $\mathbf{\Gamma}$ complete set B, then ψ maps onto δ and every $\check{\mathbf{\Gamma}}$ subset of B is bounded in the norm. In particular, every $\check{\mathbf{\Gamma}}$ wellfounded relation has length less than δ (otherwise the coding lemma gives an unbounded $\check{\mathbf{\Gamma}}$ subset of B). Also from pwo($\mathbf{\Gamma}$), every $\mathbf{\Gamma}$ set is a δ union of $\mathbf{\Delta}$ sets. It follows that every $\exists^{\omega^\omega}\mathbf{\Gamma}$ set is a δ union of $\exists^{\omega^\omega}\mathbf{\Delta} \subseteq \check{\mathbf{\Gamma}}$ sets. So write $S = \bigcup_{\beta<\delta} S_\beta$, where each $S_\beta \in \check{\mathbf{\Gamma}}$. For $\beta < \delta$ let $\leq_\beta$ be the prewellordering on S_β defined by

$$\begin{aligned}(x_1, y_1, z_1) \leq_\beta (x_2, y_2, z_2) &\leftrightarrow (x_1, y_1, z_1), (x_2, y_2, z_2) \in S_\beta \wedge \phi(y_1) \leq \phi(y_2)\\ &\leftrightarrow (x_1, y_1, z_1), (x_2, y_2, z_2) \in S_\beta \wedge y_1 \in U_{x_2}\\ &\leftrightarrow (x_1, y_1, z_1), (x_2, y_2, z_2) \in S_\beta \wedge z_2 \notin U_{x_1}\end{aligned}$$

Thus, $\leq_\beta$ can be written as the intersection of $S_\beta \times S_\beta$ with a $\mathbf{\Gamma}$ set or with a $\check{\mathbf{\Gamma}}$ set. In particular, $\leq_\beta \in \check{\mathbf{\Gamma}}$, and has length less than δ. A similar computation shows the strict part $<_\beta$ of the prewellordering to be in $\check{\mathbf{\Gamma}}$ as well.

This, however gives a one-to-one map of δ^+ into $\delta \times \delta$, a contradiction. Namely, given $\alpha < \delta^+$ let $\pi_0(\alpha)$ be the least ordinal $\beta < \delta$ such that there is an $(x, y, z) \in S_\beta$ with $\phi(y) = \alpha$. Let $\pi_1(\alpha)$ be the rank of any $(x, y, z) \in S_\beta$ with $\phi(y) = \alpha$ in the prewellordering $\leq_\beta$. It is easy to check that this is well-defined and that $\alpha \mapsto (\pi_0(\alpha), \pi_1(\alpha))$ is one-to-one.

Case II. $\mathbf{\Gamma}$ is closed under $\exists^{\omega^\omega}$.

In this case $\mathbf{\Gamma}$ is closed under real quantification, countable unions and intersections. Define C as in case I. If there is a $\mathbf{\Gamma}$ wellfounded relation of length δ^+, then using the coding as in case I gives a $\mathbf{\Gamma}$ set S as in that case. We still have that every $\check{\mathbf{\Gamma}}$ wellfounded relation has length less than δ, and we thus get a contradiction exactly as in case I. So suppose every $\mathbf{\Gamma}$ wellfounded relation has length less than δ^+. From the coding lemma there are $\mathbf{\Gamma}$ wellfounded relations of any length less than δ^+, so δ^+ is the supremum of the lengths of the $\mathbf{\Gamma}$ wellfounded relations. From this, the coding lemma easily implies that δ^+ is regular. Consider the integer game where I plays out $w \in \omega^\omega$ and II plays out $(x, y, z) \in (\omega^\omega)^3$. II wins the run iff (where U' is universal for $\mathbf{\Gamma}$ subsets of $(\omega^\omega)^2$):

$$U'_w \text{ is wellfounded } \rightarrow ((x, y, z) \in C \wedge \phi(y) > |U'_w|),$$

where $|U'_w|$ denotes the rank of the relation U'_w. I cannot have a winning strategy, as this would give a $\mathbf{\Sigma}^1_1$ set of codes of $\mathbf{\Gamma}$ wellfounded relations whose lengths were unbounded in δ^+, and from this we would get a $\mathbf{\Gamma}$ wellfounded relation of length δ^+ (in fact there can be no $\mathbf{\Gamma}$ set of codes of $\mathbf{\Gamma}$ wellfounded relations having lengths unbounded in δ^+). Let τ be a winning strategy for II. Define the relation

$$w_1 \ll w_2 \leftrightarrow (U'_{w_1}, U'_{w_2} \text{ are wellfounded}) \wedge y_2 \notin U_{x_1},$$

where $\tau(w_1) = (x_1, y_1, z_1)$ and $\tau(w_2) = (x_2, y_2, z_2)$. From the closure of $\mathbf{\Gamma}$ under quantifiers it follows easily that $\ll \in \check{\mathbf{\Gamma}}$, and from the regularity of δ^+ an easy argument shows that $\ll$ has length δ^+ [for $\alpha < \delta^+$ let $f(\alpha) < \delta^+$

be least such that for some w with U'_w wellfounded and $|U'_w| = \alpha$ we have $f(\alpha) = \phi(y)$ where $\tau(w) = (x, y, z)$. Let $C \subseteq \delta^+$ be c.u.b. and closed under f. By a straightforward induction check that for $\alpha \in C$ and w with $|U'_w| = \alpha$, we have $|w|_{\ll} \geq \gamma$, where α is the γ^{th} element of C.] Thus we have produced a $\check{\mathbf{\Gamma}}$ wellfounded relation of length δ^+, a contradiction.

We have shown that there is no strictly increasing sequence of $\mathbf{\Gamma}$ sets of length $\delta(\mathbf{\Gamma})^+$. The argument for decreasing sequences is similar in each case (we use now for C the set of all (x, y, z) such that x codes some A_α, $y \in A_\alpha - A_{\alpha+1}$, and z codes $A_{\alpha+1}$). ⊣

By a *limit Suslin cardinal* we mean a Suslin cardinal which is the supremum of the smaller Suslin cardinals. A limit Suslin cardinal is necessarily a limit cardinal. By a *successor Suslin cardinal* we mean a Suslin cardinal κ which is the least Suslin cardinal greater than some Suslin cardinal λ. κ may or may not be a successor cardinal in this case. From theorem 3.1 it follows that every Suslin cardinal is either a limit Suslin cardinal or a successor Suslin cardinal.

3.6 Lemma *Let κ be a Suslin cardinal and assume $S(\kappa)$ is closed under $\forall^{\omega^\omega}$. Then κ is a regular limit of Suslin cardinals and* scale($S(\kappa)$).

Proof. In this case $S(\kappa)$ is closed under real quantification and thus also countable unions and intersections. So $\mathbf{\Delta} = \mathbf{\Delta}(S(\kappa))$ is closed under real quantification, countable unions and intersections. Thus, $S(\kappa)$ is at the base of a type IV hierarchy. Let $\delta = o(\mathbf{\Delta})$. Since $S(\kappa)$ is closed under $\wedge$, $\vee$, an argument using the coding lemma shows that δ is regular. [If δ were singular, then the coding lemma would give a $S \in \mathbf{\Delta}$ consisting of pairs (x, y) coding Lipschitz continuous function f_x, f_y with $f_x^{-1}(A) = f_y^{-1}(B)$, where A is a $S(\kappa)$ complete set and B is $\check{S}(\kappa)$ complete. So, (x, y) codes a $\mathbf{\Delta}$ set of some Wadge rank $|(x, y)| < \delta$. Also, $\{|(x, y)| \colon (x, y) \in S\}$ will be cofinal in δ. Let

$$\begin{aligned} D =& \{(x, y, z) \colon (x, y) \in S \wedge f_x(z) \in A\} \\ &= \{(x, y, z) \colon (x, y) \in S \wedge f_y \notin B\}. \end{aligned}$$

But then $D \in \mathbf{\Delta}$ yet every set in $\mathbf{\Delta}$ is Wadge reducible to D, a contradiction.]

We cannot have $\delta > \kappa$ as then there is a $\mathbf{\Delta}$ prewellordering of length κ, and by the coding lemma every subset of κ could be coded in $\mathbf{\Delta}$, and so $S(\kappa)$ would be contained in $\mathbf{\Delta}$. So, $\delta \leq \kappa$. On the other hand, from lemma 3.4 there is a κ strictly increasing sequence of $S(\kappa)$ sets. From theorem 3.5 we have $\kappa < \delta^+$, so $\kappa \leq \delta$. Thus, $\kappa = \delta$. In particular, κ is regular.

Let λ be the supremum of the Suslin cardinals which are less than κ. We show that $\lambda = \kappa$. Suppose $\lambda < \kappa$. From lemma 3.4 there is a κ strictly increasing sequence of sets in $S(\lambda)$ (note: λ is actually a Suslin cardinal by theorem 3.1, but we don't need this here. Note that $S(\lambda)$ is properly contained in $\boldsymbol{\Delta}$ from the regularity of δ, even if λ is not a Suslin cardinal). Within the projective hierarchy over $S(\lambda)$ we may find a non-selfdual $\boldsymbol{\Gamma}_0$ closed under $\exists^{\omega^\omega}$ and $\mathrm{pwo}(\boldsymbol{\Gamma}_0)$, and so by lemma 2.18, $\boldsymbol{\Gamma}_0$ is closed under wellordered unions. Then $S(\kappa) \subseteq \boldsymbol{\Gamma}_0$, a contradiction (as $\boldsymbol{\Gamma}_0 \subseteq \boldsymbol{\Delta}$). Thus, κ is a limit Suslin cardinal.

Since $\mathrm{cof}(\kappa) > \omega$, every $S(\kappa)$ set is a κ union of sets in $\boldsymbol{\Delta}$. Thus, $\boldsymbol{\Delta}$ is not closed under wellordered unions and so either $\mathrm{pwo}(S(\kappa))$ or $\mathrm{pwo}(\check{S}(\kappa))$. We cannot have $\mathrm{pwo}(\check{S}(\kappa))$ as then $\check{S}(\kappa)$ would be closed under wellordered unions and then $S(\kappa) \subseteq \check{S}(\kappa)$, a contradiction. So, $\mathrm{pwo}(S(\kappa))$. Thus, $S(\kappa)$ is closed under wellordered unions. Hence, $S(\kappa) = \bigcup_\kappa \boldsymbol{\Delta}$. To see now $\mathrm{scale}(S(\kappa))$, let $A \in S(\kappa) - \check{S}(\kappa)$ and let T be a tree on $\omega \times \kappa$ with $A = p[T]$. For $x \in A$, let $\phi_0(x)$ be the least $\alpha < \kappa$ such that $x \in p[T \restriction \alpha]$. For $i > 0$ let

$$\phi_i(x) = \langle \phi_0(x), \ell_0^{\phi_0(x)}, \dots, \ell_i^{\phi_0(x)}(x) \rangle,$$

where $\ell_i^\beta(x)$ is the i^{th} coordinate of the left-most branch of $(T \restriction \beta)_x$. Here $\langle \beta, \alpha_0, \dots, \alpha_i \rangle$ denotes the rank of the tuple $(\beta, \alpha_0, \dots, \alpha_i)$ in lexicographic ordering on those tuples satisfying $\beta \geq \max\{\alpha_0, \dots, \alpha_i\}$. It is easy to check that $\{\phi_i\}$ is a scale on A with all norms into κ. Moreover, each of the norms ϕ_i is an $S(\kappa)$-norm as the norm relations $\leq_i^*$, $<_i^*$ are easily expressible as κ unions of $\boldsymbol{\Delta}$ sets. ⊣

We consider first Suslin cardinals of uncountable cofinality. First we consider the successor Suslin cardinals.

3.7 Lemma *Let κ be a successor Suslin cardinal with* $\mathrm{cof}(\kappa) > \omega$. *Let λ be the largest Suslin cardinal less than κ. Then $\kappa = \lambda^+$ and* $\mathrm{cof}(\lambda) = \omega$. *Furthermore, $S(\kappa)$ has the scale property and $S(\kappa) = \exists^{\omega^\omega} \check{S}(\lambda)$. Also, κ is regular.*

Proof. Let $A \in S(\kappa) - \check{S}(\kappa)$, and let $\{\phi_i\}$ be a regular scale on A with norms into κ. From lemma 3.4, there is a κ strictly increasing sequence of λ-Suslin sets. We cannot have $\mathrm{pwo}(S(\lambda))$ as then $S(\lambda)$ would be closed under wellordered unions by lemma 2.18, and so A would be in $S(\lambda)$. So, $\mathrm{pwo}(\check{S}(\lambda))$. From theorem 3.5 applied to $\check{S}(\lambda)$ it follows that $\kappa < \delta(\check{S}(\lambda))^+$, and thus $\kappa \leq \delta(\check{S}(\lambda))$. Every $\boldsymbol{\Delta}(\check{S}(\lambda))$ prewellordering is in $S(\lambda)$ and so by the Kunen-Martin theorem has length $< \lambda^+$. So, $\kappa \leq \lambda^+$ and thus $\kappa = \lambda^+$.

Since $\kappa \leq \delta(\check{S}(\lambda))$, there is an $S(\lambda)$ wellfounded relation of length λ. From the coding lemma it follows that $S(\lambda)$ is closed under λ unions. Suppose that $\mathrm{cof}(\lambda) > \omega$. Then every $S(\lambda)$ set is a λ union of $\boldsymbol{\Delta}(S(\lambda))$ sets. Thus $S(\lambda) = \bigcup_\lambda \boldsymbol{\Delta}(S(\lambda))$. By Martin's argument this give $\mathrm{pwo}(S(\lambda))$, a contradiction. So, $\mathrm{cof}(\lambda) = \omega$. Since $\mathrm{pwo}(\check{S}(\lambda))$ we also have $\mathrm{pwo}(\exists^{\omega^\omega} \check{S}(\lambda))$ and so from lemma 2.18, $\exists^{\omega^\omega} \check{S}(\lambda)$ is closed under wellordered unions. Note that $S(\lambda)$

is not closed under $\forall^{\omega^\omega}$ as otherwise by lemma 3.6 λ would be regular. It follows that $S(\lambda) \subseteq \exists^{\omega^\omega} \check{S}(\lambda)$. Every $S(\kappa)$ set is a wellordered union of $S(\lambda)$ sets, and thus $S(\kappa) \subseteq \exists^{\omega^\omega} \check{S}(\lambda)$. To show the other inclusion it suffices to show that $\check{S}(\lambda) \subseteq S(\kappa)$, and this is immediate by Wadge's lemma. So, $S(\kappa) = \exists^{\omega^\omega} \check{S}(\lambda)$.

Since pwo$(\check{S}(\lambda))$ and $S(\kappa) = \exists^{\omega^\omega} \check{S}(\lambda)$ we have pwo$(S(\kappa))$. Thus, $S(\kappa)$ is closed under wellordered unions by lemma 2.18. It follows that $S(\kappa) = \bigcup_\kappa \mathbf{\Delta}(S(\kappa))$. Since $\kappa = \lambda^+$, cof$(\kappa) > \omega$, and so the argument at the end of lemma 3.6 shows that every $S(\kappa)$ set admits a scale all of whose norm relations can be written as κ unions of $\mathbf{\Delta}(S(\kappa))$ sets. Thus, scale$(S(\kappa))$.

Finally, from $\kappa = \delta(S(\lambda))$ and the Kunen-Martin theorem it follows that κ is the supremum of the lengths of the $S(\lambda)$ wellfounded relations. The coding lemma implies that for any non-selfdual pointclass $\mathbf{\Gamma}$ closed under $\exists^{\omega^\omega}$, $\wedge$, the supremum of the lengths of the $\mathbf{\Gamma}$ prewellorderings is a regular cardinal. Thus, κ is regular. ⊣

Next we consider the limit Suslin cardinals of uncountable cofinality.

3.8 Lemma *Let κ be a limit Suslin cardinal with* cof$(\kappa) > \omega$. *Then* scale$(S(\kappa))$. *Furthermore, if $\mathbf{\Delta} = \bigcup_{\lambda<\kappa} S(\lambda)$ and $\mathbf{\Gamma}$ is the corresponding Steel pointclass, then $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$. Also,* scale$(\mathbf{\Gamma})$.

Proof. Let $\mathbf{\Delta} = \bigcup_{\lambda<\kappa} S(\lambda)$. Thus $\mathbf{\Delta}$ is selfdual and closed under real quantification, countable unions and intersections (for countable unions and intersections we use cof$(\kappa) > \omega$). Let $\delta = \delta(\mathbf{\Delta}) = o(\mathbf{\Delta})$. Let $\mathbf{\Gamma}$ be the corresponding Steel pointclass, that is, $\mathbf{\Delta} = \mathbf{\Delta}(\mathbf{\Gamma})$ and $\mathbf{\Gamma}$ is closed under $\forall^{\omega^\omega}$. Similar to an earlier argument, we cannot have $\kappa < \delta$ as then the coding lemma would compute $S(\kappa) \subseteq \mathbf{\Delta}$. So, $\delta \leq \kappa$. Suppose $\delta < \kappa$. From lemma 3.4 there is a strictly increasing sequence $\{A_\alpha\}_{\alpha<\kappa}$ of $\mathbf{\Delta}$ sets of length κ. For each $\beta < \delta$ let $S_\beta \subseteq \kappa$ consist of those α such that $o(A_\alpha) = \beta$. If all the S_β had size $< \delta$, then since $\kappa = \bigcup_{\beta<\delta} S_\beta$, we would have a map from $\delta \times \delta$ onto κ, a contradiction. Fix β so that $|S_\beta| \geq \delta$. Thus we have a δ-length strictly increasing sequence of sets $\{B_\gamma\}_{\gamma<\delta}$ of Wadge degree $\leq \beta$. Within $\mathbf{\Delta}$ we can find a non-selfdual pointclass $\mathbf{\Gamma}_0$ closed under $\wedge$, $\vee$ and closed under wellordered unions and properly containing the sets of Wadge degree $\leq \beta$ (from lemma 2.18). The prewellordering associated to the B_γ (i.e., $x \prec y \leftrightarrow \exists \eta_1 < \eta_2\ (x \in B_{\eta_1} \wedge y \in B_{\eta_2} \wedge y \notin B_{\eta_1})$) is a $\mathbf{\Gamma}_0$ prewellordering of length δ, a contradiction (recall δ is the supremum of the $\mathbf{\Delta}$ wellfounded relations). So, $\kappa = \delta$.

Recall also pwo$(\mathbf{\Gamma})$ (c.f. [37]). Thus pwo$(\exists^{\omega^\omega}\mathbf{\Gamma})$, and by lemma 2.18 $\exists^{\omega^\omega}\mathbf{\Gamma}$ is closed under wellordered unions. Since cof$(\kappa) > \omega$, every $S(\kappa)$ set is a union of $\mathbf{\Delta}$ sets, and so $S(\kappa) \subseteq \exists^{\omega^\omega}\mathbf{\Gamma}$. Since $S(\kappa)$ is closed under $\exists^{\omega^\omega}$, it follows from Wadge's lemma that either $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$ or $S(\kappa) = \check{\mathbf{\Gamma}}$. We claim the first possibility holds. To see this, let $A \in S(\kappa) - \check{S}(\kappa)$, and let $A = p[T]$ with T a tree on $\omega \times \kappa$. From theorem 3.2, T is weakly homogeneous. Thus, there is a homogeneous tree T' on $\omega \times \kappa$ such that $x \in A \leftrightarrow \exists y\ \langle x, y\rangle \in p[T']$ (here $x, y \mapsto \langle x, y\rangle$ denotes our coding function). Let $\mathbf{\Gamma}'$ denote the pointclass of sets which are projections of homogeneous trees on $\omega \times \kappa$. Thus, $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}'$. From lemma 3.3, $\mathbf{\Gamma}'$ is closed under $\forall^{\omega^\omega}$. If $\mathbf{\Gamma}$ is not closed under $\exists^{\omega^\omega}$, then the facts that $S(\kappa) \subseteq \exists^{\omega^\omega}\mathbf{\Gamma}$ and $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}'$ where $\mathbf{\Gamma}'$ is closed under $\forall^{\omega^\omega}$ imply that $\mathbf{\Gamma} = \mathbf{\Gamma}'$ and $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$. If $\mathbf{\Gamma}$ is closed under real quantification, then $S(\kappa) \subseteq \exists^{\omega^\omega}\mathbf{\Gamma} = \mathbf{\Gamma}$ (and also $\mathbf{\Gamma} = \mathbf{\Gamma}'$) Thus in all cases $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$.

Since pwo($\mathbf{\Gamma}$) we also have pwo($S(\kappa)$), and so $S(\kappa)$ is closed under well-ordered unions. Since $\mathrm{cof}(\kappa) > \omega$, the argument at the end of lemma 3.6 shows that every $S(\kappa)$ sets admits a scale all of whose norm relations can be written as κ unions of $\mathbf{\Delta}$ sets, and thus are $S(\kappa)$ relations. Thus, scale($S(\kappa)$).

It remains to show scale($\mathbf{\Gamma}$). We use an argument similar to one of [35]. Let $A \in \mathbf{\Gamma}$. Let $U \subseteq (\omega^\omega)^2$ be universal $\mathbf{\Sigma}^1_1$. Define $B = \{y \colon U_y \subseteq A\}$. As $\mathbf{\Gamma}$ is closed under unions with $\mathbf{\Sigma}^1_1$ sets (from theorem 2.1 of [35]), $B \in \mathbf{\Gamma}$. Let $B = p[T]$ where T is a tree on $\omega \times \kappa$. Let S be a tree on $(\omega)^3$ with $p[S] = \{(x, y) \colon U(y, x)\}$. Let V be the tree on $(\omega)^3 \times \kappa$ defined by:

$$(s, t, u, \vec{\alpha}) \in V \leftrightarrow (s, t, u) \in S \wedge (t, \vec{\alpha}) \in T.$$

Clearly $A = p[V]$. We identify V with a tree V' on $\omega \times \kappa$ by ordering the triples $(a, b, \alpha) \in \omega \times \omega \times \kappa$ by reverse lexicographic order (i.e., order by α first). Let V'' be the slight modification of V' (using $\mathrm{cof}(\kappa) > \omega$) so that for $(s, \vec{\alpha}) \in V''$, $\alpha_0 \geq \max\{\alpha_1, \dots, \alpha_{\mathrm{lh}(s)}\}$. Let $\{\phi_i\}$ be the very-good scale derived from V'', so each ϕ_i maps into κ. To show that $\leq^*_{\phi_i} \in \mathbf{\Gamma}$ (and similarly for $<^*_{\phi_i}$) it suffices to show that $\leq^*_{\phi_i}$ can be written as a $\mathbf{\Sigma}^1_1$-bounded κ-length union of $\mathbf{\Delta}$ sets. For $\alpha < \kappa$ let

$$C_\alpha = \{(x, y) \colon x \in A \wedge \phi_i(x) = \alpha \wedge \neg(y \in A \wedge \phi_i(y) \leq \alpha)\}.$$

Clearly $C_\alpha \in \mathbf{\Delta}$ (it is a Boolean combination of α-Suslin sets). Suppose $S \subseteq \omega^\omega \times \omega^\omega$ is $\mathbf{\Sigma}^1_1$ and for all $(x, y) \in S$, $x \leq^*_{\phi_i} y$. In particular, $x \in A$. Let $S' = \{x \colon \exists y\ (x, y) \in S\}$. Let y be such that $U_y = S'$. Then $y \in B$. Fix $\vec{\alpha} \in \kappa^\omega$ so that $(y, \vec{\alpha}) \in p[T]$. Let $\alpha' \geq \max\{\vec{\alpha}_i\}$. Then $S' \subseteq p[V' \restriction (\omega)^3 \times \alpha']$. So for some $\alpha'' < \kappa$, $S' \subseteq \{x \colon \phi_i(x) \leq \alpha''\}$. It follows that $S \subseteq C_{\alpha''}$. Thus, $\{\phi_i\}$ is a $\mathbf{\Gamma}$-scale on A. $\dashv$

3.2. The Next Suslin Cardinal

The results presented so far suffice to give the theory of the Suslin cardinals up to the least κ so that $S(\kappa)$ is closed under real quantification, that is, at the base of a type IV hierarchy (we will give the details below). However, at the base of a type IV hierarchy a new method is needed since we cannot propagate the scale property upwards by periodicity. Martin [23] developed a method for analyzing the next Suslin cardinal which works not just in this case, but in general. Although we only need this construction in the case where we are at the base of a type IV hierarchy, we present Martin's method in general. The idea is to describe the pointclass where the next scale gets constructed as those sets which are "Wadge reducible to a measure on κ." We will make this statement precise later.

3.9 Definition (Martin) *Let $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ be a sequence of sets $A_\alpha \subseteq \omega^\omega$, for some $\kappa \in ON$. Then $\bar{\mathcal{A}}$ is the collection of $A \subseteq \omega^\omega$ such that for all countable $S \subseteq \omega^\omega$ there is an $\alpha < \kappa$ such that $S \cap A = S \cap A_\alpha$.*

For $\boldsymbol{\Gamma}$ a pointclass and $\kappa \in ON$ we let

$$\Lambda(\boldsymbol{\Gamma}, \kappa) = \cup\{\bar{\mathcal{A}} \colon \mathcal{A} \subseteq \boldsymbol{\Gamma} \wedge ||\mathcal{A}|| \leq \kappa\},$$

where $||\mathcal{A}||$ denotes the cardinality of $\mathcal{A}$.

Note that $\bar{\mathcal{A}}$ is the closure of $\{A_\alpha \colon \alpha < \kappa\}$ under the topology on $\mathcal{P}(\omega^\omega)$ with basic open sets of the form $N_f = \{A \colon \forall x \in \mathrm{dom}(f)\ (x \in A \leftrightarrow f(x) = 1)\}$, where $f \colon S \to \{0,1\}$ and $S \subseteq \omega^\omega$ is countable.

Following [23] we next prove a few basic facts about Λ.

3.10 Lemma *Let $\boldsymbol{\Gamma}$ be non-selfdual, closed under $\forall^{\omega^\omega}$, and* $\mathrm{pwo}(\boldsymbol{\Gamma})$*. If $\boldsymbol{\Delta}$ is not closed under real quantification, then assume also* $\mathrm{scale}(\boldsymbol{\Gamma})$*. Let $\kappa = \delta(\boldsymbol{\Delta})$. Then there is a $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \boldsymbol{\Delta}$ such that for every $A \in \Lambda(\boldsymbol{\Gamma}, \kappa)$ there is a $B \in \bar{\mathcal{A}}$ with $A \leq_w B$.*

Proof. First consider the case where $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$ (so $\boldsymbol{\Gamma}$ is at the base of a type IV hierarchy), and so also countable unions and intersections. First we show in this case that for every $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \boldsymbol{\Gamma}$ there is an $\mathcal{A}' = \{A'_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \boldsymbol{\Delta}$ such that $\bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}'$. Let W be a $\boldsymbol{\Gamma}$ complete set and ϕ a $\boldsymbol{\Gamma}$ norm on W. Let $C = \{(x,y) \colon x \in W \wedge y \in A_{\phi(x)}\}$. By the coding lemma $C \in \boldsymbol{\Gamma}$. Let $C = \bigcup_{\beta<\kappa} C_\beta$ where $C_\beta \in \boldsymbol{\Delta}$. For $\alpha, \beta < \kappa$ let $A'_{\alpha,\beta} = \{y \colon \exists x\ (x \in W \wedge \phi(x) = \alpha \wedge (x,y) \in C_\beta\}$. Then $A'_{\alpha,\beta} \in \boldsymbol{\Delta}$ and easily $\bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}'$. Next we show that there is a "universal" $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with all $A_\alpha \in \boldsymbol{\Gamma}$. That is, for all $A' \in \Lambda(\boldsymbol{\Gamma}, \kappa)$, $A' \leq_w A$ for some $A \in \bar{\mathcal{A}}$. Let $U_1 \subseteq (\omega^\omega)^2$, $U_2 \subseteq (\omega^\omega)^3$ in $\boldsymbol{\Gamma}$ be universal for $\boldsymbol{\Gamma}{\restriction}\omega^\omega$, $\boldsymbol{\Gamma}{\restriction}(\omega^\omega)^2$ respectively. For $\alpha < \kappa$ let $A_\alpha \subseteq \omega^\omega \times \omega^\omega$ be defined by

$$(x,y) \in A_\alpha \leftrightarrow \exists z, u\ [z \in W \wedge \phi(z) = \alpha \wedge U_2(x,z,u) \wedge U_1(u,y)].$$

Each A_α lies in $\boldsymbol{\Gamma}$. Given $\mathcal{A}' = \{A'_\alpha\}_{\alpha<\kappa}$ with each $A'_\alpha \in \boldsymbol{\Gamma}$, by the coding lemma there is an x_0 such that for all $\alpha < \kappa$, $A'_\alpha = (A_\alpha)_{x_0}$. Suppose

$A' \in \bar{\mathcal{A}}'$. For each Turing degree d, let $\beta(d)$ be the least $\beta < \kappa$ so that A' and A'_β agree on all $x \in d$ (i.e., $A' \cap d = A'_\beta \cap d$). Define $A \subseteq \omega^\omega \times \omega^\omega$ by:

$$(x, y) \in A \leftrightarrow \forall^* d\ [(x, y) \in A_{\beta(d)}].$$

Clearly $A' = A_{x_0}$. Also, we easily have $A \in \bar{\mathcal{A}}$. [If $S \subseteq \omega^\omega \times \omega^\omega$ is countable, then for every $(x, y) \in S$ let $d_{x,y}$ be a large enough degree so that for all $d \geq_T d_{x,y}$ we have $(x, y) \in A_{\beta(d)}$ iff $(x, y) \in A$. Then for any d above all the $d_{x,y}$ for $(x, y) \in S$ we have that A and $A_{\beta(d)}$ agree on all $(x, y) \in S$.] So $A' = A_{x_0} \leq_w A \in \bar{\mathcal{A}}$.

Suppose now we are in the case where $\mathbf{\Delta}$ is closed under real quantification, but $\mathbf{\Gamma}$ is not closed under $\exists^{\omega^\omega}$ (so $\mathbf{\Gamma}$ is at the base of a type II or III hierarchy). Again in this case $\kappa = o(\mathbf{\Delta})$. Suppose $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \exists^{\omega^\omega}\mathbf{\Gamma}$. From lemma 2.19 there is an $\exists^{\omega^\omega}\mathbf{\Gamma}$ set W and an $\exists^{\omega^\omega}$ prewellordering of W of length κ with corresponding norm ϕ say, such that for each $\alpha < \kappa$, $W_\alpha = \{x \in w \colon \phi(x) = \alpha\} \in \mathbf{\Delta}$. Let C be defined as in the first case. Then $C \in \exists^{\omega^\omega}\mathbf{\Gamma}$. We can write $C = \bigcup_{\beta<\rho} C_\beta$ where each $C_\beta \in \mathbf{\Delta}$ and $\rho \leq \kappa$ (in fact $\rho = \mathrm{cof}(\kappa)$ from lemma 2.19). For $\alpha < \kappa$, $\beta < \rho$ let as in the first case $A'_{\alpha,\beta} = \{y \colon \exists x\ (x \in W \wedge \phi(x) = \alpha \wedge (x, y) \in C_\beta)\}$. Then $A'_{\alpha,\beta} \in \mathbf{\Delta}$ and $\bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}'$. It suffices then to construct a universal $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with all $A_\alpha \in \exists^{\omega^\omega}\mathbf{\Gamma}$. This is done as in the first case, using the W, ϕ just mentioned, universal sets U_1, U_2 for $\exists^{\omega^\omega}\mathbf{\Gamma}$, and the coding lemma with respect to the pointclass $\exists^{\omega^\omega}\mathbf{\Gamma}$.

Suppose finally that $\mathbf{\Delta}$ is not closed under real quantification. Inspecting the hierarchy analysis shows that $\mathbf{\Gamma}$ is closed under countable unions and intersections. It follows that a $\mathbf{\Gamma}$-norm ϕ on a $\mathbf{\Gamma}$ complete set W has length κ. We are assuming scale($\mathbf{\Gamma}$), and it follows that any regular $\mathbf{\Gamma}$-scale on a $\mathbf{\Gamma}$-complete set has norms that map onto κ. From this and the coding lemma it follows that κ is a Suslin cardinal and $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$. Also, κ is the supremum of the $\check{\mathbf{\Gamma}}$ wellfounded relations which shows κ is regular. The same argument as in the previous case also shows here that there is a single $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with all $A_\alpha \in \exists^{\omega^\omega}\mathbf{\Gamma}$ which is universal for all such sequences. It remains to show that for such a $\mathcal{A}$ we can find a $\mathcal{A}' = \{A'_\alpha\}_{\alpha<\kappa}$ with $A'_\alpha \in \mathbf{\Delta}$ such that $\bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}'$. Let $\{\phi_n\}$ be a regular $\mathbf{\Gamma}$-scale on a $\mathbf{\Gamma}$-universal set U. Let $T \subseteq (\omega \times \kappa)^{<\omega}$ be the tree of the scale $\{\phi_n\}$. From the coding lemma there is an $\exists^{\omega^\omega}\mathbf{\Gamma}$ relation $R \subseteq \omega^\omega \times \omega^\omega$ with $\mathrm{dom}(R) = W$ and such that $R(x, y)$ implies $U_y = A_{\phi(x)}$. For $\alpha < \kappa$ let $\beta(\alpha)$ be the least reliable ordinal $> \alpha$ with respect to $\{\phi_n\}$ ($\beta(\alpha) < \kappa$ since κ is regular). As in the Becker-Kechris theorem (see [3]), there is an ordinal game $G_{\beta(\alpha)}(z)$ (defined uniformly from $\beta(\alpha)$ and $z \in \omega^\omega$) in which I plays ordinals $< \beta(\alpha)$, II plays ordinals $< \kappa$, and the game is closed for II satisfying: for all $z \in \omega^\omega$, $z \in A_\alpha$ iff II has a winning strategy in $G_{\beta(\alpha)}(z)$. For $\gamma < \kappa$, let $G_{\beta(\alpha),\gamma}(z)$ be the game played as $G_{\beta(\alpha)}(z)$ except now II's ordinal moves are restricted to be $< \gamma$. Let $A_{\beta(\alpha),\gamma}$ be the set of z such that II has a winning strategy in $G_{\beta(\alpha),\gamma}(z)$. Thus, $A_{\beta(\alpha)} = \bigcup_\gamma A_{\beta(\alpha),\gamma}$. Finally, for $\delta < \kappa$ let $A_{\beta(\alpha),\gamma,\delta}$ be the set of z such that I does not win the open game $G_{\beta(\alpha),\gamma}(z)$ with ordinal $< \delta$ (that is, it is not the case that the empty node has rank $< \delta$ in the rank analysis of the open game). Clearly for any countable $S \subseteq \omega^\omega$ and any α, there are γ, $\delta < \kappa$ such that $A_\alpha \cap S = A_{\beta(\alpha),\gamma,\delta} \cap S$. Also, each $A_{\beta(\alpha),\gamma,\delta} \in \mathbf{\Delta}$ since $\mathbf{\Delta}$ is closed under $< \kappa$ unions and intersections by Martin's argument (c.f. the proof of theorem 2.15). We use here the fact that the games $G_{\beta(\alpha)}(z)$ are "uniformly" closed for II, that is, the set of all $(\vec{\eta}, z)$ such that $\vec{\eta}$ is a winning run for II in $G_{\beta(\alpha)}(z)$ is closed in $\kappa^\omega \times \omega^\omega$. If we let the A'_α enumerate the $A_{\beta(\alpha),\gamma,\delta}$ then $\bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}'$. ⊣

Note that if $\mathbf{\Delta}$ is any selfdual class closed under $\wedge$, and κ is any cardinal, then $\Lambda(\mathbf{\Delta}, \kappa)$ is closed under $\wedge$, $\vee$, and $\neg$. For example, to see closure under $\wedge$, given $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$, and $\mathcal{B} = \{B_\alpha\}_{\alpha<\kappa}$ and $A \in \bar{\mathcal{A}}$, $B \in \bar{\mathcal{B}}$, consider the sequence $\mathcal{C} = \{A_\alpha \cap B_\beta \colon \alpha, \beta < \kappa\}$. Easily $A \cap B \in \bar{\mathcal{C}}$. In particular, for any Levy class $\mathbf{\Gamma}$ and any κ, $\Lambda(\mathbf{\Delta}(\mathbf{\Gamma}), \kappa)$ is closed under $\wedge$, $\vee$, $\neg$. Thus as an immediate corollary to lemma 3.10 we have the following.

3.11 Corollary *Under the hypotheses of lemma 3.10,* $\Lambda(\mathbf{\Gamma}, \kappa)$ *is closed under* $\wedge$*,* $\vee$*, and* $\neg$*.*

The next lemma shows that if $\mathbf{\Gamma}$ is closed under real quantification, then so will be Λ.

3.12 Lemma *Let* $\mathbf{\Gamma}$ *be non-selfdual, closed under* $\forall^{\omega^\omega}$, $\exists^{\omega^\omega}$*, and* pwo($\mathbf{\Gamma}$)*. Let* $\kappa = \delta(\mathbf{\Delta}) = o(\mathbf{\Delta})$*. Then* $\Lambda(\mathbf{\Gamma}, \kappa)$ *is closed under* $\forall^{\omega^\omega}$, $\exists^{\omega^\omega}$*.*

Proof. Let $A \subseteq \omega^\omega \times \omega^\omega$ be in $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa)$. Fix $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \mathbf{\Delta}$ and with $A \in \bar{\mathcal{A}}$ (here each $A_\alpha \subseteq \omega^\omega \times \omega^\omega$). Define $x \in B \leftrightarrow \exists y\ (x, y) \in A$. View every real z as coding a countable $s_z \subseteq \omega^\omega$ and a $f_z \colon s_z \to \{0, 1\}$. For d a Turing degree, let $\alpha_z(d)$ be the least ordinal $< \kappa$, if one exists, such that $\forall x \in s_z\ (f_z(x) = 1 \leftrightarrow \exists y \le d\ (x, y) \in A_\alpha)$. Let $C = \{z \colon \forall^* d\ \alpha_z(d) \text{ is defined}\}$. A straightforward computation using the closure of $\mathbf{\Gamma}$ shows that $C \in \mathbf{\Gamma}$. Let ϕ be the norm on C corresponding to the prewellordering $z_1 \preceq z_2 \leftrightarrow \forall^* d\ (\alpha_{z_1}(d) \le \alpha_{z_2}(d))$. A straightforward computation shows that the norm relation $\le^*_\phi$, $<^*_\phi$ are in $\mathbf{\Gamma}$, so ϕ is a $\mathbf{\Gamma}$-norm on C. For example,

$$\begin{aligned} z_1 \le^*_\phi z_2 \leftrightarrow \forall^* d\ \exists \alpha < \kappa\ \forall \beta < \alpha\ [&\forall x \in s_{z_1}\ (f_{z_1}(x) = 1 \leftrightarrow \exists y \le d\ (x, y) \in A_\alpha) \\ &\wedge \neg \forall x \in s_{z_2}\ (f_{z_2}(x) = 1 \leftrightarrow \exists y \le d\ (x, y) \in A_\beta)] \end{aligned}$$

This is in $\mathbf{\Gamma}$ using the closure of $\mathbf{\Delta}$ under $< \kappa$ intersections. Thus ϕ has length κ.

For $z \in C$, define $B_{\phi(z)}$ by:

$$x \in B_{\phi(z)} \leftrightarrow \forall^* d\ \exists y \le d\ (x, y) \in A_{\alpha_z(d)}.$$

A straightforward computation as above shows $B_{\phi(z)} \in \boldsymbol{\Delta}$. It suffices to show that $B \in \bar{\mathcal{B}}$ where $\mathcal{B} = \{B_{\phi(z)} \colon z \in C\}$. Let $s \subseteq \omega^\omega$ be countable. Define $f \colon s \to \{0,1\}$ by $f(x) = 1$ iff $x \in B$. Let z code s, f. Since $A \in \bar{\mathcal{A}}$ it follows that $z \in C$, so $\phi(z)$ is defined. Let d_0 be a large enough degree so that for all $x \in s$ we have $x \in B$ iff $\exists y \leq d_0\ (x,y) \in A$. For any $d \geq d_0$ and $x \in s$ we then have:

$$\begin{aligned} x \in B &\leftrightarrow \exists y \leq d\ (x,y) \in A \\ &\leftrightarrow \exists y \leq d\ (x,y) \in A_{\alpha_z(d)}, \end{aligned}$$

which shows that for $x \in s$, $x \in B$ iff $x \in B_{\phi(z)}$. ⊣

We next make precise the statement that $\Lambda(\boldsymbol{\Gamma}, \kappa)$ is the pointclass of sets "Wadge reducible to a measure on κ." Let $\boldsymbol{\Gamma}$, κ be as in lemma 3.10. From the coding lemma, every subset of κ by be coded within the pointclass $\exists^{\omega^\omega}\boldsymbol{\Gamma}$. To make this precise, let $U \subseteq \omega^\omega \times \omega^\omega$ be universal for $\exists^{\omega^\omega}\boldsymbol{\Gamma}$. Let ψ be an $\exists^{\omega^\omega}\boldsymbol{\Gamma}$-norm on a $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ set W of length κ (the existence of ψ follows from lemmas 2.18 and 2.19). For $z \in \omega^\omega$, define $B_z \subseteq \kappa$ by $\alpha \in B_z$ iff $\exists x \in W\ (\psi(x) = \alpha \wedge U(z,x))$. By the coding lemma, every subset of κ is of the form B_z for some z. We now define the code set C_μ of a measure μ on κ.

3.13 Definition *Let $\boldsymbol{\Gamma}$, κ be as in lemma 3.10. For μ a measure on κ define $C_\mu = \{z \colon \mu(A_z) = 1\}$.*

3.14 Lemma *Let $\boldsymbol{\Gamma}$, κ be as in lemma 3.10. Then $A \in \Lambda(\boldsymbol{\Gamma}, \kappa)$ iff there is a measure μ on κ such that $A \leq_w C_\mu$.*

Proof. First suppose μ is a measure on κ and we show $C_\mu \in \Lambda(\boldsymbol{\Gamma}, \kappa)$. For $\alpha < \kappa$ define $A_\alpha = \{z \colon \alpha \in A_z\}$. Clearly $A_\alpha \in \exists^{\omega^\omega}\boldsymbol{\Gamma}$. It suffices to observe that $C_\mu \in \bar{\mathcal{A}}$, where $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ (using the fact that $\Lambda(\exists^{\omega^\omega}\boldsymbol{\Gamma}, \kappa) = \Lambda(\boldsymbol{\Gamma}, \kappa)$ from the proof of lemma 3.10). Let $s \subseteq \omega^\omega$ be countable. Let $\alpha(s)$ be the least element of $\bigcap_{z \in s} A'_z$, where $A'_z = A_z$ if $z \in C_\mu$ and otherwise $A'_z = \kappa - A_z$. Then for all $z \in s$ we have $z \in C_\mu$ iff $z \in A_{\alpha(s)}$. So, $C_\mu \in \Lambda(\boldsymbol{\Gamma}, \kappa)$.

Suppose next that $A \in \Lambda(\boldsymbol{\Gamma}, \kappa)$. Fix $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with $A \in \bar{\mathcal{A}}$. For d a Turing degree let $f(d)$ be the least ordinal less than κ such that for all $z \in d$, $z \in A$ iff $z \in A_{f(d)}$. Define $\mu = f(\nu)$ where ν is the Martin measure on the Turing degrees. Consider the relation $R(w,x) \leftrightarrow (x \in W \wedge w \in A_{\psi(x)})$, where W, ψ are defined just before definition 3.13. From the coding lemma, $R \in \exists^{\omega^\omega}\boldsymbol{\Gamma}$. From the *s-m-n* theorem there is a continuous function $h \colon \omega^\omega \to \omega^\omega$ such that $U_{h(w)} = \{x \in W \colon w \in A_{\psi(x)}\}$, and thus $B_{h(w)} = \{\alpha \colon w \in A_\alpha\}$. We then have that for all $w \in \omega^\omega$, $w \in A$ iff $\forall^* d\ (w \in A_{f(d)})$ iff $\forall^*_\mu \alpha\ (w \in A_\alpha)$ iff $h(w) \in C_\mu$. So, $A \leq_w C_\mu$. ⊣

We next show that the pointclass $\Lambda(\boldsymbol{\Gamma}, \kappa)$ is where the next scale is constructed. We first show the upper bound.

3.15 Lemma *Let $\mathbf{\Gamma}$ be non-selfdual, closed under $\forall^{\omega^\omega}$, and* pwo($\mathbf{\Gamma}$)*. Assume also $\exists^{\omega^\omega}\mathbf{\Gamma}$ has the scale property with norms into $\kappa \doteq \delta(\mathbf{\Gamma})$. Assume also that there is a Suslin cardinal greater than κ. Then every set in $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$ admits a scale with each norm a $\Lambda(\mathbf{\Gamma}, \kappa)$-norm.*

Proof. In all cases we have $\mathrm{cof}(\kappa) > \omega$. If $A \in \exists^{\omega^\omega}\mathbf{\Gamma}$ then there is a tree T on $\omega \times \kappa$ with $A = p[T]$ by hypothesis. Since $\mathrm{cof}(\kappa) > \omega$ we may assume T has the property that if $(s, \vec{\alpha}) \in T$ with $\vec{\alpha} = (\alpha_0, \dots, \alpha_{n-1})$, then $\alpha_0 > \max\{\alpha_1, \dots, \alpha_{n-1}\}$. From theorem 3.2, T is weakly homogeneous. The homogeneous tree construction then produces a tree T' with $p[T'] = \omega^\omega - A \doteq B$. Let $\{\phi_n\}$ be the scale on B associated to T'. For $x \in B$, $\phi_n(x)$ is of the form $[f_x]_\mu$ where f_x is the rank function on T_x and μ is a measure on κ^j for some j. Our property of T gives that $f_x(\vec{\alpha}) < \kappa$ for every $\vec{\alpha} \in T_x$. It suffices to show that the norm relations $\leq^*_{\phi_n}$, $<^*_{\phi_n}$ are in $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa)$. We consider the case $\phi = \phi_0$ which is of the form $\phi(x) = [f_x(\alpha)]_\mu$ (that is, $j = 1$). The general case is similar. For α, $\beta < \kappa$, let

$$A_{\alpha,\beta} = \{(x, y) \colon |\alpha|_{T_x} \leq \beta \wedge \neg(|\alpha|_{T_y} \leq \beta)\},$$

where $|\alpha|_{T_x} \leq \beta$ means α is in the wellfounded part of T_x and has rank $\leq \beta$. We claim that each $A_{\alpha,\beta}$ is in $\mathbf{\Delta} = \mathbf{\Delta}(\mathbf{\Gamma})$. If $\mathbf{\Gamma}$ is closed under real quantification or $\mathbf{\Delta}$ is not closed under real quantification, then this follows from the fact that $\mathbf{\Delta}$ is closed under $< \kappa$ unions and intersections. The remaining case is when $\mathbf{\Gamma}$ is at the base of a type II or III hierarchy. Since $\mathbf{\Delta}$ is closed under real quantification, we may apply the coding lemma to a suitable non-selfdual pointclass $\mathbf{\Gamma}_0 \subseteq \mathbf{\Delta}$ closed under $\exists^{\omega^\omega}$ to code functions $h \colon (\alpha + 1)^{<\omega} \to \beta$. From this and the closure of $\mathbf{\Delta}$ under quantifiers we easily compute $A_{\alpha,\beta} \in \mathbf{\Delta}$.

Suppose $s = \{(x_n, y_n) \colon n \in \omega\}$ is a countable set of pairs. For each n such that $x_n \leq^*_\phi y$, let B_n be a μ measure one set such that for all $\alpha \in B_n$, $f_x(\alpha) \leq f_y(\alpha)$ (we allow here the possibility that $\vec{\alpha}$ is in the illfounded part of f_y). For other n, let B_n be such that for all $\vec{\alpha} \in B_n$ we have either α is in the illfounded part of T_x or else $f_y(\alpha) < f_x(\alpha)$. Fix $\alpha \in \bigcap_n B_n$. Let β be large enough so that for all n, if $f_{x_n}(\alpha)$ is defined, then $f_{x_n}(\alpha) < \beta$. Then for all n we have $x_n \leq^*_\phi y_n$ iff $(x_n, y_n) \in A_{\alpha,\beta}$. This shows $\leq^*_\phi \in \bar{\mathcal{A}}$ where $\mathcal{A} = \{A_{\alpha,\beta}\}$. A similar argument works for $<^*_\phi$. ⊣

We next get the lower bound.

3.16 Lemma *Let $\mathbf{\Gamma}$, κ be as in lemma 3.15. Let A be $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$-complete. Then A does not admit a scale all of whose norm relations are Wadge reducible to some $B \in \Lambda(\mathbf{\Gamma}, \kappa)$.*

Proof. Fix $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \mathbf{\Delta}$ and which is universal for $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa)$ (i.e., every $A \in \Lambda$ is Wadge reducible to some $B \in \bar{\mathcal{A}}$). This is possible from lemma 3.10. View every real r as coding a continuous function $r\colon (\omega)^2 \times (\omega^\omega)^3 \to \omega^\omega$. Define $D \subseteq \omega^\omega \times \omega^\omega$ by:

$$D(x,y) \leftrightarrow \exists r \leq_T x\ \forall^* d\ \exists \alpha < \kappa\ \forall n,m \in \omega\ (y(n) = m \leftrightarrow \exists z \leq_T d\ \forall w \leq_T d\ r(n,m,x,z,w) \in A_\alpha).$$

From the closure of $\exists^{\omega^\omega}\mathbf{\Gamma}$ under wellordered unions it follows easily that $D \in \forall^{\omega^\omega}\exists^{\omega^\omega}\mathbf{\Gamma}$. Also, each section D_x of D is countable since it can be wellordered (for fixed $r \leq_T x$, the wellordering is induced by the map $y \to [f]_\nu$ where ν is the Martin measure on the degrees and $f(d)$ is the least $\alpha < \kappa$ satisfying the definition above). Consider $\neg D \in \exists^{\omega^\omega}\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$. Since we are assuming every $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$ set has a scale all of whose norms are reducible to some $B \in \Lambda$, a standard argument shows that $\neg D$ has a uniformizing function $f\colon \omega^\omega \to \omega^\omega$ such that the relation $R(n,m,x) \leftrightarrow f(x)(n) = m$ is in $\Game\Lambda$. To see this, write $\neg D(x,y) \leftrightarrow \exists u\ E(x,y,u)$ with $E \in \forall^{\omega^\omega}\check{\mathbf{\Gamma}}$. Let $\{\phi_n\}$ be an excellent scale on E with all norm relations $\leq^*_{\phi_n}$ Wadge reducible to $B \in \Lambda$ (in particular $\phi_n(x,y,u)$ determines $(x{\restriction}n{+}1, y{\restriction}n{+}1, z{\restriction}n{+}1)$). This is possible since Λ is closed under $\wedge$, $\vee$. Let f be the canonical uniformizing function for $\neg D$ from this scale (that is, for some u, $(x, f(x), u)$ has minimal ϕ_n norm for all n). Then $f(x)(n) = m$ iff I wins the game $G^{n,m}_x$ where I and II play out $z = \langle y, u\rangle$, $w = \langle y', u'\rangle$ respectively and I wins the run iff $y(n) = m$ and $(x,y,z) \leq^*_{\phi_n} (x,y',z')$. Fix now a $B \in \Lambda$ and a real r coding a continuous function so that for all n,m,x,z,w,

$$\begin{aligned} f(x)(n) = m &\leftrightarrow \text{I wins the game } G^{n,m}_x \\ &\leftrightarrow \Game\langle z,w\rangle\ r(n,m,x,z,w) \in B \\ &\leftrightarrow \forall^* d\ \exists z \leq_T d\ \forall w \leq_T d\ r(n,m,x,z,w) \in B. \end{aligned}$$

Fix now $x \geq_T r$ and let $y = f(x)$, so $\neg D(x,y)$. On the other hand, for d a Turing degree let $\alpha(d) < \kappa$ be least such that B and $A_{\alpha(d)}$ agree on reals in d. Then for any n, m we have $y(n) = m$ iff $\forall^* d\ \exists z \leq_T d\ \forall w \leq_T d\ r(n,m,x,z,w) \in A_{\alpha(d)}$. Intersecting countably many cones gives that

$$\forall^* d\ \forall n,m\ (y(n) = m \leftrightarrow \exists z \leq_T d\ \forall w \leq_T d\ r(n,m,x,z,w) \in A_{\alpha(d)}).$$

Since $r \leq_T x$, this shows that $D(x,y)$, a contradiction. ⊣

Lemmas 3.15 and 3.16 show that $\Lambda(\mathbf{\Gamma},\kappa)$ is precisely where the norms relations for the next Suslin class are constructed. We record this in the following theorem.

3.17 Theorem *Let $\mathbf{\Gamma}$ be non-selfdual, closed under $\forall^{\omega^\omega}$, and* pwo($\mathbf{\Gamma}$). *Assume also $\exists^{\omega^\omega}\mathbf{\Gamma}$ has the scale property with norms into $\kappa \doteq \delta(\mathbf{\Gamma})$. Assume also that there is a Suslin cardinal greater than κ. Then every $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$ set admits a scale all of whose norm relations lie in the pointclass $\Lambda = \Lambda(\mathbf{\Gamma},\kappa)$. Furthermore, there is no scale on a $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$ complete set all of whose norm relations are Wadge reducible to some $B \in \Lambda$.*

It is immediate from theorem 3.17 that $\mathrm{cof}(o(\Lambda)) = \omega$. When $\mathbf{\Gamma}$ is closed under real quantifiers we can say a bit more.

3.18 Lemma *Let $\mathbf{\Gamma}$ be non-selfdual, closed under real quantification, and* $\mathrm{scale}(\mathbf{\Gamma})$. *Assume there is Suslin cardinal greater than $\kappa = o(\mathbf{\Gamma})$. Then $\lambda \doteq o(\Lambda)$ (where $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa)$) is the least Suslin cardinal greater than κ. Furthermore, every set in Λ admits a scale all of whose norm relations lie in Λ.*

Proof. Recall in this case that Λ is a selfdual class closed under real quantification. If $A \in \check{\mathbf{\Gamma}}$ and $\{\phi_n\}$ is the scale on A as in lemma 3.15 (i.e., from the homogeneous tree construction) then each norm relation $\leq_n^*$ is in Λ and so has length less than $\delta(\Lambda) = o(\Lambda)$. Thus, A is λ-Suslin. On the other hand, if A were λ'-Suslin where $\lambda' < \lambda$, then from the coding lemma and the closure of Λ under real quantifiers a straightforward computation shows that for some $B \in \Lambda$, $\leq_n^*$ is Wadge reducible to B for all n, a contradiction to lemma 3.16 [Since $\delta(\mathbf{\Delta}) = o(\mathbf{\Delta})$ we can find a non-selfdual $\mathbf{\Gamma}_0 \subseteq \mathbf{\Delta}$ closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$, with $\mathrm{pwo}(\mathbf{\Gamma}_0)$ and such that there is a $\mathbf{\Gamma}_0$ prewellordering of length λ'. Then each $\leq_n^*$ is in $\Sigma_2(\mathbf{\Gamma}_0)$.] Thus, λ is the next Suslin cardinal after κ.

Suppose now $B \in \Lambda$, and consider again the scale $\{\phi_n\}$ on $A \in \check{\mathbf{\Gamma}} - \mathbf{\Gamma}$ as in lemma 3.15. Let $\mathbf{\Gamma}_0 \subseteq \Lambda$ be a non-selfdual pointclass containing B and $\omega^\omega - B$. Let $\mathbf{\Gamma}_1 \subseteq \Lambda$ be non-selfdual, closed under $\exists^{\omega^\omega}$, $\mathrm{pwo}(\mathbf{\Gamma}_1)$, and $\mathbf{\Gamma}_1$ properly contains $\mathbf{\Gamma}_0$. Let $\lambda_1 < \lambda$ be greater than the supremum of the lengths of the $\mathbf{\Gamma}_1$ prewellorders. Fix n so that ϕ_n has length $> \lambda_1$ (by the previous paragraph). For α less than the length of ϕ_n, let $A_\alpha = \{x \in A \colon \phi_n(x) \leq \alpha\}$. If all the A_α are in $\mathbf{\Gamma}_0$, then from lemma 2.18 we would have a $\mathbf{\Gamma}_1$ prewellordering of length $\geq \lambda_1$, a contradiction. Fix α so that $A_\alpha \notin \mathbf{\Gamma}_0$. Then $B \leq_w A_\alpha$. The scale $\{\phi_n\}$ restricted to A_α is a scale on A_α with all the norm relations in Λ. Since $B \leq_w A_\alpha$, B also admits such a scale. ⊣

3.3. The Classification of the Suslin Cardinals

We are now ready to classify the Suslin cardinals and scale property assuming AD. This is the content of the next theorem. The reader should recall the definition of the Steel pointclass from remark 2.17

3.19 Theorem *Let κ be a limit of Suslin cardinals, and suppose there is Suslin cardinal greater than κ. Then κ is a Suslin cardinal and one of the following cases applies.*

<u>Case I</u>. $\mathrm{cof}(\kappa) = \omega$.

Let $\mathbf{\Sigma}_0 = \bigcup_\omega S_{<\kappa}$, where $S_{<\kappa} = \bigcup_{\rho<\kappa} S(\rho)$. Define $\mathbf{\Pi}_0 = \check{\mathbf{\Sigma}}_0$, and for $i > 0$, define as usual $\mathbf{\Sigma}_i = \exists^{\omega^\omega} \mathbf{\Pi}_{i-1}$ and $\mathbf{\Pi}_i = \forall^{\omega^\omega} \mathbf{\Sigma}_{i-1}$. Then $\mathbf{\Sigma}_0$ has the *scale property with norms into κ. Also, $S(\kappa) = \mathbf{\Sigma}_1$. Let $\delta_{2i+1} = \delta(\mathbf{\Pi}_{2i+1})$. Then for all i,* scale$(\mathbf{\Pi}_{2i+1})$ *and* scale$(\mathbf{\Sigma}_{2i+2})$ *with scales into δ_{2i+1}. δ_{2i+1} is a Suslin cardinal and $S(\delta_{2i+1}) = \mathbf{\Sigma}_{2i+2}$. Also, $\delta_{2i+1} = (\lambda_{2i+1})^+$, where λ_{2i+1} is a Suslin cardinal of cofinality ω, and $\lambda_1 = \kappa$. $S(\lambda_{2i+1}) = \mathbf{\Sigma}_{2i+1}$. The sequence δ_1, λ_3, $\delta_3, \ldots$ enumerates the first ω Suslin cardinals greater than κ.*

Case II. cof$(\kappa) > \omega$ *and the Steel pointclass $\mathbf{\Gamma}_0$ is not closed under $\exists^{\omega^\omega}$. Let $\mathbf{\Sigma}_0 = \exists^{\omega^\omega}\mathbf{\Gamma}_0$, $\mathbf{\Pi}_0 = \check{\mathbf{\Sigma}}_0$, and for $i > 0$ define the $\mathbf{\Sigma}_i$, $\mathbf{\Pi}_i$ as usual. Then* scale$(\mathbf{\Gamma}_0)$ *and* scale$(\mathbf{\Sigma}_0)$ *with norms into κ. Let $\delta_{2i+1} = \delta(\mathbf{\Pi}_{2i+1})$. Then for all i,* scale$(\mathbf{\Pi}_{2i+1})$ *and* scale$(\mathbf{\Sigma}_{2i+2})$ *with scales into δ_{2i+1}. δ_{2i+1} is a Suslin cardinal and $S(\delta_{2i+1}) = \mathbf{\Sigma}_{2i+2}$. Also, $\delta_{2i+1} = (\lambda_{2i+1})^+$, where λ_{2i+1} is a Suslin cardinal of cofinality ω. $S(\lambda_{2i+1}) = \mathbf{\Sigma}_{2i+1}$. The sequence λ_1, δ_1, λ_3, $\delta_3, \ldots$ enumerates the first ω Suslin cardinals greater than κ.*

Case III. cof$(\kappa) > \omega$ *and the Steel pointclass $\mathbf{\Gamma}_0$ is closed under $\exists^{\omega^\omega}$. $\mathbf{\Gamma}_0$ has the scale property with norms into κ. Let $\Lambda = \Lambda(\mathbf{\Gamma}_0, \kappa)$. Let $\mathbf{\Sigma}_0 = \bigcup_\omega \Lambda$, $\mathbf{\Pi}_0 = \check{\mathbf{\Sigma}}_0$, and for $i > 0$ define $\mathbf{\Sigma}_i$, Π_i as usual. Then for all i,* scale$(\mathbf{\Pi}_{2i+1})$ *and* scale$(\mathbf{\Sigma}_{2i+2})$ *with scales into δ_{2i+1}. δ_{2i+1} is a Suslin cardinal and $S(\delta_{2i+1}) = \mathbf{\Sigma}_{2i+2}$. Also, $\delta_{2i+1} = (\lambda_{2i+1})^+$, where λ_{2i+1} is a Suslin cardinal of cofinality ω. $S(\lambda_{2i+1}) = \mathbf{\Sigma}_{2i+1}$. The sequence λ_1, δ_1, λ_3, $\delta_3, \ldots$ enumerates the first ω Suslin cardinals greater than κ, and $\lambda_1 = o(\Lambda)$.*

Proof. Consider first case I, that is, cof$(\kappa) = \omega$. Let $A \in \mathbf{\Sigma}_0$, and write $A = \bigcup_i A_i$ with $A_i \in S_{<\kappa}$. Let $\{\phi^i_n\}_{n\in\omega}$ be a scale on A_i of length $\gamma_i < \kappa$. From the coding lemma, all of the norm relations $\leq^*_{\phi^i_n}$, $<^*_{\phi^i_n}$ are in $S_{<\kappa}$. Define the norms ψ_n on A as follows. $\psi_0(x) = \mu i\ (x \in A_i)$, $\psi_{k+1}(x) = \langle \phi_0(x), \phi_k^{\phi_0(x)}(x)\rangle$, where $\langle\alpha,\beta\rangle$ denotes the rank of (α,β) in the lexicographic ordering on $(\gamma_i)^2$. This is easily checked to be a scale on A with norms into κ, and is clearly a $\mathbf{\Sigma}_0$-scale. Thus, scale$(\mathbf{\Sigma}_0)$. From the coding lemma an easy computation show $S(\kappa) \subseteq \mathbf{\Sigma}_1$. Also, $\mathbf{\Sigma}_1 \subseteq S(\kappa)$ as $S(\kappa)$ is closed under $\exists^{\omega^\omega}$ and countable intersections. Thus, $S(\kappa) = \mathbf{\Sigma}_1$. By periodicity, scale$(\mathbf{\Pi}_{2i+1})$, scale$(\mathbf{\Sigma}_{2i})$. By definition of δ_{2i+1}, every regular $\mathbf{\Pi}_{2i+1}$ scale has length $\leq \delta_{2i+1}$. In particular, every $\mathbf{\Pi}_{2i+1}$, and hence every $\mathbf{\Sigma}_{2i+2}$ set is δ_{2i+1}-Suslin. A straightforward computation from the coding lemma shows $S(\delta_{2i+1}) \subseteq \mathbf{\Sigma}_{2i+2}$, so $S(\delta_{2i+1}) = \mathbf{\Sigma}_{2i+2}$. Every $\mathbf{\Pi}_{2i+1}$ set admits a $\mathbf{\Pi}_{2i+1}$-scale with norms into δ_{2i+1}, and the standard argument transferring the scale property to $\exists^{\omega^\omega}\mathbf{\Pi}_{2i+1}$ shows that every $\mathbf{\Sigma}_{2i+2}$ set admits a $\mathbf{\Sigma}_{2i+2}$-scale with norms into δ_{2i+1} (that is, the lengths of the norms do not increase with this transfer). Since $\mathbf{\Pi}_{2i+1}$ is closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$ and pwo$(\mathbf{\Pi}_{2i+1})$, another standard argument (c.f. the proof of theorem 2.14) shows that δ_{2i+1} is the supremum of the lengths of the $\mathbf{\Sigma}_{2i+1}$ wellfounded relations, and from the coding lemma this must be a regular cardinal. So, δ_{2i+1} is regular. Clearly δ_{2i+1} cannot be a limit Suslin cardinal (there are only finitely many Levy classes between $\mathbf{\Sigma}_o$ and $\mathbf{\Sigma}_{2i+1}$). From lemma 3.7, $\delta_{2i+1} = \lambda^+_{2i+1}$ for some Suslin cardinal λ_{2i+1} with cof$(\lambda_{2i+1}) = \omega$. Also from that lemma, $\mathbf{\Sigma}_{2i+2} = \exists^{\omega^\omega} \check{S}(\lambda_{2i+1})$, and thus $S(\lambda_{2i+1}) = \mathbf{\Sigma}_{2i+1}$ (this also follows from the fact there is only one Levy class closed under $\exists^{\omega^\omega}$ between $\mathbf{\Sigma}_{2i}$ and $\mathbf{\Sigma}_{2i+2}$). As we have now accounted for all the Levy classes closed under $\exists^{\omega^\omega}$, we must have $\delta_1 = \kappa^+$ and δ_1, λ_3, $\delta_3, \dots$ enumerates the next ω Suslin cardinals after κ.

Consider next case II. From lemma 3.8 we have scale($\mathbf{\Sigma}_0$) with norms into κ. Since $\mathbf{\Sigma}_0$ is not closed under $\forall^{\omega^\omega}$ (as $\mathbf{\Pi}_0 \subseteq \forall^{\omega^\omega} \mathbf{\Sigma}_0$) we may propagate the scale property by periodicity to $\mathbf{\Pi}_{2i+1}$, $\mathbf{\Sigma}_{2i+2}$. As in the previous case, δ_{2i+1} is a Suslin cardinal, $S(\delta_{2i+1}) = \mathbf{\Sigma}_{2i+2}$, and $\delta_{2i+1} = \lambda_{2i+1}^+$, where λ_{2i+1} is a Suslin cardinal of cofinality ω. In this case, since $\text{cof}(\kappa) > \omega$, we have $\lambda_1 > \kappa$. From lemma 3.7 again we have that $S(\lambda_{2i+1}) = \mathbf{\Sigma}_{2i+1}$. As we have again accounted for all the Levy classes closed under $\exists^{\omega^\omega}$, the sequence λ_1, δ_1, λ_3, $\delta_3 \dots$ enumerates the next ω Suslin cardinals after κ. scale($\mathbf{\Gamma}_0$) follows from lemma 3.8.

Consider now case III. From lemma 3.8, $\mathbf{\Gamma}_0$ has the scale property with norms into κ, and also $S(\kappa) = \mathbf{\Gamma}_0$. From lemma 3.18, $\mathbf{\Sigma}_0$ has the scale property with norms into $\lambda \doteq o(\Lambda)$. Recall in this case that Λ is selfdual, closed under real quantification, and $\text{cof}(\lambda) = \omega$. From lemma 3.18, λ is the next Suslin cardinal after κ, and $S(\lambda) = \mathbf{\Sigma}_1$ ($\mathbf{\Sigma}_1 \subseteq S(\lambda)$ follows from the closure of $S(\lambda)$ under $\exists^{\omega^\omega}$ and countable intersections, and $S(\lambda) \subseteq \mathbf{\Sigma}_1$ follows by a straightforward computation using the coding lemma and the closure of $\mathbf{\Sigma}_1$ under countable unions and intersections). Since λ is the next Suslin cardinal after κ and δ_1 is regular, we must have $\delta_1 > \lambda$. Since every $\mathbf{\Sigma}_1$ wellfounded relation has length $< \lambda^+$ by the Kunen-Martin theorem, it follows that $\delta_1 \leq \lambda^+$, and so $\delta_1 = \lambda^+$. In particular, $\lambda_1 = \lambda$. The remaining arguments are exactly as in the previous cases. ⊣

4. Trivial Descriptions: A Theory of ω_1

We assume AD + DC throughout §4.

Our goal in this section is to present a "theory of ω_1," using only techniques that will generalize to higher levels. Starting from the weak partition relation on ω_1 (proved in the last section), we prove the strong partition relation on ω_1, calculate $\boldsymbol{\delta}^1_3$, and prove the weak partition relation on $\boldsymbol{\delta}^1_3$. This represents the first step in the inductive analysis of the projective ordinals. We also use these techniques to present a proof of the Kechris-Martin theorem on $\mathbf{\Pi}^1_3$. These results are not new (c.f. [9, 33]), but our proofs do not rely on the theory of indiscernibles for L (as did the original proofs) but rather on a direct analysis on the measures on ω_1 (and the ω_n). The idea of using an analysis of measures to provide a good coding for sets was first used by Kunen (see [33]) in the original proof of the weak partition relation on $\boldsymbol{\delta}^1_3$.

As we mentioned in the introduction, one of our motivations in this section is to introduce and use the terminology of "descriptions," even though the concept at this level is rather trivial and could be dispensed with. This way, the arguments at the higher levels will have the same general form, though the concept of description will become non-trivial. This will free us, in the next section, to concentrate on this point.

Our first job is to analyze the measures on ω_1, from which the strong partition relation on ω_1 will follow.

Recall that if $T \subseteq \rho^{<\omega}$ is a tree on ρ and $\alpha < \rho$, then $T\restriction\alpha = T \cap \alpha^{<\omega}$ denotes the tree restricted to α, and if β is in the well-founded part of $T\restriction\alpha$, then $|(T\restriction\alpha)(\beta)|$ denotes the rank of β in $T\restriction\alpha$. Also, if T is a tree on $\alpha \times \rho$, then $T_x = \{\vec{s} \in \rho^{<\omega} \colon (x\restriction \mathrm{lh}(\vec{s}), \vec{s}) \in T\}$ is the section of the tree at x.

Let WO $\subseteq \omega^\omega$ be the standard set of codes for well-orderings, that is $x \in$ WO iff $\prec_x \doteq \{(n,m) \colon x(\langle n,m\rangle) = 1\}$ is a well-ordering of ω. Let WF $\supseteq$ WO be the set of codes of well-founded, transitive relations on ω. That is, $x \in$ WF iff $\prec_x$ is a well-founded and transitive relation. Both WO, WF are in $\Pi^1_1 - \boldsymbol{\Sigma}^1_1$. For $x \in$ WF, let $|x| = |\prec_x| = \sup_{n \in \mathrm{dom}(\prec_x)} (|n|_{\prec_x} + 1)$.

The finite exponent partition relation on ω_1 easily implies that the c.u.b. filter on ω_1 is a normal measure, and is the unique normal measure on ω_1. We let W^m_1 denote the m-fold product of this normal measure. Equivalently, $A \subseteq (\omega_1)^m$ has measure one with respect to W^m_1 iff there is a c.u.b. $C \subseteq \omega_1$ such that $(C)^m \subseteq A$. Note that we regard W^m_1 as a measure on the set of tuples $(\alpha_0, \dots, \alpha_{m-1})$ for which $\alpha_0 < \cdots < \alpha_{m-1}$. With this convention, we may safely write the ordinals in the tuple in any order (which will be notationally convenient later).

Recall the construction of the Kunen tree on ω_1:

4.1 Lemma (Kunen) *There is a tree T on $\omega \times \omega_1$ such that for all $f \colon \omega_1 \to \omega_1$, there is an $x \in \omega^\omega$ such that T_x is well-founded and for all $\omega \le \alpha < \omega_1$ we have $f(\alpha) \le |T_x\restriction\alpha|$.*

Proof. Let S be a recursive tree on $\omega \times \omega$ with $p[S]$ a $\boldsymbol{\Sigma}^1_1$-complete set. Define the tree U on $\omega \times \omega_1$ by:

$$((a_0, \dots, a_{n-1}), (\alpha_0, \dots, \alpha_{n-1})) \in U \leftrightarrow \forall i, j < n \ (a_{\langle i,j\rangle} = 1 \to \alpha_i < \alpha_j)$$
$$\wedge\ \forall i, j, k < n \ (a_{\langle i,j\rangle} = 1 \wedge a_{\langle j,k\rangle} = 1 \to a_{\langle i,k\rangle} = 1)$$

Clearly, $p[U] =$ WF. Let V be the tree on $\omega \times \omega \times \omega_1 \times \omega \times \omega$ defined by:

$$(\vec{s}, \vec{a}, \vec{\alpha}, \vec{b}, \vec{c}) \in V \leftrightarrow (\vec{a}, \vec{\alpha}) \in U \wedge (\vec{b}, \vec{c}) \in S$$
$$\wedge\ \exists\sigma \text{ extending } \vec{s}\ (\sigma(\vec{a}) = \vec{b}),$$

where we view every real σ as coding a strategy for II in some reasonable manner. Suppose $f \colon \omega_1 \to \omega_1$. Consider the game where I, II play out x, y, and II wins iff $[x \in \mathrm{WF} \to S_y$ is wellfounded $\wedge |S_y| > \sup_{\beta \leq |x|} f(\beta)]$. II easily wins by boundedness, noting that

$$\sup\{|S_y| \colon S_y \text{ is wellfounded}\} = \omega_1,$$

as otherwise $p[S]$ would be Borel. Let σ be winning for II. Note that for all $\alpha < \omega_1$, if $x \in \mathrm{WF}$ and $|x| = \alpha$, then $U_x \restriction \alpha$ is ill-founded. It follows that V_σ is well-founded and for all $\alpha \geq \omega$, we have $|V_\sigma \restriction \alpha| > f(\alpha)$. It is now easy to code the 2^{nd}, 3^{rd}, 4^{th}, and 5^{th} coordinates of V into the second coordinate of a tree T (say by weaving the values of the four components; this does not decrease rank, that is, $|T_\sigma \restriction \alpha| \geq |V_\sigma \restriction \alpha|$ for all σ, α). ⊣

We note that the Kunen tree T is Δ^1_1 in the codes. By this we mean that there are Σ^1_1, Π^1_1 relations $S(n, a, x)$, $R(n, a, x)$ such that for all x with $x_0, \ldots, x_{n-1} \in \mathrm{WO}$, we have

$$S(n, a, x) \leftrightarrow R(n, a, x) \leftrightarrow ((a_0, \ldots, a_{n-1}), (|x_0|, \ldots, |x_{n-1}|)) \in T.$$

This follows immediately from the definition of T (see lemma 4.1).

Although the difference is not large, it is sometimes to more convenient to deal with linear orderings rather than trees. The following theorem shows that we may modify the Kunen tree so as to make this possible. Recall that if $s \in \omega^n$, then $T_s = \{\vec{\alpha} \in \omega_1^n : (s, \vec{\alpha}) \in T\}$.

4.2 Theorem *There is a function $s \to T(s)$ which assigns to each $s \in \omega^{<\omega}$ a well-ordering of a subset of ω_1 with the following properties. If t extends s, then $T(t) \supseteq T(s)$. For $x \in \omega^\omega$, let $T(x) = \bigcup_n T(x \restriction n)$, so $T(x)$ is a linear order. Then for any $f : \omega_1 \to \omega_1$, there is an $x \in \omega^\omega$ such that $T(x)$ is a well-ordering and for all $\alpha \geq \omega$, $f(\alpha) < |T(x) \restriction \alpha|$. Furthermore, the map $s \to T(s)$ is Δ^1_1 in the codes. That is, there are Σ^1_1, Π^1_1 relations $S(n, a, x, y)$, $R(n, a, x, y)$ such that for all $x, y \in$ WO, we have*

$$S(n, a, x, y) \leftrightarrow R(n, a, x, y) \leftrightarrow [(|x|, |y|) \in T(a_0, \ldots, a_{n-1})].$$

Proof. We modify the Kunen tree T as follows. Fix a bijection $\pi : (\omega_1)^{<\omega} \to \omega_1$ such that for all $\alpha_0, \ldots, \alpha_{n-1} < \omega$, we have $\pi(\alpha_0, \ldots, \alpha_{n-1}) < \omega$. For $s \in \omega^{<\omega}$, let $T(s)$ be the well-ordering defined by

$$\alpha \; T(s) \; \beta \leftrightarrow \pi^{-1}(\alpha), \pi^{-1}(\beta) \in T_s \wedge (\pi^{-1}(\alpha) <^{\mathrm{KB}}_{T_s} \pi^{-1}(\beta)),$$

where $<^{\mathrm{KB}}_{T_s}$ denotes the Kleene-Brouwer ordering on T_s. For $x \in \omega^\omega$, let $T(x) = \bigcup_n T(x \restriction n)$. Clearly $T(x)$ is a linear ordering, and is a well-ordering iff T_x is well-founded. Suppose now $f : \omega_1 \to \omega_1$. Let $C \subseteq \omega_1$ be the c.u.b. set of ordinals closed under π. Note that $\omega \in C$. For $\alpha \geq \omega$, let $l(\alpha)$ be the greatest element of C which is less than or equal to α. Define

$$f'(\alpha) = \sup\{f(\beta) : l(\beta) = l(\alpha)\}.$$

Let $x \in \omega^\omega$ be such that T_x is well-founded and for all $\alpha \geq \omega$, $|T_x\restriction\alpha| > f'(\alpha)$. We claim that for all $\alpha \geq \omega$, $|T(x)\restriction\alpha| > f(\alpha)$. Note that π^{-1} applied to $T(x)\restriction\alpha$ contains the entire tree $T_x\restriction l(\alpha)$. Thus, $|T(x)\restriction\alpha| \geq |T_x\restriction l(\alpha)| > f'(l(\alpha)) \geq f(\alpha)$. We may also choose the bijection π so that π is Δ^1_1 in the codes, and it is then straightforward to check that $s \rightarrow T(s)$ is Δ^1_1 in the codes. ⊣

In the future, we will use the notation T_s for the Kunen "tree," regardless of whether we are using the tree T_s or the linear ordering $T(s)$. The meaning will be clear from the context.

For the rest of §4, T will denote the Kunen tree of lemma 4.1.

Note that for every $f\colon \omega_1 \rightarrow \omega_1$, the equivalence class $[f]_{W^1_1}$ may be coded by a pair (x, β) where $x \in \omega^\omega$, $\beta < \omega_1$. By this we mean $\forall^*_{W^1_1}\alpha\ f(\alpha) = |(T_x\restriction\alpha)(\beta)|$. To see that x and β exist, use normality and the fact that $\forall^*\alpha$ $\exists\beta < \alpha\ f(\alpha) = |(T_x\restriction\alpha)(\beta)|$.

4.3 Definition *A level* -1*, or* trivial description*,* d *is simply a positive natural number* $d \in \omega$*. We let* $\mathcal{D}^{-1} = \omega - \{0\}$ *be the set of trivial descriptions. The interpretation function* h *assigns to each* $d \in \mathcal{D}^{-1}$ *an ordinal by:* $h(d) = d$*. We say a trivial description* d *is well-defined, or satisfies condition D, with respect to a measure* W^m_1 *iff* $d \leq m$*. If* d *is well-defined with respect to* W^m_1 *and* $\beta_1 < \cdots < \beta_m < \omega_1$*, we define* $h(\beta_1, \ldots, \beta_m; d) = \beta_d$*. If* $g\colon \omega_1 \rightarrow \omega_1$*, and* d *is well-defined with respect to* W^m_1*, we define an ordinal* $(g; W^m_1; d)$*. This is represented with respect to* W^m_1 *by the function which assigns to* $\beta_1, \ldots, \beta_m$ *the ordinal* $(g; \beta_1, \ldots, \beta_m; d) \doteq g(h(\beta_1, \ldots, \beta_m; d)) = g(\beta_d)$*.*

Clearly, $(g; W^m_1; d)$ only depends on $[g]_{W^1_1}$. In this way, the trivial descriptions, together with the equivalence classes of functions with respect to the normal measure on ω_1 generate equivalence classes with respect to the family of measures W^m_1. Note that for $g = \mathrm{id}$, the identity function on ω_1, we have $(\mathrm{id}; \vec{\beta}; d) = \beta_d$. We introduce a "lowering operator" $\mathcal{L}$ on $\mathcal{D}^{-1}$.

4.4 Definition *We define* $\mathcal{L}(d) = d - 1$ *for* $d > 1$*. If* $d = 1$*, we do not define* $\mathcal{L}(d)$*, and say* $d = 1$ *is minimal with respect to* $\mathcal{L}$*.*

By definition, there is a unique $d \in \mathcal{D}^{-1}$ which is minimal with respect to $\mathcal{L}$.

As a warm-up, we use this terminology to recast one familiar proof of the computation $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$. In this context, our "main technical lemma" is the following (the reader will note that the lemma corresponds to a well-known property of indiscernibles for L).

4.5 Lemma *Let* $f\colon (\omega_1)^m \rightarrow \omega_1$*,* d *be well-defined with respect to* W^m_1*, and assume* $[f]_{W^m_1} < (\mathrm{id}; W^m_1; d)$*. If* d *is non-minimal with respect to* $\mathcal{L}$*, then* *there is a* $g\colon \omega_1 \to \omega_1$ *such that* $[f]_{W_1^m} < (g; W_1^m; \mathcal{L}(d))$*. If* d *is minimal with respect to* $\mathcal{L}$*, then* $\exists \alpha < \omega_1\ [f]_{W_1^m} < \alpha$*.*

Proof. We have $\forall^*_{W_1^m} \beta_1, \dots, \beta_m\ f(\vec{\beta}) < (\mathrm{id}; \beta_1, \dots, \beta_m; d) = \beta_d$. Consider the case d non-minimal with respect to $\mathcal{L}$. Consider the partition $\mathcal{P}$, where we partition ordinals $\beta_1 < \cdots < \beta_m < \omega_1$ with the extra ordinal $\beta_{d-1} < \gamma < \beta_d$ according to whether $\gamma > f(\beta_1, \dots, \beta_m)$. Clearly, on the homogeneous side the stated property holds. Let $C \subseteq \omega_1$ be c.u.b. and homogeneous for $\mathcal{P}$. Let $g(\alpha) =$ the least element of C greater than α. Then $\forall^*_{W_1^m} \beta_1, \dots, \beta_m\ f(\vec{\beta}) < g(\beta_{d-1}) = (g; \beta_1, \dots, \beta_m; \mathcal{L}(d))$, thus $[f]_{W_1^m} < (g; W_1^m; \mathcal{L}(d))$. The case where d is minimal with respect to $\mathcal{L}$ is similar. ⊣

We let $<$ be the transitive relation on $\mathcal{D}^{-1}$ generated by the relation $\mathcal{L}(d) < d$. Of course, $<$ is just the usual ordering on the positive integers. We let $|d|$ denote the rank of d in this ordering, so $|d| = d - 1$.

Lemma 4.5 and the analysis of functions $f\colon \omega_1 \to \omega_1$ with respect to the normal measure W_1^1 on ω_1 (i.e., the Kunen tree construction) suffice to compute upper bound for $j_{W_1^m}(\omega_1)$, where $j_{W_1^m}$ denotes the ultrapower embedding corresponding to the measure W_1^m. This is made explicit in the following theorem.

4.6 Theorem *Let* d *be defined with respect to* W_1^m*. Then* $(\mathrm{id}; W_1^m; d) \leq \omega_{|d|+1}$*.*

Proof. By induction on $|d|$. If $|d| = 0$ (i.e., $d = 1$), then $(\mathrm{id}; W_1^m; d) \leq \omega_1$ from lemma 4.5. Otherwise, let $\alpha < (\mathrm{id}; W_1^m; d)$. From lemma 4.5, $\exists g\colon \omega_1 \to \omega_1$ such that $\alpha < (g; W_1^m; \mathcal{L}(d))$. Recall T is the Kunen tree of lemma 4.1. Let $x \in \omega^\omega$ be such that T_x is well-founded and $\forall^* \beta\ g(\beta) < |T_x \restriction \beta|$. Let $|T_x|$ also denote the function $\beta \to |T_x \restriction \beta|$. Thus, $\alpha < (|T_x|; W_1^m; \mathcal{L}(d))$. T_x also induces a bijection π between $(\mathrm{id}; W_1^m; \mathcal{L}(d))$ and $(|T_x|; W_1^m; \mathcal{L}(d))$. Namely, if $\delta < (\mathrm{id}; W_1^m; \mathcal{L}(d))$, then $\pi(\delta) < (|T_x|; W_1^m; \mathcal{L}(d))$ is defined by

$$\forall^*_{W_1^m} \beta_1, \dots, \beta_m\ [\pi(\delta)(\vec{\beta}) = |(T_x \restriction (\mathrm{id}; \vec{\beta}; \mathcal{L}(d)))(\delta(\vec{\beta}))|]$$

Thus, $(g; W_1^m; d) < (\mathrm{id}; W_1^m; \mathcal{L}(d))^+$, and so $(\mathrm{id}; W_1^m; d) \leq (\mathrm{id}; W_1^m; \mathcal{L}(d))^+$. By induction, $(\mathrm{id}; W_1^m; \mathcal{L}(d)) \leq \omega_{|\mathcal{L}(d)|+1} = \omega_{|d|}$, so $(\mathrm{id}; W_1^m; d) \leq \omega_{|d|+1}$. ⊣

As a corollary, we have an upper bound for $\boldsymbol{\delta}^1_3$.

4.7 Corollary $\boldsymbol{\delta}^1_3 \leq \omega_{\omega+1}$.

Proof. The homogeneous tree analysis, which we will not reproduce here, shows that every $\boldsymbol{\Pi}^1_2$ set is λ_3-Suslin, where where $\lambda_3 \leq \sup j_\nu(\omega_1)$, the supremum ranging over measures ν occurring in the homogeneous tree on a $\mathbf{\Pi}^1_1$-complete set; that is the measures W_1^m. [The homogeneous tree construction is described in detail in [18]. The definition of a homogeneous tree is given in §5, and more general arguments are presented there as well. In particular, the arguments immediately after definition 5.1 suffice to prove the above claim.] From the Kunen-Martin theorem, it follows that $\boldsymbol{\delta}^1_3 \leq [\sup_m j_{W_1^m}(\omega_1)]^+$. Now, a small variation in the proof of lemma 4.5 shows that $j_{W_1^m}(\omega_1) \leq (\mathrm{id}; W_1^m; \tilde{d})^+$, where $\tilde{d} \doteq m$ is the *maximal* description defined with respect to W_1^m. Thus, $j_{W_1^m}(\omega_1) \leq \omega_{m+1}$, and so $\lambda_3 \leq \omega_\omega$. ⊣

We will get the lower bound for $\boldsymbol{\delta}^1_3$ as a consequence of a general result of Martin. However, we first need the strong partition relation on ω_1.

4.1. Analysis of Measures on $\boldsymbol{\delta}^1_1$

The next theorem is our analysis of an arbitrary measure ν on ω_1. The key idea is to exploit a pressing down argument with respect to ν.

4.8 Theorem *Let ν be a measure on ω_1. Then there are finitely many reals $x_0, \dots, x_n$ with T_{x_i} well-founded for $0 \leq i \leq n$, and an ordinal $\alpha < \omega_1$ such that for all $A \subseteq \omega_1$:*

$$\nu(A) = 1 \leftrightarrow \forall^*_{W_1^{n+1}} \beta_0, \dots, \beta_n \ (h^\alpha_{x_0,\dots,x_n}(\beta_0, \dots, \beta_n) \in A),$$

where for $0 \leq i \leq n$, $\delta_i \doteq h^\alpha_{x_0,\dots,x_i}(\beta_0, \dots, \beta_i)$ is defined inductively by: $\delta_i = |(T_{x_i} \restriction \beta_i)(\delta_{i-1})|$, and $\delta_{-1} = \alpha$. In particular, ν is equivalent to W_1^{n+1} for some $n \in \omega$.

Proof. We may assume that ν is non-principal. Let $g_0 \colon \omega_1 \to \omega_1$ satisfy:

1. There is a ν measure one set A such that $g_0 \restriction A$ is monotonically increasing (i.e., if $\alpha < \beta$ are in A, then $g_0(\alpha) \leq g_0(\beta)$).
2. There does not exist a ν measure one set A such that $g_0 \restriction A$ is constant.
3. If $[g]_{W_1^1} < [g_0]_{W_1^1}$, then g does not satisfy (1) and (2).

g_0 exists, since the identity function satisfies (1), (2). Consider the measure $g_0(\nu)$ (that is, $g_0(\nu)(A) = 1$ iff $\nu(g_0^{-1}(A)) = 1$). If C is c.u.b. , then $g_0(\nu)(C) = 1$, as otherwise $p \circ g_0$ violates the minimality of g_0, where $p(\alpha) =$ the largest element of C less than or equal to α. Thus, $g_0(\nu) = W_1^1$. Fix A of ν measure one such that $g_0 \restriction A$ is monotonically increasing. Define $h(\alpha) = \sup\{\beta \in A \colon g_0(\beta) \leq \alpha\}$. Clearly $h \colon \omega_1 \to \omega_1$. Let x_0 be such that T_{x_0} is well-founded and $\forall^*_{W_1^1} \beta \ h(\beta) < |T_{x_0} \restriction \beta|$. Thus, $\forall^*_\nu \alpha \ \alpha < h(g_0(\alpha)) \leq |T_{x_0} \restriction g_0(\alpha)|$. Let $r_0 \colon \omega_1 \to \omega_1$ be such that $\forall^*_\nu \alpha \ r_0(\alpha) < g_0(\alpha)$ and

$\forall^*_\nu \alpha\ [\alpha = |(T_{x_0} \restriction g_0(\alpha))(r_0(\alpha))|]$. Let $d_0 = d_1 = 1$, which are defined with respect to W^1_1. For α in the ν measure one set such that $r_0(\alpha) < g_0(\alpha)$ is defined, we define

$$\begin{aligned} h(g_0(\alpha); (W^1_1; x_0; d_0, d_1); r_0(\alpha)) =& |(T_{x_0} \restriction h(g_0(\alpha); d_0))(r_0(\alpha))| \\ =& |(T_{x_0} \restriction g_0(\alpha))(r_0(\alpha))|. \end{aligned}$$

We have thus produced a tuple $(\langle W^1_1; x_0; d_0, d_1\rangle, g_0, r_0)$ satisfying the following:

1. d_0, d_1 are defined with respect to W^1_1 and T_{x_0} is well-founded.
2. $g_0(\nu) = W^1_1$.
3. $\forall^*_\nu \alpha\ r_0(\alpha) < h(g_0(\alpha); d_1)$
4. $\forall^*_\nu \alpha\ [\alpha = h(g_0(\alpha); (W^1_1; x_0; d_0, d_1); r_0(\alpha))]$.

With this first step as motivation, we make the following definitions.

4.9 Definition *A level-1 complex is a tuple of the form*

$$\mathcal{C} = \langle W^m_1; x_0, \ldots, x_{n-1}; d_0, \ldots, d_n\rangle$$

where $m \geq 1$, $n \in \omega$ (if $n = 0$, then no x_i appear). If $n \geq 1$, then each d_i is defined with respect to W^m_1 for $0 \leq i \leq n$, $d_0 > d_1 > \cdots > d_{n-1} \geq d_n$, and each T_{x_i} is well-founded for $0 \leq i \leq n-1$.

If $\mathcal{C}$ as above is a complex with $n \geq 1$, $\gamma < \omega_1$, and $\beta_1 < \ldots \beta_m < \omega_1$, define $h(\beta_1, \ldots, \beta_m; \mathcal{C}; \gamma) = |(T_{x_0} \restriction h(\vec{\beta}; d_0))(\alpha_1)|$, where $\alpha_1 = |(T_{x_1} \restriction h(\vec{\beta}; d_1))(\alpha_2)|$, ..., $\alpha_{n-1} = |(T_{x_{n-1}} \restriction h(\vec{\beta}; d_{n-1}))(\alpha_n)|$, and $\alpha_n = \gamma$. If $n = 0$, then we define $h(\vec{\beta}; \mathcal{C}; \gamma) = \gamma$.

Written out directly, the equation for h is:

$$h(\beta_1, \ldots, \beta_m; \mathcal{C}; \gamma) = |T_{x_0} \restriction \beta_{d_0}(|T_{x_1} \restriction \beta_{d_1}(\ldots |T_{x_{n-1}} \restriction \beta_{d_{n-1}}(\gamma)| \ldots)).$$

The definition of $h(\vec{\beta}; \mathcal{C}; \gamma)$ is actually only on the W^m_1 measure one set of $\vec{\beta}$ such that $h(\vec{\beta}; d_0) > \alpha_1$, etc. Off this measure one set, we leave $h(\vec{\beta}; \mathcal{C}; \gamma)$ undefined. Note that the last description d_n is not used in the definition of $h(\vec{\beta}; \mathcal{C}; \gamma)$; its role is to provide a bound for the r function in the following definition (the r stands for "remainder," it represents, roughly speaking, the part of the measure that has not yet been analyzed).

We abstract the general step of the analysis into the following definition.

4.10 Definition *A situation for ν is a triple $(\mathcal{C}, g, r)$ satisfying the following:*

1. *$\mathcal{C} = \langle W_1^m; x_0, \ldots, x_{n-1}; d_0, \ldots, d_n\rangle$ is a complex (defined immediately above).*

2. *$g\colon \omega_1 \to (\omega_1)^m$, and $g(\nu) = W_1^m$.*

3. *$r\colon \omega_1 \to \omega_1$, and $\forall^*_\nu \alpha\ r(\alpha) < h(g(\alpha); d_n)$.*

4. *$\forall^*_\nu \alpha\ [\alpha = h(g(\alpha); \mathcal{C}; r(\alpha))]$.*

Among all situations for ν, we now choose one with minimal value for $[\alpha \to h(g(\alpha); d_n)]_{W_1^1}$ (that is, which minimizes the bound for the function r). We denote this situation by $(\mathcal{C}; g; r)$, where $\mathcal{C} = \langle W_1^m; x_0,\ldots,x_{n-1}; d_0,\ldots,d_n\rangle$.

We claim that r is constant ν almost everywhere, that is, $\exists\gamma < \omega_1\ \forall^*_\nu\alpha\ r(\alpha) = \gamma$. Granting this, it follows from (2), (4) that $\nu(A) = 1$ iff $\exists$ c.u.b. $C \subseteq \omega_1\ \forall\beta_1, \ldots, \beta_m \in C\ h(\vec{\beta}; \mathcal{C}; \gamma) \in A$. This gives the theorem via a minor cosmetic change: if $m > n$ we replace W_1^m by W_1^n, and replace $d_0, \ldots, d_{n-1}$ by $n, n-1, \ldots, 1$, which gives the same measure (we eliminate the coordinates of W_1^m not used in the definition of $h(\vec{\beta}; \mathcal{C}; \gamma)$).

We consider the case where d_n is minimal, that is, $d_n = 1$, the other case being similar (in fact, as we remarked above, there is no loss of generality in assuming $m = n$ and $d_0, \ldots, d_{n-1} = n, \ldots, 1$, in which case $d_n = 1$). We consider two cases.

First assume that there is a ν measure one set A such that for $\alpha \in A$ there is a $\delta < h(g(\alpha); d_n)$ (that is, $\delta < \beta_1$, where $g(\alpha) = \beta_1, \ldots, \beta_m$) such that for all $\alpha' \in A$ with $g(\alpha') = g(\alpha)$ we have $r(\alpha') < \delta$. Fix such a measure one set A, and by (3) we may assume $\forall\alpha \in A\ r(\alpha) < h(g(\alpha); d_n)$. By (2), let C be c.u.b. such that for all $\beta_1 < \cdots < \beta_m \in C\ \exists\alpha \in A\ g(\alpha) = (\beta_1, \ldots, \beta_m)$. Thus,

$$\forall\beta_1 < \cdots < \beta_m \in C\ \exists\delta < \beta_1\ \sup\{r(\alpha)\colon \alpha \in A \wedge g(\alpha) = (\beta_1, \ldots, \beta_m)\} < \delta.$$

By normality, δ is constant on a W_1^m measure one set, and by countable additivity of ν, r is constant on a ν measure one set.

Suppose now such a measure one set does not exist. It follows that for any A of ν measure one that $\forall^*_\nu\alpha \in A\ \forall\delta < \beta_1\ \exists\alpha' \in A\ [g(\alpha') = g(\alpha) \wedge r(\alpha') > \delta]$, where $g(\alpha) = (\beta_1, \ldots, \beta_m)$. Let $g'\colon \omega_1 \to \omega_1$ satisfy the following:

1. $\forall^*_\nu\alpha\ g'(\alpha) < h(g(\alpha); d_n)$. In other words, $g'(\alpha) < \beta_1$, where $g(\alpha) = (\beta_1, \ldots, \beta_m)$.

2. There is a ν measure one set A such that if $\alpha_1, \alpha_2 \in A$, $g(\alpha_1) = g(\alpha_2)$, and $r(\alpha_1) \le r(\alpha_2)$, then $g'(\alpha_1) \le g'(\alpha_2)$.

3. For any A of ν measure one we have $\forall^*_\nu \alpha \in A\ \forall\delta < \beta_1\ \exists\alpha' \in A\ [g(\alpha') = g(\alpha) \wedge g'(\alpha') > \delta]$, where $g(\alpha) = (\beta_1, \dots, \beta_m)$.

4. If $[g'']_\nu < [g']_\nu$, then g'' does not satisfy (1)-(3).

g' exists, since r satisfies (1)-(3). Also, $g'(\nu) = W^1_1$, since if there were a c.u.b. C such that $\forall^*_\nu \alpha\ g'(\alpha) \notin C$, then $p \circ g'$ would violate the minimality of g', where $p(\alpha) =$ the largest element of C less than or equal to α. Let $\tilde{g}(\alpha) = g'(\alpha)^\frown g(\alpha)$, so $\tilde{g}(\nu) = W^{m+1}_1$. Fix a ν measure one set A witnessing (1), (2) for g'. Consider the partition $\mathcal{P}$: we partition $\beta_0 < \delta < \beta_1 < \beta_2 < \cdots < \beta_m$ according to whether

$$\sup \{r(\alpha) \colon \alpha \in A \wedge g(\alpha) = (\beta_1, \dots, \beta_m) \wedge g'(\alpha) \leq \beta_0\} < \delta.$$

Using (2), (3) it follows that on the homogeneous side the stated property holds. Let C be c.u.b and homogeneous for $\mathcal{P}$, and let $N_C(\alpha) =$ the least element of C greater than α. Thus, $\forall^*_\nu \alpha\ r(\alpha) < N_C(g'(\alpha))$. Let x_n be such that T_{x_n} is well-founded and $\forall^*_{W^1_1} \beta\ N_C(\beta) < |T_{x_n} \restriction \beta|$. Define $r' \colon \omega_1 \to \omega_1$ so that $\forall^*_\nu \alpha\ r'(\alpha) < g'(\alpha)$ and $\forall^*_\nu \alpha\ [r(\alpha) = |(T_{x_n} \restriction g'(\alpha))(r'(\alpha))|$. Let $d'_0 = d_0 + 1, \dots, d'_{n-1} = d_{n-1} + 1$ (to maintain the correspondence between the appropriate coordinates of W^m_1 and W^{m+1}_1), and let $d'_n = d'_{n+1} = 1$. Let $\mathcal{C}' = \langle W^{m+1}_1; x_0, \dots, x_n; d'_0, \dots, d'_n, d'_{n+1} \rangle$. Note then that $\forall^*_\nu \alpha\ [\alpha = h(\tilde{g}(\alpha); \mathcal{C}'; r'(\alpha))]$. Since $\forall^*_\nu \alpha\ h(\tilde{g}(\alpha); d'_n) < h(g(\alpha); d_n)$, it follows that $(\mathcal{C}'; \tilde{g}; r')$ violates the minimality of $(\mathcal{C}; g; r)$, a contradiction. ⊣

4.2. The Strong Partition Relation on ω_1

We now convert this analysis of measures to a coding of the subsets of ω_1. As we mentioned before, this idea is due to Kunen. Throughout §4.2, T continues to denote the Kunen tree of lemma 4.1.

First we code (enough) c.u.b. sets. If $\sigma \in \omega^\omega$, let

$$C_\sigma = \{\alpha < \omega_1 \colon \alpha > \omega \wedge \forall\beta < \alpha\ (T_\sigma \restriction \beta \text{ is well-founded } \wedge |T_\sigma \restriction \beta| < \alpha)\}.$$

For any σ, C_σ is a closed subset of ω_1, and if T_σ is well-founded then C_σ is also unbounded. Also, for all c.u.b. $C \subseteq \omega_1$, there is a $\sigma \in \omega^\omega$ such that T_σ is well-founded and $C_\sigma \subseteq C$. For we may choose σ so that for all infinite $\beta < \omega_1$, $|T_\sigma \restriction \beta| > N_C(\beta)$.

4.11 Definition *Suppose $\mathcal{C} = \langle W^m_1; x_0, \dots, x_{n-1}; d_0, \dots, d_n \rangle$ is a complex, T_σ is well-founded, and $\gamma < \omega_1$. If $n \geq 1$, we define (with h as in definition 4.9)*

$$S_{\sigma,\mathcal{C},\gamma} = \{h(\beta_1, \dots, \beta_m; \mathcal{C}; \gamma) \colon \vec{\beta} \in (C_\sigma)^m \wedge \gamma < \beta_1 \wedge h(\vec{\beta}; \mathcal{C}; \gamma) \geq \beta_m\}.$$

For $n = 0$ we define $S_{\sigma,\mathcal{C},\gamma} = \{\gamma\}$. We say $S \subseteq \omega_1$ is simple if it is of the form $S_{\sigma,\mathcal{C},\gamma}$ for some $\sigma, \mathcal{C}, \gamma$.

4.12 Theorem *Every $A \subseteq \omega_1$ is a countable union of simple sets.*

Proof. Let $A \subseteq \omega_1$, and suppose the theorem fails for A. Let $\mathcal{I} \subseteq \mathcal{P}(A)$ denote the countably additive ideal of $I \subseteq A$ such that $I = \bigcup_i S_i$ is a countable union of simple sets $S_i \subseteq A$. Thus, $A \notin \mathcal{I}$. Also, $\mathcal{I}$ contains all singletons as every $\{\gamma\}$ is simple. By AD, every countably additive filter (dual to an ideal $\mathcal{I}$) on an ordinal $< \Theta$ (identified here with A) can be extended to a measure. [One way to see this: by the coding lemma, let $\pi\colon \omega^\omega \to \mathcal{I}$ be onto. For d a Turing degree, let $f(d) =$ least element of $A - \bigcup_{x \in d} \pi(x)$. If μ is the Martin measure on the degrees, then $f(\mu)$ is a measure on A with $f(\mu)(I) = 0$ for all $I \in \mathcal{I}$].

Let ν be a measure on A extending the filter dual to $\mathcal{I}$. Since $\nu(A) = 1$, by theorem 4.8 we have some simple set $S \subseteq A$ with $\nu(S) = 1$. This contradicts $S \in \mathcal{I}$. $\dashv$

We view each real z as coding countably many reals z_n, each of which codes reals σ_n, w_n, and a sequence $\mathcal{C}_n = \langle W_1^m; x_0, \ldots, x_{t-1}; d_0, \ldots, d_t\rangle$ which satisfies the definition of a complex except that we do not require the T_{x_i} to be well-founded (we call this a *partial complex*). To each z_n we associate a set A_{z_n} defined as follows. If $w_n \notin \mathrm{WO}$ (the set of codes for well-orderings of ω), we set $A_{z_n} = \emptyset$. Otherwise, set

$$\alpha \in A_{z_n} \leftrightarrow \exists \beta_1 < \cdots < \beta_m \leq \alpha\ [\beta_1 > |w_n| \wedge \{\forall i\ \beta_i \in C_{\sigma_n} \wedge \alpha = h(\vec{\beta}; \mathcal{C}_n; |w_n|)\}].$$

We define here $h(\vec{\beta}; \mathcal{C}_n; \gamma)$ for the partial complex $\mathcal{C}_n$ similarly to definition 4.9: $h(\vec{\beta}; \mathcal{C}_n; \gamma) = |T_{x_0} \restriction h(\vec{\beta}; d_0)(\alpha_1)|$, where $\alpha_1 = |T_{x_1} \restriction h(\vec{\beta}; d_1)(\alpha_2)|$, etc., provided α_1 is in the well-founded part of $T_{x_0} \restriction h(\vec{\beta}; d_0)$, α_2 is in the well-founded part of $T_{x_1} \restriction h(\vec{\beta}; d_1)$, etc. If some α_{i+1} is not in the well-founded part of $T_{x_i} \restriction h(\vec{\beta}; d_i)$, we leave $h(\vec{\beta}; \mathcal{C}_n; \gamma)$ undefined. Thus, for all $z \in \omega^\omega$, the sets $A_{z_n} \subseteq \omega_1$ are defined. We set $A_z = \bigcup_{n \in \omega} A_{z_n}$.

The following theorem says that this coding is reasonable.

4.13 Theorem *The coding $z \to A_z$ satisfies the following:*

1. $\forall A \subseteq \omega_1\ \exists z\ A = A_z$.
2. $\forall \alpha < \omega_1\ \{z \colon \alpha \in A_z\} \in \mathbf{\Delta}^1_1$.

Proof. (1) follows immediately from theorem 4.12. (2) is a straightforward computation using the facts that $\forall\beta < \omega_1\ \{\sigma\colon \beta \in C_\sigma\} \in \mathbf{\Delta}^1_1$, and $\forall\beta,\gamma,\delta < \omega_1\ \{z\colon |(T_z{\restriction}\beta)(\gamma) \le \delta\} \in \mathbf{\Delta}^1_1$. ⊣

If we view functions $F\colon \omega_1 \to \omega_1$ as subsets of $(\omega_1)^2$, and use our coding above (identifying $(\omega_1)^2$ with ω_1), it is not quite good enough to witness that ω_1 is ω_1-reasonable. To get this, we must make a small modification to our coding, essentially modifying only the first step of the proof of theorem 4.8. We sketch the changes that need to be made.

We code binary relations $F \subseteq \omega_1 \times \omega_1$ as follows. Every real z codes countably many reals z_n, each of which codes reals σ_n, w^1_n, w^2_n, and a partial complex $\mathcal{C}_n = \langle W^m_1; x_0, \ldots, x_{t-1}; d_0, \ldots, d_t\rangle$. Set

$$\begin{aligned}(\alpha,\beta) \in F_z \leftrightarrow &\exists n\ [w^1_n, w^2_n \in \mathrm{WO} \wedge |w^1_n| < \alpha \wedge |w^2_n| < \alpha \wedge \\ &\quad \exists\beta_1 < \cdots < \beta_m \le \alpha\ [\beta_1 > \max\{|w^1_n|, |w^2_n\} \wedge \forall i\ \beta_i \in C_{\sigma_n} \\ &\quad \wedge\ \alpha = h(\vec{\beta}; \mathcal{C}; |w^1_n|) \wedge \beta = h(\vec{\beta}; \mathcal{C}; |w^2_n|] \\ &\wedge \forall n' \in \omega\ [\{w^1_{n'}, w^2_{n'} \in \mathrm{WO} \wedge |w^1_{n'}| < \alpha \wedge |w^2_{n'}| < \alpha \wedge \\ &\quad \exists\beta'_1 < \cdots < \beta'_m \le \alpha\ [\beta_1 > \max\{w^1_{n'}, w^2_{n'}\} \wedge \forall i\ \beta'_i \in C_{\sigma_{n'}} \\ &\quad \wedge\ \alpha = h(\vec{\beta}'; \mathcal{C}; |w^1_{n'}|)\} \to h(\vec{\beta}'; \mathcal{C}; |w^2_{n'}|) = \beta]\end{aligned}$$

The main difference now is that the β_i are required to be $\le \alpha$ (rather than $\le \max\{\alpha, \beta\}$). Note also that if $F_z(\alpha,\beta)$ and $F_z(\alpha,\beta')$, then $\beta = \beta'$.

The analog of theorem 4.12 becomes:

4.14 Theorem *For every function $F\colon \omega_1 \to \omega_1$, $\exists z\ F = F_z$.*

Proof. Fix $F\colon \omega_1 \to \omega_1$. Let $X = \{(\alpha,\beta)\colon \beta = F(\alpha)\}$. Let $\mathcal{I}$ be the countably additive ideal on X consisting of countable unions of sets I such that $I \subseteq S \subseteq F$ for some *simple* S, that is, $S = S_{\sigma,\mathcal{C},\gamma_1,\gamma_2}$ for some complex $\mathcal{C}$, well-founded T_σ, and $\gamma_1, \gamma_2 < \omega_1$, where

$$\begin{aligned}(\alpha,\beta) \in S_{\sigma,\mathcal{C},\gamma_1,\gamma_2} \leftrightarrow &\exists\beta_1 < \cdots < \beta_m \le \alpha\ [\beta_1 > \max\{\gamma_1,\gamma_2\} \\ &\wedge \forall i(\beta_i \in C_{\sigma_n}) \wedge \alpha = h(\vec{\beta}; \mathcal{C}; \gamma_1) \wedge \beta = h(\vec{\beta}; \mathcal{C}; \gamma_2)]\end{aligned}$$

We finish as in theorem 4.12 provided we show that every measure ν on X is generated by simple sets. To do this, we need only modify the first step in the proof of theorem 4.8.

Let $\pi(\alpha,\beta) = \alpha$ be the projection onto the first coordinate. Let $\nu' = \pi(\nu)$. Define g_0 exactly as in theorem 4.8, using ν'. Let A be a ν' measure one set on which g_0 is monotonically increasing, and define $h(\alpha) = \sup\{\max(\beta,\gamma)\colon \beta \in A \wedge \gamma = F(\beta) \wedge g_0(\beta) \le \alpha\}$. Let T_{x_0} be well-founded and $\forall^*_{W^1_1}\gamma\ h(\gamma) < |T_{x_0}\restriction\gamma|$. Thus, $\forall^*_\nu(\alpha,\beta)\ \max(\alpha,\beta) < |T_{x_0}\restriction g_0(\alpha)|$. Let $r_0, s_0\colon \omega_1 \to \omega_1$ be such that $\forall^*_\nu(\alpha,\beta)\ r_0(\alpha), s_0(\alpha) < g_0(\alpha)$ and

$$\forall^*_\nu(\alpha,\beta)\ [\alpha = |(T_{x_0}\restriction g_0(\alpha))(r_0(\alpha))| \wedge \beta = |(T_{x_0}\restriction g_0(\alpha))(s_0(\alpha))|].$$

The argument then proceeds as in theorem 4.8. ⊣

This coding suffices for the strong partition relation on ω_1.

4.15 Theorem *The coding $z \to F_z$ witnesses ω_1 is reasonable relative to the pointclass $\mathbf{\Sigma}^1_1$.*

Proof. (1), (2) in definition 2.30 are immediate, and (3) is a straightforward computation using the definition of F_z. (4) follows from the fact that if $A \in \mathbf{\Sigma}^1_1$, $\alpha < \omega_1$ and $\forall z \in A\ \exists\beta\ F_z(\alpha,\beta)$, then there is a $\mathbf{\Sigma}^1_1$ relation $\prec$ such that for all $z_1, z_2 \in A$, $F_{z_1}(\alpha) < F_{z_2}(\alpha) \leftrightarrow z_1 \prec z_2$. ⊣

4.16 Corollary $\omega_1 \to (\omega_1)^{\omega_1}$.

We now obtain the lower bound for $\boldsymbol{\delta}^1_3$, again, using only techniques that will generalize. We use the following general theorem of Martin.

4.17 Theorem (Martin) *Assume $\kappa \to (\kappa)^\kappa$. Then for any measure ν on κ, the ultrapower $j_\nu(\kappa)$ is a cardinal.*

Proof. Toward a contradiction, fix ν such that $j_\nu(\kappa)$ is not a cardinal, and let $F\colon j_\nu(\kappa) \to \lambda$ be a bijection, where $\lambda < j_\nu(\kappa)$. Consider the partition $\mathcal{P}$: we partition $f, g\colon \kappa \to \kappa$ of the correct type with $f(\alpha) < g(\alpha) < f(\alpha+1)$ according to whether $F([f]_\nu) < F([g]_\nu)$. It is clear by wellfoundedness that we cannot have a c.u.b. set homogeneous for the contrary side of the partition. Let $C \subseteq \kappa$ be c.u.b. and homogeneous for $\mathcal{P}$. Fix an ordinal θ with $\lambda < \theta < j_\nu(\kappa)$, and let $[h]_\nu = \theta$. Define $h'\colon \kappa \to C$ inductively by: $h'(\alpha) =$ the $\omega\cdot(h(\alpha)+1)^{st}$ element in C greater than $\sup_{\beta<\alpha} h'(\beta)$. Since κ is regular, this is well defined. We then produce an order preserving map H from θ into λ, a contradiction. Namely, if $\delta < \theta$, let $[f_\delta]_\nu = \delta$, with $f_\delta < h$ everywhere. Define $f'_\delta\colon \kappa \to C$ by: $f'_\delta(\alpha) =$ the $\omega\cdot(f_\delta(\alpha)+1)^{st}$ element of C greater than $\sup_{\beta<\alpha} h'(\beta)$. Then set $H(\delta) = F([f'_\delta]_\nu)$. It is easy to see that this is well-defined and order-preserving from θ into λ. ⊣

4.18 Corollary $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$.

Proof. Define $\pi : j_{W_1^m}(\omega_1) \to j_{W_1^{m+1}}(\omega_1)$ by: $\pi([f]_{W_1^m}) = [f']_{W_1^{m+1}}$ where $f'(\alpha_1,\dots,\alpha_{m+1}) = f(\alpha_1,\dots,\alpha_m)$. This gives an embedding from $j_{W_1^m}(\omega_1)$ into $[\mathrm{id}_{m+1}]_{W_1^{m+1}}$, where $\mathrm{id}_{m+1}(\alpha_1,\dots,\alpha_{m+1}) = \alpha_{m+1}$. Thus, $j_{W_1^m}(\omega_1) < j_{W_1^{m+1}}(\omega_1)$. From theorem 4.17 and the previous upper bound, it follows that $j_{W_1^n}(\omega_1) = \omega_{n+1}$. Using the coding lemma to code functions $f\colon (\omega_1)^m \to \omega_1$, and also our coding of c.u.b. sets, it is easy to compute the prewellordering corresponding to the ultrapower $j_{W_1^m}(\omega_1)$ as $\mathbf{\Delta}^1_3$ (in fact, a more careful computation shows that the prewellordering lies in the pointclass $\Game\,\omega\cdot m\text{–}\mathbf{\Pi}^1_1$). Thus, $\boldsymbol{\delta}^1_3 \geq \sup_m j_{W_1^m}(\omega_1) = \omega_\omega$. Since $\boldsymbol{\delta}^1_3$ is regular, $\boldsymbol{\delta}^1_3 \geq \omega_{\omega+1}$. ⊣

4.3. The Weak Partition Relation on $\boldsymbol{\delta}^1_3$

Starting from the weak partition relation on $\boldsymbol{\delta}^1_1 = \omega_1$, we have computed $\boldsymbol{\delta}^1_3$ and proved the strong partition relation on $\boldsymbol{\delta}^1_1$. To complete the cycle, we establish now the weak partition relation on $\boldsymbol{\delta}^1_3$. As we mentioned before, this is a result of Kunen (see [33]). Again, we wish to use only methods and terminology that will generalize. Nevertheless, the proof closely parallels Kunen's. An important concept which is introduced here is that of a *tree of uniform cofinalities*; this plays a central role in the general inductive analysis as well.

First, we analyze possible uniform cofinalities with respect to the measures W_1^m. This analysis is well known, but we emphasize we use only the weak partition relation on ω_1 and the Kunen tree analysis.

4.19 Lemma *Let $f\colon (\omega_1)^m \to \omega_1$, and assume $\forall^*_{W_1^m}\alpha_1,\dots,\alpha_m\ f(\vec\alpha)$ is a limit ordinal. Then either f has uniform cofinality ω almost everywhere with respect to W_1^m, or there is an i, $1 \leq i \leq m$, such that $f(\vec\alpha)$ has uniform cofinality α_i almost everywhere. Also, each of these uniform cofinalities is distinct, that is, these cases are mutually exclusive.*

4.20 Remark *The uniform cofinalities other than ω are thus described by the descriptions d defined with respect to W_1^m. The lemma also holds for any $f\colon (\omega_1)^m \to \Theta$ assuming* $\mathrm{AD} + V = L(\mathbb{R})$ *(see §6).*

Proof. Fix $f\colon (\omega_1)^m \to \omega_1$, and call a pair (S, l) a *liftup* to f provided $S\colon (\omega_1)^m \to \omega_1$, $l\colon \{(\alpha_1,\dots,\alpha_m,\beta)\colon \beta < S(\vec\alpha)\} \to \omega_1$ and $\forall^*_{W_1^m}\vec\alpha\ f(\vec\alpha) = \sup\{l(\vec\alpha,\beta)\colon \beta < S(\vec\alpha)\}$. Fix a liftup (S, l) for which $[S]_{W_1^m}$ is minimal. If S is constant almost everywhere, then easily f has uniform cofinality ω. Let $1 \leq d \leq m$ be minimal so that $\forall^*_{W_1^m}\vec\alpha\ S(\vec\alpha) \leq h(\vec\alpha; d) = \alpha_d$. If equality holds almost everywhere we are done, as then $f(\vec\alpha)$ has uniform cofinality α_d almost everywhere. Otherwise, there is a x with T_x well-founded such that

$\forall^*_{W_1^m} \vec{\alpha}\ S(\vec{\alpha}) < |T_x \restriction \alpha_{d-1}|$ ($d > 1$ now). For almost all $\vec{\alpha}$, set $S'(\vec{\alpha}) = \alpha_{d-1}$, and for $\beta < \alpha_{d-1}$ define $l'(\vec{\alpha}, \beta) = \sup\{l(\vec{\alpha}, \gamma) \colon \gamma < |(T_x \restriction \alpha_{d-1})(\beta)|\}$ if $|(T_x \restriction \alpha_{d-1})(\beta)| < S(\vec{\alpha})$, and $= 0$ otherwise. Then (S', l') is a liftup to f with $[S']_{W_1^m} < [S]_{W_1^m}$, a contradiction. We leave the uniqueness proof to the reader. ⊣

We introduce some useful notation.

4.21 Definition *Suppose $\pi = (n, i_2, \dots, i_n)$ is a permutation of $\{1, \dots, n\}$ beginning with n. $<^\pi$ is the well-ordering of $(\omega_1)^n$ defined by: $(\alpha_1, \dots, \alpha_n) <^\pi (\beta_1, \dots, \beta_n)$ iff $(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_n}) <_{lex} (\beta_n, \beta_{i_2}, \dots, \beta_{i_n})$. We say an n-tuple of ordinals $(\gamma_1, \dots, \gamma_n)$ has type π if it is order-isomorphic to π. By a partial permutation of n we mean a $\pi = (n, i_2, \dots, i_m)$, $m \leq n$, which can be extended to a permutation. We likewise define the ordering $<^\pi$ on $(\omega_1)^n$ in this case by: $(\alpha_1, \dots, \alpha_n) <^\pi (\beta_1, \dots, \beta_n)$ iff $(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_m}) <_{lex} (\beta_n, \beta_{i_2}, \dots, \beta_{i_m})$. We identify $\vec{\alpha}, \vec{\beta} \in \mathrm{dom}(<^\pi)$ if $\alpha_n = \beta_n$, ..., $\alpha_{i_m} = \beta_{i_m}$.*

Note that if $\pi = (n, i_2, \dots, i_m)$ is a partial permutation and $m < n$, then if $f \colon \mathrm{dom}(<^\pi) \to \omega_1$ is order-preserving, $f(\alpha_1, \dots, \alpha_n)$ depends only on $\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_m}$.

If $\pi = (n, i_2, \dots, i_k)$, $\pi' = (m, j_2, \dots, j_l)$ are partial permutations, we say π' *extends* π provided $m \geq n$, $l > k$, and $(m, j_2, \dots, j_k)$ is order-isomorphic to π. If, in addition, $l = k + 1$, we say π' is an *immediate extension* of π.

4.22 Definition *If $\pi = (n, i_2, \dots, i_n)$ is a permutation and $f \colon <^\pi \to \omega_1$ is order-preserving, we define the "m^{th} invariant" $f(m) \colon (\omega_1)^m \to \omega_1$, for $m \leq n$ by:*

$$f(m)(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_m}) = \sup\{f(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_m}, \beta_{i_{m+1}}, \dots, \beta_{i_n}) \colon (\alpha_n, \dots, \beta_{i_n}) \text{ has type } \pi\}$$

For example, if $\pi = (3, 1, 2)$, and $f \colon \mathrm{dom}(<_\pi) \to \omega_1$ is order-preserving, then $f(2)(\alpha_1, \alpha_3) = \sup\{f(\alpha_1, \alpha_2, \alpha_3) \colon \alpha_1 < \alpha_2 < \alpha_3\}$. Recall our convention that for $g \colon (\omega_1)^2 \to \omega_1$ and $\alpha_1 < \alpha_3$, we write $g(\alpha_1, \alpha_3)$ interchangeably with $g(\alpha_3, \alpha_1)$, etc.

4.23 Lemma *Let $f \colon (\omega_1)^n \to \omega_1$ with $\omega_n < [f]_{W_1^n}$. Then there is an $m \leq n$ and a partial permutation $\pi = (n, i_2, \dots, i_m)$ such that for some c.u.b. $C \subseteq \omega_1$ and any $\vec{\alpha}, \vec{\beta} \in (C)^n$, $f(\vec{\alpha}) < f(\vec{\beta})$ iff $(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_m}) <_{lex} (\beta_n, \beta_{i_2}, \dots, \beta_{i_m})$.*

Proof. Suppose $(n, i_2, \dots, i_k)$ have been defined such that there is a c.u.b. $C \subseteq \omega_1$ such that $\forall \vec{\alpha}, \vec{\beta} \in (C)^n$, $(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_k}) <_{lex} (\beta_n, \beta_{i_2}, \dots, \beta_{i_k}) \to$

$f(\vec{\alpha}) < f(\vec{\beta})$. For each $i \in \{1, \dots, n\} - \{n, i_1, \dots, i_k\}$, consider the partition $\mathcal{P}_i$: we partition ordinals $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ with $\alpha_n = \beta_n$, $\dots, \alpha_{i_k} = \beta_{i_k}$, $\alpha_i < \beta_i < \alpha_{i+1}$, and $\alpha_{j-1} < \beta_j < \alpha_j$ for all other j. We partition according to whether $f(\vec{\alpha}) < f(\vec{\beta})$.

If all the partitions $\mathcal{P}_i$ are homogeneous for the contrary side, it is easy to see that on a c.u.b. set, if $(\alpha_n, \dots, \alpha_{i_k}) = (\beta_n, \dots, \beta_{i_k})$, then $f(\vec{\alpha}) = f(\vec{\beta})$, and we are done.

Otherwise, it is easy to see that there is a unique i such that $\mathcal{P}_i$ is homogeneous for the stated side. We may then extend $(n, i_2, \dots, i_k)$ to $(n, i_2, \dots, i_k, i_{k+1})$, setting $i_{k+1} = i$. Continuing, we establish the theorem. ⊣

Lemmas 4.19, 4.23 completely analyze the "type" of a function $f\colon (\omega_1)^n \to \omega_1$. If $\pi = (n, i_2, \dots, i_m)$ is a partial permutation of n, we say $f\colon (\omega_1)^n \to \omega_1$ is of type π if f is order-preserving from $<^\pi$ to ω_1, of uniform cofinality ω, and is everywhere discontinuous, that is, for $\vec{\alpha} \in (\omega_1)^n$, $f(\vec{\alpha}) > \sup\{f(\vec{\beta})\colon \vec{\beta} <^\pi \vec{\alpha}\}$. We say f is of type (π, s) if f is order-preserving from $<^\pi$ and for $\vec{\alpha} \in (\omega_1)^n$ of limit rank in $<^\pi$, $f(\vec{\alpha}) = \sup\{f(\vec{\beta})\colon \vec{\beta} <^\pi \vec{\alpha}\}$ (and for $\vec{\alpha}$ of successor rank, $f(\vec{\alpha})$ has uniform cofinality ω). Finally, if $\pi' = (n+1, j_2, \dots, j_{m+1})$ is a partial permutation extending π, we say f is of type (π, π') if f is order-preserving with respect to $<^\pi$, everywhere discontinuous, and $f(\alpha_{n+1}, \alpha_{j_2}, \dots, \alpha_{j_m})$ has uniform cofinality $\{\beta\colon (\alpha_{n+1}, \alpha_{j_2}, \dots, \alpha_{j_m}, \beta)$ has type $\pi'\}$ if this set has limit order-type (and otherwise has uniform cofinality ω). We say f has type π, (π, s) or (π, π') almost everywhere if there is a c.u.b. C such that $f \restriction (C)^n$ is order-preserving with respect to $<^\pi$, and $f \restriction (C)^n$ has the appropriate uniform cofinality and continuity properties.

Lemmas 4.19, 4.23 thus say that if $f\colon (\omega_1)^n \to \omega_1$, $[f] > \omega_n$, and $\forall^* \vec{\alpha}\, f(\vec{\alpha})$ is a limit ordinal, then f has type π, (π, s), or (π, π') almost everywhere, for some partial permutation(s) π, π'.

The next lemma is simple but important. It says we may change the values of functions on measure zero sets so that they are everywhere ordered correctly.

4.24 Lemma (sliding lemma) *Suppose $f\colon (\omega_1)^m \to \omega_1$, $g\colon (\omega_1)^n \to \omega_1$ have types $\pi_1 = (m, i_2, \dots, i_k)$, $\pi_2 = (n, j_2, \dots, j_l)$ almost everywhere respectively. Suppose $r \le \min(k, l)$ is such that $[f(r-1)] = [g(r-1)]$, but $[f(r)] < [g(r)]$. Then there are f', g' of types π_1, π_2 with $[f'] = [f]$, $[g'] = [g]$, $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$, $\mathrm{ran}(g') \subseteq \mathrm{ran}(g)$, and such that for all $\vec{\alpha} \in (\omega_1)^m$, $\vec{\beta} \in (\omega_1)^n$, $g(\vec{\beta}) > f(\vec{\alpha})$ iff $(\beta_n, \beta_{j_2}, \dots, \beta_{j_r}) \ge_{lex} (\alpha_m, \alpha_{i_2}, \dots, \alpha_{i_r})$.*

Proof. Note that $(m, i_2, \dots, i_r)$, $(n, j_2, \dots, j_r)$ are order-isomorphic, say to the permutation $\pi = (r, k_2, \dots, k_r)$, by uniqueness of the uniform cofinality and the fact that $[f(r-1)] = [g(r-1)]$. Let $C_1 \subseteq \omega_1$ be c.u.b. such that

$f \restriction (C_1)^m$ is order-preserving with respect to $<^{\pi_1}$ and of uniform cofinality ω, and similarly for g. Let $l \colon \omega_1 \to \omega_1$, and $C \subseteq C_1$ be c.u.b. and closed under l such that

$$\forall \vec{\alpha} \in (C)^r \; f(r)(\alpha_r, \alpha_{k_2}, \ldots, \alpha_{k_r}) < g(r)(\alpha_r, \alpha_{k_2}, \ldots, \alpha_{k_r}) \\ < f(r)(\alpha_r, \alpha_{k_2}, \ldots, l(\alpha_{k_r})).$$

Let $p(\alpha) =$ the $\omega \cdot \alpha^{th}$ element of C. Define

$$f'(\alpha_1, \ldots, \alpha_m) = f(p(\alpha_1), \ldots, p(\alpha_m)),$$

and similarly

$$g'(\alpha_1, \ldots, \alpha_n) = g(p(\alpha_1), \ldots, p(\alpha_n)).$$

It is easy to check that f', g' have the desired properties. $\dashv$

A useful special case of the lemma is that if $f \colon (\omega_1)^n \to \omega_1$ has type π almost everywhere, then there is an f' of type π with $[f'] = [f]$ and $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$. A small variation of the argument shows this is also true for functions f of type (π, s) or (π, π') almost everywhere.

We now define the notion of a tree of uniform cofinalities, which plays an important role in the projective hierarchy analysis. The concept is similar to that of a homogeneous tree. Roughly speaking, for each node in this tree we allow as extensions all possible uniform cofinalities with respect to the measure which is associated to that node (these uniform cofinalities in turn define new measures).

In the following definition, we let $(s), (\omega)$ be two formal symbols (the first stands for "sup", and the second for "uniform cofinality ω").

4.25 Definition *A type-1 tree of uniform cofinalities (of depth n) is a function $\mathcal{R}$ satisfying the following:*

1. *$\langle p_1, i_1 \rangle \in \mathrm{dom}(\mathcal{R})$ for $0 \leq i_1 \leq a$ for some integer a, and $p_1 =$ the unique permutation of length 1, namely $p_1 = (1)$. For $i_1 = 0$, $\mathcal{R}(\langle p_1, i_1 \rangle) = (s)$, and for $i_1 > 0$, $\mathcal{R}(\langle p_1, i_1 \rangle)$ is either (ω), or a permutation p_2 of length 2 (hence $p_2 = (2,1)$). Also, $\langle p_1, i_1 \rangle$ is maximal in $\mathrm{dom}(\mathcal{R})$ iff $\mathcal{R}(\langle p_1, i_1 \rangle) = (s)$ or (ω).*

2. *In general, $\mathrm{dom}(\mathcal{R})$ consists of tuples $\langle p_1, i_1, \ldots, i_{m-1}, p_m, i_m \rangle$, $m \leq n$, and such a tuple is maximal in $\mathrm{dom}(\mathcal{R})$ iff $\mathcal{R}(\langle p_1, i_1, \ldots, p_m, i_m \rangle) = (s)$ or (ω) (these are the only values permitted, therefore, if $m = n$). $\mathcal{R}(\langle p_1, i_1, \ldots, p_m, i_m \rangle) = (s)$ iff $i_m = 0$. If $\mathcal{R}(\langle p_1, i_1, \ldots, p_m, i_m \rangle) \neq (s)$ or (ω), then $\mathcal{R}(\langle p_1, i_1, \ldots, p_m, i_m \rangle)$ is a permutation p_{m+1} immediately extending p_m. In this case, we have $(\langle p_1, i_1, \ldots, p_m, i_m, p_{m+1}, i_{m+1} \rangle) \in \mathrm{dom}(\mathcal{R})$ for some integers $0 \leq i_{m+1} \leq a$ ($a \geq 0$ and depends on $\langle p_1, i_1, \ldots, p_m, i_m, p_{m+1} \rangle$).*

For $\mathcal{R}$ a type-1 tree of uniform cofinalities, we define $<^{\mathcal{R}}$ to be the lexicographic ordering on sequences $\langle \alpha_1, i_1, \ldots, i_{m-1}, \alpha_m, i_m \rangle$, $m \leq n$, satisfying:

1. $\alpha_1, \ldots, \alpha_m < \omega_1$
2. $(\alpha_1, \ldots, \alpha_m)$ is of type p_m, where $(p_1, \ldots, p_m)$ is the unique sequence such that $\langle p_1, i_1, \ldots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R})$.

We say a sequence $\langle \alpha_1, i_1, \ldots, i_{m-1}, \alpha_m, i_m \rangle$ with $(\alpha_1, \ldots, \alpha_m)$ as in (2) is of type $\langle p_1, i_1, \ldots, p_m, i_m \rangle$.

To each tree of uniform cofinalities $\mathcal{R}$ we associate a measure $M^{\mathcal{R}}$. First, say a function $f\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ is of *type* $\mathcal{R}$ if it is order-preserving, $f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle)$ has uniform cofinality ω if $\mathcal{R}(\langle p_1, i_1, \ldots, p_m, i_m \rangle) = (\omega)$ or $\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle$ has successor rank in $<^{\mathcal{R}}$ (for $\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle$ having type $\langle p_1, i_1, \ldots, p_m, i_m \rangle$), and otherwise

$$f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle) = \sup \{ f(\vec{s})\colon \vec{s} <^{\mathcal{R}} \langle \alpha_1, i_, \ldots, \alpha_m, i_m \rangle \}.$$

Define A to have has measure one with respect to $M^{\mathcal{R}}$ iff $\exists$ c.u.b. $C \subseteq \omega_1$ $\forall f\colon \mathrm{dom}(<^{\mathcal{R}}) \to C$ of type $\mathcal{R}$ $(\ldots, \alpha^{\langle p_1, i_1, \ldots, p_m, i_m \rangle}, \ldots) \in A$ where $\alpha^{\langle p_1, i_1, \ldots, p_m, i_m \rangle}$ is represented with respect to W_1^m by:

$$f^{\langle p_1, i_1, \ldots, p_m, i_m \rangle}(\alpha_1, \ldots, \alpha_m) = f(\langle \alpha_1, i_1, \ldots, \alpha_m, i_m \rangle).$$

(Recall our notational convention; the $\alpha_1, \ldots, \alpha_m$ here are not written in increasing order).

Thus, each $M^{\mathcal{R}}$ is a measure on $(\omega_\omega)^{<\omega}$, and these measures generalize somewhat the measures occurring in the homogeneous tree construction on a $\mathbf{\Pi}^1_2$-complete set. We note that the requirement that $\langle p_1, i_1, \ldots, p_m, i_m \rangle$ be non-maximal in $\mathrm{dom}(\mathcal{R})$ if $\mathcal{R}(\langle p_1, i_1, \ldots, p_m, i_m \rangle) \neq (s)$ or (ω) causes no essential loss of generality as far as specifying functions of type $\mathcal{R}$ is concerned. For suppose $\mathcal{R}(\langle p_1, i_1, \ldots, p_m, i_m \rangle) = p_{m+1}$, where p_{m+1} is the permutation $(m+1, j_2, \ldots, j_m, j_{m+1})$. Thus, p_m is order-isomorphic to $(m+1, j_2, \ldots, j_m)$. Suppose $f\colon \mathrm{dom}(<^{p_m}) \to \omega_1$ is order-preserving, f is discontinuous, that is, for $(\alpha_1, \ldots, \alpha_m)$ of type p_m we have

$$f(\alpha_1, \ldots, \alpha_m) > \sup \{ f(\vec{\beta})\colon \vec{\beta} <^{\pi} \vec{\alpha} \},$$

and for W_1^m almost $(\alpha_1, \ldots, \alpha_m)$ of type p_m we have $f(\vec{\alpha})$ has uniform cofinality $\{\beta : (\alpha_1, \ldots, \alpha_m, \beta)$ has type $p_{m+1}\}$. By definition, there is an

$$f'\colon \{(\alpha_1, \ldots, \alpha_m, \beta)\colon (\alpha_1, \ldots, \alpha_m, \beta) \text{ has type } p_{m+1}\} \to \omega_1$$

which is order-preserving with respect to $<^{p_{m+1}}$, $f'(\vec{\alpha}, \beta) = \sup_{\beta' < \beta} f'(\vec{\alpha}, \beta')$ for limit β, and f' induces f in that $f(\vec{\alpha}) = \sup_{\beta} f'(\vec{\alpha}, \beta)$ almost everywhere. Furthermore, it is not difficult to see that $[f']_{W_1^{m+1}}$ is uniquely determined from $[f]_{W_1^m}$. This shows that there is no harm in adding $\langle p_1, i_1, \ldots, p_m, i_m, p_{m+1}, 0\rangle$ to $\mathrm{dom}(\mathcal{R})$.

If $\mathcal{R}$ is a type-1 tree of uniform cofinalities, and $f_1, f_2\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ are of type $\mathcal{R}$, we write $[f_1] = [f_2]$ to mean for all $\langle p_1, i_1, \ldots, p_k, i_k\rangle \in \mathrm{dom}(\mathcal{R})$ we have $[f_1^{\langle p_1,i_1,\ldots,p_k,i_k\rangle}]_{W_1^k} = [f_2^{\langle p_1,i_1,\ldots,p_k,i_k\rangle}]_{W_1^k}$.

We say the type-1 tree of uniform cofinalities $\mathcal{R}'$ *extends* $\mathcal{R}$ if there is a length preserving injection $\rho\colon \mathrm{dom}(\mathcal{R}) \to \mathrm{dom}(\mathcal{R}')$ such that $\mathcal{R}(\vec{s}) = \mathcal{R}'(\rho(\vec{s}))$ for all $\vec{s} \in \mathrm{dom}(\mathcal{R})$. We usually just say $\vec{s} \in \mathrm{dom}(\mathcal{R})$ is "identified" with $\rho(\vec{s}) \in \mathrm{dom}(\mathcal{R}')$. We say $\mathcal{R}'$ is an immediate extension of $\mathcal{R}$ is there is exactly one sequence $\vec{s}$ not ending in 0 in $\mathrm{dom}(\mathcal{R}') - \mathrm{dom}(\mathcal{R})$.

We say $\mathcal{R}'$ is a *partial extension* of $\mathcal{R}$ if $\mathcal{R}'$ has a unique extra sequence $\vec{s} = \langle q_1, j_1, \ldots, q_l, j_l^*\rangle$ in $\mathrm{dom}(\mathcal{R}')$, and $j_l^* \neq 0$. More precisely, this means $\langle q_1, j_1, \ldots, q_l, j_l\rangle \in \mathrm{dom}(\mathcal{R})$ for $0 \leq j_l \leq a$ and we have $\langle q_1, j_1, \ldots, q_l, j_l\rangle \in \mathrm{dom}(\mathcal{R}')$ for $0 \leq j_l \leq a+1$. For $j_l < j_l^*$, we identify $\langle q_1, j_1, \ldots, q_l, j_l\rangle \in \mathrm{dom}(<^{\mathcal{R}})$ with $\langle q_1, j_1, \ldots, q_l, j_l\rangle \in \mathrm{dom}(<^{\mathcal{R}'})$, and for $j_l \geq j_l^*$, we identify $\langle q_1, j_1, \ldots, q_l, j_l\rangle \in \mathrm{dom}(<^{\mathcal{R}})$ with $\langle q_1, j_1, \ldots, q_l, j_l+1\rangle \in \mathrm{dom}(<^{\mathcal{R}'})$. We leave $\mathcal{R}'(\vec{s})$ undefined (more formally, in order to still have $\vec{s} \in \mathrm{dom}(\mathcal{R}')$, we require $\mathcal{R}'(\vec{s})$ to be a formal symbol (u) for "undefined"). We define $<^{\mathcal{R}'}$ as for immediate extensions. Thus, after identifying $\mathrm{dom}(<^{\mathcal{R}})$ with a subset of $\mathrm{dom}(<^{\mathcal{R}'})$ we have

$$\mathrm{dom}(<^{\mathcal{R}'}) = \mathrm{dom}(<^{\mathcal{R}}) \cup \{\langle \alpha_1, j_1, \ldots, \alpha_l, j_l^*\rangle \colon (\alpha_1, \ldots, \alpha_l) \text{ has type } q_l\}.$$

Finally, we say $g'\colon \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ is of *semi-type* $\mathcal{R}'$ if g' is order-preserving with respect to $<^{\mathcal{R}'}$, and the function $g\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ it induces by restriction is of type $\mathcal{R}$. Note that g' being of semi-type $\mathcal{R}'$ imposes no restriction on the uniform cofinality of the component function $g'^{\langle q_1,j_1,\ldots,q_l,j_l^*\rangle}$.

Lemma 4.24 generalizes to level-1 trees of uniform cofinalities as follows.

4.26 Lemma *Let $\mathcal{R}$ be a level-1 tree of uniform cofinalities, and suppose $f\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ is of type $\mathcal{R}$. Suppose $\vec{s} = \langle p_1, i_1, \ldots, p_k, i_k\rangle$ is non-maximal in $\mathrm{dom}(\mathcal{R})$, and $\mathcal{R}(\vec{s}) = p_{k+1}$. Suppose $\delta \in ON$ and for W_1^{k+1} almost all $(\alpha_1, \ldots, \alpha_{k+1})$ of type p_{k+1} we have $\delta(\vec{\alpha}) < f(\langle \alpha_1, i_1, \ldots, \alpha_k, i_k\rangle)$. Let i_{k+1}^* be maximal such that $\langle p_1, i_1, \ldots, p_{k+1}, i_{k+1}^*\rangle \in \mathrm{dom}(\mathcal{R})$. Let $\mathcal{R}'$ be the partial extension of $\mathcal{R}$ with extra sequence $\langle p_1, i_1, \ldots, p_{k+1}, i_{k+1}^*+1\rangle$ in its domain. Then there is an f' of semi-type $\mathcal{R}'$ such that:*

1. *$[f' \restriction \mathrm{dom}(<^{\mathcal{R}})] = [f]$. That is, for each $\langle q_1, j_1, \ldots, q_l, j_l\rangle \in \mathrm{dom}(\mathcal{R})$, $\forall^*_{W_1^l}(\alpha_1, \ldots, \alpha_l)\ f'(\langle \alpha_1, q_1, \ldots, \alpha_l, q_l\rangle) = f(\langle \alpha_1, q_1, \ldots, \alpha_l, q_l\rangle)$.*

2. *$\mathrm{ran}(f') \subseteq \mathrm{ran}(f) \cup (\mathrm{ran}(f))'$.*

3. *$\forall^*_{W_1^{k+1}}(\alpha_1, \ldots, \alpha_{k+1})\ \delta(\vec{\alpha}) < f'(\langle \alpha_1, i_1 \ldots, \alpha_{k+1}, i_{k+1}^*\rangle)$.*

Proof. Fix a representing function $\vec{\alpha} \to \delta(\vec{\alpha})$ for δ, and fix $b\colon \omega_1 \to \omega_1$ and a c.u.b. $C \subseteq \omega_1$ closed under b such that for all $(\alpha_1, \dots, \alpha_{k+1})$ of type p_{k+1} in C, $\delta(\vec{\alpha}) < f(\langle \alpha_1, i_1, \dots, \alpha_k, i_k, b(\alpha_{k+1}), 0\rangle)$. Let D be the set of closure points of C, and define $l(\alpha) = \alpha^{\text{th}}$ element of D. For $\langle \alpha_1, j_1, \dots, \alpha_l, j_l\rangle \in \operatorname{dom}(<^{\mathcal{R}})$, define

$$f'(\langle \alpha_1, j_1, \dots, \alpha_l, j_l\rangle) = f(\langle l(\alpha_1), j_1, \dots, l(\alpha_l), j_l\rangle).$$

Define also

$$f'(\langle \alpha_1, i_1, \dots, i_k, \alpha_{k+1}, i^*_{k+1}\rangle) = f(\langle l(\alpha_1), i_1, \dots, l(\alpha_k), i_k, \beta, 0\rangle),$$

where β is the ω^{th} element of C greater than $l(\alpha_{k+1})$. It is easy to check that f' satisfies (1)-(3) above, and also satisfies all of the requirements for being of type $\mathcal{R}$ except for the requirement that $f'(\vec{s})$ have uniform cofinality ω for $\vec{s}$ in certain measure zero sets. It is easy, however, to redefine $f'(\vec{s})$ at these points to guarantee this last requirement. ⊣

4.27 Definition *A generalized trivial (or type -1) description defined relative to a measure W_1^m and a type-1 tree of uniform cofinalities $\mathcal{R}$ is a sequence $d = \langle d_1, i_1, \dots, d_k, i_k\rangle$ where each d_i is a trivial description defined relative to W_1^m (i.e., $1 \le d_i \le m$), and for some (uniquely determined) $\langle p_1, i_1, \dots, p_k, i_k\rangle \in \operatorname{dom}(\mathcal{R})$ we have p_k is order-isomorphic to $(d_1, d_2, \dots, d_k)$. We order the generalized descriptions lexicographically.*

We extend the interpretation function h to generalized descriptions. If $d = \langle d_1, i_1, \dots, d_k, i_k\rangle$ is defined relative to W_1^m and $\mathcal{R}$, $f\colon \operatorname{dom}(<^{\mathcal{R}}) \to \omega_1$ is of type $\mathcal{R}$, and $\beta_1 < \cdots < \beta_m$, then we define

$$h(f; \vec{\beta}; d) = f(\langle h(\vec{\beta}; d_1), i_1, \dots, h(\vec{\beta}; d_k), i_k\rangle).$$

We extend also the lowering operation $\mathcal{L}$ to generalized descriptions as follows.

4.28 Definition *Suppose $d = \langle d_1, i_1, \dots, d_k, i_k\rangle$ is defined relative to W_1^m and $\mathcal{R}$. Let $\langle p_1, i_1, \dots, p_k, i_k\rangle \in \operatorname{dom}(\mathcal{R})$, where p_k is order-isomorphic to $(d_1, \dots, d_k)$. We define $\mathcal{L}(d)$ through the following cases:*

1. *If $\mathcal{R}(\langle p_1, i_1, \dots, p_k, i_k\rangle) = (\omega)$, $\mathcal{L}(d) = \langle d_1, i_1, \dots, d_k, i_k - 1\rangle$ (note here that $i_k > 0$).*
2. *If $i_k = 0$, then $\mathcal{L}(d) = \langle d_1, i_1, \dots, \mathcal{L}(d_k), i^*_k\rangle$ provided $\mathcal{L}(d_k)$ is defined (i.e., $d_k > 1$) and $(d_1, \dots, \mathcal{L}(d_k))$ is order-isomorphic to p_k. Here i^*_k is maximal such that $\langle p_1, i_1, \dots, p_k, i^*_k\rangle \in \operatorname{dom}(\mathcal{R})$. Otherwise, set $\mathcal{L}(d) = \langle d_1, i_1, \dots, d_{k-1}, i_{k-1} - 1\rangle$ (note that $i_{k-1} > 0$) provided $k > 1$. For $k = 1$ in this case (and thus $d = \langle d_1, 0\rangle$, with $d_1 = 1$), we declare d to be $\mathcal{L}$-minimal, and do not define $\mathcal{L}(d)$.*

3. *Suppose now* $i_k > 0$ *and* $\mathcal{R}(\langle p_1, i_1, \ldots, p_k, i_k\rangle) = p_{k+1}$. *Let* $1 \le j \le k$ *be such that if* $\alpha_{d_j - 1} < \beta < \alpha_{d_j}$, *then* $(\alpha_{d_1}, \ldots, \alpha_{d_k}, \beta)$ *is order-isomorphic to* p_{k+1}. *If* $d^*_{k+1} \doteq d_j - 1 \notin \{d_1, \ldots, d_k\}$, *set* $\mathcal{L}(d) = \langle d_1, i_1, \ldots, d_k, i_k, d^*_{k+1}, i^*_{k+1}\rangle$, *where* i^*_{k+1} *is maximal such that* $\langle p_1, i_1, \ldots p_k, i_k, p_{k+1}, i^*_{k+1}\rangle \in \mathrm{dom}(\mathcal{R})$. *If* $d_j - 1 = 0$ *or* $d_j - 1 \in \{d_1, \ldots, d_k\}$, *set* $\mathcal{L}(d) = \langle d_1, i_1, \ldots, d_k, i_k - 1\rangle$.

The significance of this definition is embodied in the following lemma.

4.29 Lemma *Suppose* $d = \langle d_1, i_1, \ldots, d_k, i_k\rangle$ *is defined relative to* $W_1^m, \mathcal{R}$, *and* $f\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ *has type* $\mathcal{R}$. *Let* $\delta \in ON$ *be such that* $\forall^*_{W_1^m}\vec{\beta}\ \delta(\vec{\beta}) < h(f; \vec{\beta}; d)$. *Suppose* d *is non-minimal with respect to* $\mathcal{L}$. *Then there is an* $f'\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ *of type* $\mathcal{R}$ *such that* $[f'] = [f]$ *and* $\forall^*_{W_1^m}\vec{\beta}\ \delta(\vec{\beta}) < N_{f'}(h(f'; \vec{\beta}; \mathcal{L}(d)))$.

Proof. The result follows easily from lemma 4.26 in all cases. For example, suppose $d = \langle d_1, i_1, \ldots, d_k, i_k\rangle$, and $\mathcal{L}(d) = \langle d_1, i_1, \ldots, d_k, i_k, d^*_{k+1}, i^*_{k+1}\rangle$, where $d^*_{k+1} = d_j - 1$. Thus,

$$\forall^*_{W_1^m}\vec{\alpha}\ \exists\beta < \alpha_{d_j}\ \delta(\vec{\alpha}) < f(\langle h(\vec{\alpha}; d_1), i_1, \ldots, h(\vec{\alpha}; d_k), i_k, \beta, 0\rangle).$$

Thus, $\exists l\colon \omega_1 \to \omega_1$ such that

$$\forall^*_{W_1^m}\vec{\alpha}\ \delta(\vec{\alpha}) < f(\langle h(\vec{\alpha}; d_1), i_1, \ldots, h(\vec{\alpha}; d_k), i_k, l(\alpha_{d_j - 1}), 0\rangle).$$

From lemma 4.26 there is an f' of type $\mathcal{R}$ with $[f'] = [f]$ (and in fact with $\mathrm{ran}(f') \subseteq \mathrm{ran}(f) \cup \mathrm{ran}(f)'$), and

$$\forall^*\vec{\alpha}\ N_{f'}(\langle h(\vec{\alpha}; d_1), i_1, \ldots, h(\vec{\alpha}; d_k), i_k, \alpha_{d_j j-1}, i^*_{k+1}\rangle) > f(\langle h(\vec{\alpha}; d_1), i_1, \ldots, h(\vec{\alpha}; d_k), i_k, l(\alpha_{d_j-1}), 0\rangle).$$

f' is as desired. ⊣

4.30 Definition *A level-2 complex* $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \ldots, x_{n-1}; d_0, \ldots, d_n\rangle$ *is a sequence where* $\mathcal{R}$ *is a type-1 tree of uniform cofinalities,* $T_{x_0}, \ldots, T_{x_{n-1}}$ *are well-founded, and* $d_0, \ldots, d_n$ *are generalized trivial descriptions defined relative to* W_1^m.

For $\mathcal{C}$ a complex as above, $f\colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$, $\vec{\beta} \in (\omega_1)^m$, and $\gamma < \omega_1$, we define $h(f; \vec{\beta}; \mathcal{C}; \gamma) = |T_{x_0}\restriction h(f; \vec{\beta}; d_0)(\alpha_1)|$, where $\alpha_1 = |T_{x_1}\restriction h(f; \vec{\beta}; d_1)(\alpha_2)|$, $\ldots, \alpha_{n-1} = |T_{x_{n-1}}\restriction h(f; \vec{\beta}; d_{n-1})(\alpha_n)|$, and $\alpha_n = \gamma$. For f of type $\mathcal{R}$, we let $h(f; W_1^m; \mathcal{C}; \gamma) < \omega_{m+1}$ be the ordinal represented with respect to W_1^m by the function $\vec{\beta} \to h(f; \vec{\beta}; \mathcal{C}; \gamma)$. Note that d_n is not used in the definition of $h(f; \vec{\beta}; \mathcal{C}; \gamma)$ (it plays a role in the definition of a situation below).

The following theorem analyzes the measures on ω_ω.

4.31 Theorem *Let ν be a measure on ω_ω. Then there is a measure μ on ω_1 and a complex $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \ldots, x_{n-1}; d_0, \ldots, d_n \rangle$ such that for all $A \subseteq \omega_\omega$:*

$$\nu(A) = 1 \leftrightarrow \forall^*_\mu \gamma \; \exists \textit{ c.u.b. } C \subseteq \omega_1 \; \forall f \colon \operatorname{dom}(<^{\mathcal{R}}) \to C \textit{ of type } \mathcal{R} \quad [h(f; W_1^m; \mathcal{C}; \gamma) \in A].$$

Proof. The proof is similar to that of theorem 4.8. Fix the measure ν, and let m be least such that $\nu(\omega_{m+1}) = 1$. We may assume $m \geq 1$.

4.32 Definition *A situation for ν is a triple $(\mathcal{C}, g, r)$ consisting of a complex $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \ldots, x_{n-1}; d_0, \ldots, d_n \rangle$ (same m as above), and functions g, r with domain ω_{m+1} satisfying:*

1. *$g \colon \omega_{m+1} \to \omega_{m+1}^{<\omega}$ and $g(\nu) = M^{\mathcal{R}}$, the measure associated with $\mathcal{R}$.*
2. *$r \colon \omega_{m+1} \to \omega_{m+1}$ and $\forall^*_\nu \alpha \; r(\alpha) < h(g(\alpha); W_1^m; d_n)$.*
3. *$\forall^*_\nu \alpha \; [\alpha = h(g(\alpha); W_1^m; \mathcal{C}; r(\alpha))]$.*

Note that in (2), (3), when we write $h(g(\alpha); W_1^m; d_n)$, for example, we mean $h(g; W_1^m; d_n)$ where $g \colon \operatorname{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ represents $g(\alpha)$.

We fix now a situation $(\mathcal{C}; g; r)$ for ν with minimal value for

$$[\alpha \to h(g(\alpha); W_1^m; d_n)]_\nu,$$

where $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \ldots, x_{n-1}; d_0, \ldots, d_n \rangle$.

We claim that $\forall^*_\nu \alpha \; r(\alpha) < \omega_1$. Granting this, let $\mu = r(\nu)$. Let ν' be the measure defined by:

$$\nu'(A) = 1 \leftrightarrow \forall^*_\mu \gamma \; \exists \text{ c.u.b. } C \subseteq \omega_1 \; \forall f \colon \operatorname{dom}(<^{\mathcal{R}}) \to C \text{ of type } \mathcal{R} \quad [h(f; W_1^m; \mathcal{C}; \gamma) \in A].$$

We show that $\nu' = \nu$. Suppose not, and let $\nu'(A) = 1$, $\nu(A^c) = 1$. By the ω_2-additivity of $M^{\mathcal{R}}$ (which follows from an easy partition argument) fix a μ measure one set $D \subseteq \omega_1$, and a c.u.b. $C \subseteq \omega_1$ such that if $\gamma \in D$ and $f \colon \operatorname{dom}(<^{\mathcal{R}}) \to C$ is of type $\mathcal{R}$, then $h(f; W_1^m; \mathcal{C}; \gamma) \in A$. Since $g(\nu) = M^{\mathcal{R}}$ and $r(\nu) = \mu$, there is an $\alpha \in A^c$ such that $r(\alpha) \in D$ and $g(\alpha)$ is representable by $f \colon \operatorname{dom}(<^{\mathcal{R}}) \to C$ of type $\mathcal{R}$, and such that $\alpha = h(f; W_1^m; \mathcal{C}; r(\alpha))$. However, $h(f; W_1^m; \mathcal{C}; r(\alpha)) \in A$, a contradiction.

To fix notation, say $d_n = \langle d_1^n, i_1, \ldots, d_k^n, i_k \rangle$. We may assume d_n is non-minimal with respect to $\mathcal{L}$, since otherwise $\forall^*_\nu \alpha \; r(\alpha) < h(g(\alpha); W_1^m; d_n) = \omega_1$. We proceed to violate the minimality of $(\mathcal{C}; g; r)$. From lemma 4.29 we have that $\forall^*_\nu \alpha \; \exists f \colon \operatorname{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ such that $[f] = g(\alpha)$ and $\forall^*_{W_1^m} \vec{\beta} \; r(\alpha)(\vec{\beta}) < N_f(h(f; \vec{\beta}; \mathcal{L}(d_n)))$.

Define a partial extension $\mathcal{R}'$ of $\mathcal{R}$ as follows. If

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_k^n, i_k - 1\rangle,$$

then $\mathcal{R}'$ has the extra sequence $\langle d_1^n, i_1, \ldots, d_k^n, i_k\rangle$ in its domain. We identify $\langle d_1^n, \ldots, i\rangle \in \mathrm{dom}(\mathcal{R})$ with $\langle d_1^n, \ldots, i\rangle \in \mathrm{dom}(\mathcal{R}')$ for $i < i_k$, and with $\langle d_1^n, \ldots, i+1\rangle \in \mathrm{dom}(\mathcal{R}')$ for $i \geq i_k$. If $i_k = 0$ and

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, \mathcal{L}(d_k^n), i_k^*\rangle,$$

then $\mathcal{R}'$ has the extra sequence $\langle d_1^n, i_1, \ldots, \mathcal{L}(d_k^n),\ i_k^* + 1\rangle$ in its domain. If $i_k = 0$ and

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_{k-1}^n, i_{k-1} - 1\rangle,$$

then $\mathcal{R}'$ has the extra sequence $\langle d_1^n, i_1, \ldots, d_{k-1}^n, i_{k-1}\rangle$ in its domain. Finally, if

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_k^n, i_k, d_{k+1}^*, i_{k+1}^*\rangle,$$

then $\mathcal{R}'$ has the extra sequence $\langle d_1^n, i_1, \ldots, d_k^n, i_k, d_{k+1}^*,\ i_{k+1}^* + 1\rangle$ in its domain. Thus, in all cases the extra sequence is inserted according to lemma 4.29. In all cases, let d_n^* be the generalized description corresponding to the extra sequence in the partial complex $\mathcal{R}'$. For example, if

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_k^n, i_k - 1\rangle,$$

then $d_n^* = d_n = \langle d_1^n, i_1, \ldots, d_k^n, i_k\rangle$, and if

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_k^n, i_k, d_{k+1}^*, i_{k+1}^*\rangle,$$

then $d_n^* = \langle d_1^n, i_1, \ldots, d_{k+1}^*, i_{k+1}^* + 1\rangle$.

To unify notation, let $\vec{s} = \langle q_1, j_1, \ldots, q_l, j_l\rangle$ be the extra sequence in $\mathrm{dom}(\mathcal{R}') - \mathrm{dom}(\mathcal{R})$. We therefore have: $\forall_\nu^* \alpha\ \exists f' \colon \mathrm{dom}(\mathcal{R}') \to \omega_1$ of semi-type $\mathcal{R}'$ inducing f of type $\mathcal{R}$ such that $[f] = g(\alpha)$ and $\forall_{W_1^m}^* \vec{\beta}\ r(\alpha)(\vec{\beta}) < N_{f'}(h(f; \vec{\beta}; \mathcal{L}(d_n)))$. As in theorem 4.8 we consider two cases.

First suppose there is a ν measure one A such that for all $\alpha \in A$ there is a $f' \colon \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ of semi-type $\mathcal{R}'$ inducing (by restriction) the function $f \colon \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ such that $[f] = g(\alpha)$ and

$$\forall \alpha' \in A\ [(g(\alpha') = g(\alpha)) \to \forall_{W_1^m}^* \vec{\beta}\ r(\alpha')(\vec{\beta}) \leq N_{f'}(h(f; \vec{\beta}; \mathcal{L}(d_n)))].$$

Consider the partition $\mathcal{P}$ where we partition f' of semi-type $\mathcal{R}'$ inducing f of type $\mathcal{R}$, and where $f'(\langle \alpha_1, j_1, \ldots, \alpha_l, j_l\rangle)$ has uniform cofinality ω, according to whether

$$\forall \alpha' \in A\ [(g(\alpha') = [f]) \to \forall_{W_1^m}^* \vec{\beta}\ r(\alpha')(\vec{\beta}) \leq N_{f'}(h(f; \vec{\beta}; \mathcal{L}(d_n)))].$$

We easily have, using lemma 4.24, that on the homogeneous side the stated property holds. Fix $C \subseteq \omega_1$ homogeneous for $\mathcal{P}$. Let T_{x_n} be well-founded and $\forall^*_{W_1^m}\beta\ |T_{x_n}\restriction\beta| > N_C(\beta)$. Then $\forall^*_\nu\alpha\ \exists f$ of type $\mathcal{R}$ with $[f] = g(\alpha)$ and $\forall^*_{W_1^m}\vec{\beta}\ r(\alpha)(\vec{\beta}) < |T_{x_n}\restriction h(f;\vec{\beta};\mathcal{L}(d_n))|$. Define r' by: $\forall^*_\nu\alpha$ if $[f] = g(\alpha)$, then

$$\forall^*_{W_1^m}\vec{\beta}\ r(\alpha)(\vec{\beta}) = |(T_{x_n}\restriction h(f;\vec{\beta};\mathcal{L}(d_n)))(r'(\alpha)(\vec{\beta}))|.$$

Let $\mathcal{C}' = \langle \mathcal{R}; W_1^m; x_0, \ldots, x_{n-1}, x_n; d_0, \ldots, d_{n-1}, \mathcal{L}(d_n), \mathcal{L}(d_n)\rangle$. Then we have $(\mathcal{C}'; g; r')$ violates the minimality of the original situation.

Suppose next that such a measure one set does not exist. Let g' satisfy the following:

1. $\forall^*_\nu\alpha\ g'(\alpha)$ is represented by an $f'\colon \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ of semi-type $\mathcal{R}'$ inducing f of type $\mathcal{R}$ representing $g(\alpha)$.
2. There is a ν measure one set A such that if $\alpha_1, \alpha_2 \in A$, $g(\alpha_1) = g(\alpha_2)$, and $r(\alpha_1) \le r(\alpha_2)$, then $[{f'_1}^{\langle q_1, j_1, \ldots, q_l, j_l\rangle}]_{W_1^l} \le [{f'_2}^{\langle q_1, j_1, \ldots, q_l, j_l\rangle}]_{W_1^l}$. Here f'_1, f'_2 represent $g'(\alpha_1)$, $g'(\alpha_2)$.
3. There does not exist a ν measure one set A such that $\forall\alpha \in A\ \exists f'$ of semi-type $\mathcal{R}'$ inducing f of type $\mathcal{R}$ representing $g(\alpha)$ and
$$\forall\alpha' \in A\ [(g(\alpha') = g(\alpha)) \to [{f'_1}^{\langle q_1, j_1, \ldots, q_l, j_l\rangle}]_{W_1^l} \le [{f'}^{\langle q_1, j_1, \ldots, q_l, j_l\rangle}]_{W_1^l}.$$
Here f'_1 represents $g'(\alpha)$.
4. For all $[g'']_\nu < [g']_\nu$, g'' does not satisfy (1)-(3).

Easily g' is well-defined [to satisfy (1)-(3), let $g'(\alpha)$ be least such that it is representable by an f' of semi-type $\mathcal{R}$ inducing f representing $g(\alpha)$ and such that $\forall^*_{W_1^m}\vec{\beta}\ r(\alpha)(\vec{\beta}) \le N_{f'}(h(f;\vec{\beta};\mathcal{L}(d_n)))$]. From (4) it follows by a pressing down argument that for any c.u.b. $C \subseteq \omega_1$, $\forall^*_\nu\alpha\ g'(\alpha)$ is represented by some $f'\colon \mathrm{dom}(<^{\mathcal{R}'}) \to C$ of semi-type $\mathcal{R}'$. By countable additivity of ν, there is an immediate extension $\mathcal{R}''$ of $\mathcal{R}$ extending $\mathcal{R}'$ (that is, $\mathcal{R}''(\vec{s})$ is now defined) such that $\forall^*_\nu\alpha\ g'(\alpha)$ is representable by f' of type $\mathcal{R}''$. Fix a ν measure one set A on which (1), (2) above hold. Consider the partition $\mathcal{P}$: we partition f' of type $\mathcal{R}''$ with the "extra values" $h(\langle\alpha_1, j_1, \ldots, \alpha_l, j_l\rangle)$, for $(\alpha_1, \ldots, \alpha_l)$ of type q_l, of uniform cofinality ω inserted between $f'(\langle\alpha_1, j_1, \ldots, \alpha_l, j_l\rangle)$ and $N_{f'}(f'(\langle\alpha_1, j_1, \ldots, \alpha_l, j_l\rangle))$ according to whether

$$\forall\alpha' \in A\ [(g'(\alpha') = [f']) \to \forall^*\vec{\beta}\ r(\alpha')(\vec{\beta}) \le N_{f'}(f';\vec{\beta};d^*_n)].$$

From (2), (3) and the definition of A it follows that on the homogeneous side the stated property holds. Fix $C \subseteq \omega_1$ homogeneous for $\mathcal{P}$, and x_n with T_{x_n} well-founded such that $\forall^*_{W_1^m}\beta\ |T_{x_n}\restriction\beta| > N_C(\beta)$. Define r' so that

$$\forall^*_\nu\alpha\ r'(\alpha) < h(f'; W_1^m; d^*_n)$$

and

$$\forall^*_\nu \alpha \; \forall^*_{W_1^m} \vec{\beta} \; r(\alpha)(\vec{\beta}) = |(T_{x_n} \restriction h(f'; \vec{\beta}; d_n^*))(r'(\alpha)(\vec{\beta}))|,$$

where $[f'] = g'(\alpha)$. Let $\mathcal{C}' = \langle \mathcal{R}''; W_1^m; x_0, \ldots, x_n; \pi(d_1), \ldots, \pi(d_{n-1}), d_n^*, d_n^* \rangle$, where $\pi(d_i)$ is the generalized description defined relative to $\mathcal{R}''$ corresponding to d_i defined relative to $\mathcal{R}$. Then $(\mathcal{C}'; g'; r')$ violates the minimality of the original situation. ⊣

From theorem 4.31, a suitable coding for the subsets of ω_ω follows, and thus the weak partition relation on $\boldsymbol{\delta}^1_3$. Since the details are now almost identical to those of theorem 4.12, we merely sketch the results.

If $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \ldots, x_{n-1}; d_0, \ldots, d_n \rangle$ is a level-2 complex, $B \subseteq \omega_1$, and $\sigma \in \omega^\omega$ codes the c.u.b. set $C_\sigma \subseteq \omega_1$, let $S_{\sigma,\mathcal{C},B} \subseteq \omega_{m+1}$ be the corresponding *simple* set defined by:

$$\alpha \in S_{\sigma,\mathcal{C},B} \leftrightarrow \exists \gamma \in B \; \exists f \colon \mathrm{dom}(<^{\mathcal{R}}) \to C_\sigma \text{ of type } \mathcal{R} \; [\alpha = h(f; W_1^m; \mathcal{C}; \gamma)].$$

Theorem 4.31 then shows that every $A \subseteq \omega_{m+1}$ is a countable union of simple sets.

We define our coding $z \to A_z \subseteq \omega_\omega$ as follows. Every real z codes countably many reals z_n, each of which codes a real σ_n, a set $B_n \subseteq \omega_1$, and a sequence $\mathcal{C}_n = \langle \mathcal{R}_n; W_1^m; x_0, \ldots, x_{t-1}; d_0, \ldots, d_t \rangle$ (here m, t depend on n) satisfying the definition of a complex, except we do not require the T_{x_i} to be well-founded (the exact manner in which B_n is coded is not important; we could use the coding of theorem 4.13, or simply use the coding lemma). For each $n \in \omega$, define $A_{z_n} \subseteq \omega_{m(n)+1}$ as follows. If σ_n does not code a c.u.b. set, or one of the T_{x_i} is ill-founded, set $A_{z_n} = \emptyset$. Otherwise, $A_{z_n} = S_{\sigma_n, \mathcal{C}_n, B_n}$. Then set $A_z = \bigcup_{n \in \omega} A_{z_n}$.

4.33 Theorem *The coding $z \to A_z \subseteq \omega_\omega$ satisfies the following:*

1. $\forall A \subseteq \omega_\omega \; \exists z \; A = A_z$.
2. $\forall \alpha < \omega_\omega \; \{z \colon \alpha \in A_z\} \in \boldsymbol{\Delta}^1_3$.

Proof. The computation in (2) is straightforward using the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ unions and intersections, and the fact that if $A_{\beta_1,\ldots,\beta_k}$ are $\boldsymbol{\Delta}^1_3$ sets for all $\vec{\beta} \in (\omega_1)^k$, then $\{x \colon \forall^*_{W_1^k} \vec{\beta} \; x \in A_{\vec{\beta}}\} \in \boldsymbol{\Delta}^1_3$ (this last fact is an easy computation using our coding of c.u.b. sets). ⊣

As a corollary we obtain the following result, due originally to Kunen.

4.34 Theorem *For all $\lambda < \boldsymbol{\delta}^1_3$, $\boldsymbol{\delta}^1_3 \to (\boldsymbol{\delta}^1_3)^\lambda$.*

Proof. Fix $\lambda < \boldsymbol{\delta}^1_3$, and a bijection $\pi\colon \omega_\omega \to \lambda$. Fix the coding $z \to A_z \subseteq (\omega_\omega)^3$ satisfying (1), (2) above (identifying $(\omega_\omega)^3$ and ω_ω). Define $z \to R_z \subseteq \lambda \times \boldsymbol{\delta}^1_3$ as follows. If $\pi(\alpha) = \delta$, set $R_z(\delta, \epsilon) \leftrightarrow \{(\beta, \gamma)\colon A_z(\alpha, \beta, \gamma)\}$ is a well-ordering of length ϵ. From the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ unions and intersections, it is easy to see that the coding $z \to R_z$ satisfies (3) in the definition of reasonable, definition 2.30. Theorem 4.33 also implies that there is a $\boldsymbol{\Delta}^1_3$ coding of the ordinals (*i.e.*, singleton sets) less than ω_ω. That is, there is a map $x \to |x| < \omega_\omega$ from ω^ω onto ω_ω such that for all $\alpha < \omega_\omega$, $\{x : |x| = \alpha\} \in \boldsymbol{\Delta}^1_3$. [In definition 4.36 below we define a better $\boldsymbol{\Delta}^1_3$ coding of ω_ω via code sets WO_m for the ordinals less than ω_m.] From the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ unions it follows that $\{(x, z) : |x| \in A_z\} \in \boldsymbol{\Delta}^1_3$. From this, it follows that (4) in the definition of reasonable is satisfied, since if $S \subseteq \{z\colon R_z \text{ is well-founded }\}$ is $\boldsymbol{\Sigma}^1_3$, then we get a $\boldsymbol{\Sigma}^1_3$ well-founded relation on ω^ω of length $\geq \sup\{|R_z|\colon z \in S\}$. ⊣

As another consequence of the analysis of measures we obtain the following result, also due originally to Kunen.

4.35 Theorem *Let $\alpha, \beta < \boldsymbol{\delta}^1_3$, and μ a measure on μ. Then $j_\mu(\beta) < \boldsymbol{\delta}^1_3$.*

proof(sketch). We may assume $\alpha = \beta = \omega_\omega$. We use the coding of subsets of $\omega_\omega \times \omega_\omega$ given by theorem 4.33. It is enough to show that the prewellordering $\preceq$ corresponding to the ultrapower relation, that is,

$$\begin{aligned} x \preceq y \leftrightarrow\ & (x, y \text{ code functions } f_x, f_y : \omega_\omega \to \omega_\omega) \\ & \wedge\ \forall^*_\mu \alpha\ \exists \beta_1, \beta_2 < \omega_\omega\ (f_x(\alpha) = \beta_1 \wedge f_y(\alpha) = \beta_2 \wedge (\beta_1 \leq \beta_2)) \end{aligned}$$

is $\boldsymbol{\Delta}^1_3$. Using the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ length unions and intersections, we see that it suffices to show that if $\{B_\alpha\}_{\alpha < \boldsymbol{\delta}^1_3}$ is a sequence of $\boldsymbol{\Delta}^1_3$ sets, then $B \doteq \{x : \forall^*_\mu \alpha\ (x \in B_\alpha)\}$ is also $\boldsymbol{\Delta}^1_3$. It clearly suffices to show that B_μ is $\boldsymbol{\Sigma}^1_3$. Let

$$\mathcal{C} = \langle \mathcal{R}; W^m_1; x_0, \ldots, x_{n-1}; d_0, \ldots, d_n \rangle$$

be a level-2 complex, and μ_1 a measure on ω_1, which generate μ, as in theorem 4.31. Let $y_0, \ldots, y_n$ be reals with T_{y_j} well-founded for all j, and $\epsilon < \omega_1$ which generate μ_1 as in theorem 4.8 (ϵ playing the role of α there). T is again the Kunen tree. Recall our coding of c.u.b. subsets of ω_1 from §4.2. We then have

$$\begin{aligned} x \in B \leftrightarrow\ & \exists \sigma_1, \sigma_2\ (T_{\sigma_1}, T_{\sigma_2} \text{ are well-founded} \\ & \wedge \forall \gamma_1 < \cdots < \gamma_n \in C_{\sigma_1}\ \forall \vec{\delta} \in (\omega_\omega)^{<\omega}\ \forall \eta < \omega_\omega\ [\text{ If } \vec{\delta} \text{ is} \\ & \text{representable by an } f : \mathrm{dom}(<_{\mathcal{R}}) \to \omega_1 \text{ of the correct type,} \\ & \text{each } f^{\langle p_1, i_1, \ldots, p_k, i_k \rangle} \text{ has range in } C_{\sigma_2} \text{ almost everywhere,} \\ & \text{and } h(f; W^m_1; \mathcal{C}; \gamma') = \eta, \text{ then } x \in B_\eta]) \end{aligned}$$

where h is as in theorem 4.31, and $\gamma' = h^{\epsilon}_{y_0,\dots,y_n}(\gamma_1,\dots,\gamma_n)$ as in theorem 4.8. Note that $h(f; W_1^m; \mathcal{C}; \gamma')$ depends only on $\vec{\delta}$. From the closure properties of $\boldsymbol{\Delta}^1_3$ again, it is enough to show that if $C_{\alpha_1,\dots,\alpha_l} \in \boldsymbol{\Delta}^1_3$ for all $\vec{\alpha} \in (\omega_1)^l$, then C defined by

$$z \in C \leftrightarrow \forall^*_{W_1^l} \vec{\alpha}\ (z \in C_{\vec{\alpha}})$$

is also $\boldsymbol{\Delta}^1_3$. This special case now follows easily by the same type of computation, using just level-1 complexes. ⊣

Starting from the weak partition relation on $\boldsymbol{\delta}^1_1 = \omega_1$, we have obtained the strong partition relation on $\boldsymbol{\delta}^1_1$, calculated $\boldsymbol{\delta}^1_3$, and obtained the weak partition relation on $\boldsymbol{\delta}^1_3$. This completes the first step in the inductive projective hierarchy analysis. We have used only techniques that will generalize (when combined with a suitable notion of description at the higher levels). We will sketch how this generalization takes place in §5.

4.4. The Kechris-Martin Theorem Revisited

We finish this section by using our theory of "trivial descriptions" to give a proof of the Kechris-Martin theorem for $\boldsymbol{\Pi}^1_3$ sets. This is an important result in descriptive set theory, although it (and its higher level analogs) are not needed for the inductive analysis of the projective sets. The proof we give follows closely the proof of Kechris and Martin, recast into the theory of trivial descriptions (their original proof appealed to the theory of indiscernibles for L). We assume AD throughout this section. We caution the reader that we will be using lightface notions in this section.

To state the theorem, we first introduce our coding for the ordinals $< \omega_\omega$. T continues to denote the Kunen tree from lemma 4.1 (or theorem 4.2).

4.36 Definition *$WO_1 = WO =$ the standard set of codes of well-orderings of ω. For $m \geq 1$,*

$$WO_{m+1} = \{\langle a, x_1, \dots, x_m\rangle : a \in WO_1 \wedge \forall i \leq m\ T_{x_i} \text{ is well-founded } \}.$$

For $y = \langle a, x_1, \dots, x_m\rangle \in WO_{m+1}$, let $|y| = [f_y]_{W_1^m}$, where $f_y : (\omega_1)^m \to \omega_1$ is defined by:

$$f_y(\beta_1,\dots,\beta_m) = |(T_{x_m}\restriction\beta_m)(\delta_{m-1})|, \text{ where } \delta_{m-1} = |(T_{x_{m-1}}\restriction\beta_{m-1})(\delta_{m-2})|, \dots, \delta_1 = |(T_{x_1}\restriction\beta_1)(\delta_0)|, \text{ and } \delta_0 = |a|.$$

Let $WO_\omega = \bigcup_m WO_m$.

Easily, $\mathrm{WO}_{m+1} \in \boldsymbol{\Pi}^1_2$ for all $m \geq 1$, and for all $\alpha < \omega_{m+1}$ there is a $y \in \mathrm{WO}_{m+1}$ with $|y| = \alpha$.

4.37 Definition *We say a relation* $R \subseteq \omega^\omega \times WO_{m+1}$, $m \geq 0$, *is* invariant in the codes *if*

$$\forall x, w_1, w_2\ [w_1, w_2 \in WO_{m+1} \wedge |w_1| = |w_2| \wedge R(x, w_1) \rightarrow R(x, w_2)].$$

In this case, we write $R(x, \alpha)$, *for* $\alpha < \omega_{m+1}$, *to denote* $\exists w \in WO_{m+1}\ [|w| = \alpha \wedge R(x, w)]$. *We similarly define* R *being invariant in the codes for* $R \subseteq WO_m$, *or* $R \subseteq \omega^\omega \times WO_m \times WO_n$, etc.

4.38 Theorem (Kechris-Martin) *Let* $R \subseteq \omega^\omega \times WO_{m+1}$, $m \geq 0$ *be* Π^1_3 *and invariant in the codes. Then* $P(x) \leftrightarrow \exists w \in WO_{m+1}\ R(x, w)$ *is also* Π^1_3.

For the sake of completeness we include first the $m = 0$ case, though the proof is unchanged here (c.f. [16]). So, let $R \subseteq \omega^\omega \times \mathrm{WO}$ be Π^1_3 and invariant in the codes. We show that $P(x) \leftrightarrow \exists w\ R(x, w) \leftrightarrow \exists w \in \Delta^1_3(x)\ R(x, w)$, which suffices (c.f. theorem 4D.3 of [32]). So fix x such that $P(x)$. Let $S(w) \leftrightarrow w \in \mathrm{WO} \wedge \forall w' \in \Delta^1_3(w)\ [|w'| \leq |w| \rightarrow \neg R(x, w')]$. By "bounded quantification" (4D.3 of [32]) $S \in \Sigma^1_3(x)$. Clearly S is invariant in the codes and codes a proper initial segment of ω_1. Relativizing to x, it suffices to show the following claim.

4.39 Claim *If* $S \subseteq WO$ *is* Σ^1_3, *invariant in the codes, and* $\sup\{|w| \colon w \in S\} = \alpha_0 < \omega_1$, *then* $\exists w^* \in \Delta^1_3 \cap WO\ (|w^*| > \alpha_0)$.

Proof. Let $S(w) \leftrightarrow \exists z\ B(w, z)$, where $B \in \Pi^1_2$. Consider the integer game where I plays out w_1, z, and II plays out w_2. II wins iff $w_2 \in \mathrm{WO} \wedge [B(w_1, z) \rightarrow |w_2| > |w_1|]$. This is a Σ^1_2 game for II, and II clearly wins, so by third periodicity II has a Δ^1_3 winning strategy τ. Then $A \doteq \tau(\omega^\omega) \subseteq \mathrm{WO}$ is $\Sigma^1_1(\tau)$, so there is a $\Delta^1_1(\tau)$ real $w^* \in \mathrm{WO}$ with $|w^*| > \sup\{|w| \colon w \in A\} \geq \sup\{|w| \colon w \in S\}$. Since $\tau \in \Delta^1_3$, $w^* \in \Delta^1_3$. ⊣

A useful consequence of the $m = 0$ case which we shall need is the following lemma.

4.40 Lemma *Let* $R \subseteq \omega^\omega \times WO$ *be* Σ^1_3 (Π^1_3, Δ^1_3) *and invariant in the codes. Then* $P(x) \leftrightarrow \forall^*_{W^1_1}\alpha\ R(x, \alpha)$ *is* Σ^1_3 (Π^1_3, Δ^1_3).

Proof. Suppose, for example, $R \in \Sigma^1_3$. We use a variation of our previous coding of c.u.b. subsets of ω_1. For any c.u.b. $C \subseteq \omega_1$, there is a strategy σ for II such that $\forall z \in \mathrm{WO}\ \sigma(z) \in \mathrm{WO}$ and $C(\sigma) \doteq \{\alpha < \omega_1 : \forall z \in \mathrm{WO}\ (|z| < \alpha \rightarrow |\sigma(z)| < \alpha\}$ is a c.u.b. subset of C. This follows by playing a simple Solovay game (I plays x, II plays y, and II wins iff $(x \in \mathrm{WO} \rightarrow y \in \mathrm{WO} \wedge |y| > N_C(|x|))$.

Then

$$\begin{aligned} P(x) \leftrightarrow & \exists\sigma\ [\forall w \in \mathrm{WO}\ (\sigma(w) \in \mathrm{WO}) \wedge \forall w \in \mathrm{WO}\ (\forall z\ (z \in \mathrm{WO} \wedge |z| < |w| \\ & \rightarrow |\sigma(z)| < |w|) \rightarrow R(x, w))] \end{aligned}$$

From the $m = 0$ case, $P \in \Sigma^1_3$. From this, the result for Π^1_3, Δ^1_3 follows immediately. $\dashv$

We recall our coding $z \to F_z \subseteq (\omega_1)^2$ of theorem 4.15, which we will need for the proof. Recall each real z codes countably many z_n, each of which codes reals σ_n, w^1_n, w^2_n, and a partial (level-1) complex

$$\mathcal{C}_n = \langle W^m_1; x_0, \dots, x_{t-1}; d_0, \dots, d_t \rangle.$$

As we noted previously, each F_z is a partial function.

The next lemma summarizes the properties of this coding we will need.

4.41 Lemma *Consider the relations defined by:*

$$\begin{aligned}
R_0(z) &\leftrightarrow \forall\beta\ \exists\gamma\ F_z(\beta,\gamma)\\
R_1(z,y) &\leftrightarrow y \in WO \wedge \exists\gamma\ F_z(|y|,\gamma)\\
R_2(z,y) &\leftrightarrow y \in WO \wedge \forall\beta \le |y|\ \exists\gamma\ F_z(\beta,\gamma)\\
R_3(z,x,y) &\leftrightarrow x,y \in WO \wedge \forall\beta \le |x|\ \exists\gamma \le |y|\ F_z(\beta,\gamma)
\end{aligned}$$

Then $R_0 \in \Pi^1_2$, and $R_1, R_2 \in \Pi^1_1$. Also, R_3 is Δ^1_1 in the codes for x, y, that is, there are Σ^1_1, Π^1_1 relations C, D such that for all z and $x, y \in WO$, $R_3(z,x,y) \leftrightarrow C(z,x,y) \leftrightarrow D(z,x,y)$.

Proof. The computations are all straightforward, as in the proof of the strong partition relation. For example, (in this equation, t, x_i refer to $\mathcal{C}_n$, and t', x'_i refer to $\mathcal{C}_{n'}$)

$$\begin{aligned}
R_1(z,y) \leftrightarrow{} & y \in \mathrm{WO} \wedge \exists n\ \{w^1_n, w^2_n \in \mathrm{WO} \wedge |w^1_n|, |w^2_n| < |y| \wedge\\
& \exists\beta_{t-1} < \cdots < \beta_0 \le |y|\ \exists\gamma_{t-1}, \dots, \gamma_1 < |y|\ \exists\delta_{t-1}, \dots, \delta_1 < |y|\\
& [\beta_{t-1} > \max(|w^1_n|, |w^2_n|) \wedge \forall i\ \beta_i \in C_{\sigma_n} \wedge\\
& |(T_{x_{t-1}} \restriction \beta_{t-1})(|w^1_n|)| = \gamma_{t-1} \wedge |(T_{x_{t-2}} \restriction \beta_{t-2})(\gamma_{t-1})| = \gamma_{t-2} \wedge \dots \wedge\\
& |(T_{x_0} \restriction \beta_0)(\gamma_1)| = |y| \wedge |(T_{x_{t-1}} \restriction \beta_{t-1})(|w^2_n|)| = \delta_{t-1}\\
& \wedge |(T_{x_{t-2}} \restriction \beta_{t-2})(\delta_{t-1})| = \delta_{t-2} \wedge \dots \wedge\\
& \delta_1 \text{ is in the wellfounded part of } (T_{x_0} \restriction \beta_0) \wedge\\
& \forall n' \in \omega[\{w^1_{n'}, w^2_{n'} \in \mathrm{WO} \wedge |w^1_{n'}|, |w^2_{n'}| < |y| \wedge\\
& \exists\beta'_{t'-1} < \cdots < \beta'_0 \le |y|\ \exists\gamma'_{t'-1}, \dots, \gamma'_1 < |y|\ \exists\delta'_{t'-1}, \dots, \delta'_1 < |y|\\
& (\forall i\ \beta'_i \in C_{\sigma_{n'}} \wedge |(T_{x'_{t'-1}} \restriction \beta'_{t'-1})(|w^1_{n'}|)| = \gamma'_{t'-1} \wedge\\
& |(T_{x'_{t'-2}} \restriction \beta'_{t'-2})(\gamma'_{t'-1})| = \gamma'_{t'-2} \wedge \dots \wedge |(T_{x'_0} \restriction \beta'_0)(\gamma'_1)| = |y| \wedge\\
& |(T_{x'_{t'-1}} \restriction \beta'_{t'-1})(|w^2_{n'}|)| = \delta'_{t'-1} \wedge |(T_{x'_{t'-2}} \restriction \beta'_{t'-2})(\delta'_{t'-1})| = \delta'_{t'-2}\\
& \wedge \dots \wedge |(T_{x'_1} \restriction \beta'_1)(\delta'_2)| = \delta'_1)\} \to |(T_{x'_0} \restriction \beta'_0)(\delta'_1)| = |(T_{x_0} \restriction \beta_0)(\delta_1)|\,]]\}
\end{aligned}$$

In the last line of this formula, "$|(T_{x'_0}\restriction\beta'_0)(\delta'_1)| = |(T_{x_0}\restriction\beta_0)(\delta_1)|$", abbreviates "$\delta'_1$ is in the well-founded part of $T_{x'_0}\restriction\beta'_0$ and $|(T_{x'_0}\restriction\beta'_0)(\delta'_1)| = |(T_{x_0}\restriction\beta_0)(\delta_1)|$". It is straightforward to check that all clauses in this formula define Δ^1_1 relations except the clauses "$y \in$ WO" and "δ_1 is in the well-founded part of $T_{x_0}\restriction\beta_0$", which are Π^1_1.

⊣

We will write "F_z is a function" in place of $R_0(z)$, "$F_z(|y|)$ is defined" in place of $R_1(z,y)$.

4.42 Lemma *The relation* $Q(x,z) \leftrightarrow [x \in WO_2 \wedge (F_z$ *is a function* $) \wedge |x| = [F_z]_{W^1_1}]$ *is* Δ^1_3.

Proof. $Q(x,z) \leftrightarrow \exists\sigma\ [T_\sigma$ is well-founded $\wedge\, \forall w \in \mathrm{WO}(|w| \in C_\sigma \rightarrow \exists v \in \mathrm{WO}\ (f_x(|w|) = |v| \wedge F_z(|w|,|v|)))]$. The $m = 0$ case shows $Q \in \Sigma^1_3$. A similar computation shows $\neg Q \in \Sigma^1_3$. ⊣

If V is a tree on $\omega \times \omega_1$ and $x \in \omega^\omega$, let $\prec_x$ denote the Kleene-Brouwer ordering on V_x. Thus, $\prec_x$ is a linear ordering, and is a well-order iff V_x is well-founded. We say $\alpha < \omega_1$ is represented in the well-founded part of $V_x\restriction\beta$ if there is an $s \in V_x\restriction\beta$ such that the initial segment determined by s in the Kleene-Brouwer ordering on $V_x\restriction\beta$ is order-isomorphic to α. For the trees used below this will be equivalent to saying that the initial segment of $\prec_x$ determined by s is order-isomorphic to α.

4.43 Lemma *Let* $R \subseteq \omega^\omega \times \omega^\omega$ *be* Π^1_2. *Then there is a tree* V *on* $\omega \times \omega_1$ *such that:*

1. V *is* Δ^1_1 *in the codes.*
2. $\forall x,y \in \omega^\omega\ [R(x,y) \leftrightarrow V_{\langle x,y\rangle}$ *is well-founded* $\leftrightarrow \forall\alpha < \omega_1$ (α *is represented in the well-founded part of* $V_{\langle x,y\rangle}\restriction\alpha)]$.
3. *The relation* $S(x,y,w) \leftrightarrow [w \in WO \wedge |w|$ *is represented in the well-founded part of* $V_{\langle x,y\rangle}\restriction|w|]$ *is* Δ^1_1 *in the codes for* w.

Proof. Let V' be the standard Shoenfield tree on $\omega \times \omega_1$ for R^c. Thus, V' is Δ^1_1 in the codes and $\forall x,y\ [R(x,y) \leftrightarrow V'_{\langle x,y\rangle}$ is well-founded $]$. Let V be a minor modification of V' such that

$$\forall x\,\forall\alpha < \omega_1\ (V_x\restriction\alpha \text{ is well-founded } \rightarrow |V_x\restriction\alpha| > \alpha).$$

[For example, code into V_x all finite decreasing chains $\beta_0 > \beta_1 > \cdots > \beta_n$]. We may assume that if $((a_0,\ldots,a_n),(\alpha_0,\ldots,\alpha_n)) \in V$, then $\alpha_0 \geq \alpha_1,\ldots,\alpha_n$. Clearly V is Δ^1_1 in the codes.

If $R(x,y)$, then $V_{\langle x,y\rangle}$ is well-founded, and by construction $\forall\alpha\ |V_{\langle x,y\rangle}\restriction\alpha| > \alpha$, hence α is represented in the well-founded part of $V_{\langle x,y\rangle}\restriction\alpha$. If $(x,y) \notin R$, then $V_{\langle x,y\rangle}$ is ill-founded, say $(\langle x,y\rangle,(\beta_0,\beta_1,\ldots)) \in [V]$. If the initial segment $I^{\vec{\alpha}}_{x,y}$ of $\prec_{\langle x,y\rangle}$ determined by $\vec{\alpha} = (\alpha_0,\ldots,\alpha_{n-1}) \in V_{\langle x,y\rangle}$ is well-founded, we must have $\alpha_0,\ldots,\alpha_n \leq \beta_0$. Thus,

$$\gamma \doteq \sup\{|\vec{\alpha}|_{\prec_{\langle x,y\rangle}} : I^{\vec{\alpha}}_{x,y} \text{ is well-founded }\} < \omega_1.$$

For $\delta > \max\{\beta_0,\gamma\}$, δ is not represented in the well-founded part of $V_{\langle x,y\rangle}\restriction\delta$.

Finally, (3) is a standard computation using (1). ⊣

4.44 Lemma *Let $W \subseteq WO_2$ be Σ^1_3, invariant in the codes, and code a bounded initial segment of ω_2. Then there is a Δ^1_3 relation $F \subseteq WO \times WO$ which is invariant in the codes, and defines a total function $F\colon \omega_1 \to \omega_1$ (i.e., $\forall\alpha < \omega_1 \exists!\beta < \omega_1\ F(\alpha,\beta)$) such that $[F]_{W^1_1} > |x|$ for all $x \in W$.*

Proof. Define $W'(w) \leftrightarrow \exists x \in \mathrm{WO}_2\ [W(x) \wedge (w$ codes a function $F_w\colon \omega_1 \to \omega_1) \wedge (|x| = [F_w]_{W^1_1})]$. From lemma 4.42, $W' \in \Sigma^1_3$. W' is also invariant in the sense that if w,w' code functions $F_w, F_{w'}$, $[F_w]_{W^1_1} = [F_{w'}]_{W^1_1}$, and $W'(w)$, then $W'(w')$. Let $W'(w) \leftrightarrow \exists y\ R(w,y)$, where $R \in \Pi^1_2$. Let V be a tree on $\omega \times \omega_1$ as in lemma 4.43, in particular $R(w,y) \leftrightarrow V_{\langle w,y\rangle}$ is well-founded.

Say a real w is α-good if $F_w(\alpha)$ is defined, and say w is $\leq\alpha$-good if it is α' good for all $\alpha' \leq \alpha$. Say a pair (w,y) is α-good if w is α-good and α is represented in the well-founded part of $V_{\langle w,y\rangle}\restriction\alpha$.

Consider the integer game G where I plays out reals w_1, y, and II plays out w_2, and II wins the run iff there exists an $\eta_0 < \omega_1$ such that either:

1. $\forall\eta < \eta_0\ (w_1,y), w_2$ are η-good, (w_1,y) is not η_0-good, and w_2 is η_0-good.

   or

2. $\forall\eta \leq \eta_0\ (w_1,y), w_2$ are η-good, and $F_{w_1}(\eta_0) < F_{w_2}(\eta_0)$.

Using lemmas 4.41, 4.43, G is a Σ^1_2 game for II. II easily wins the game, by playing any w^* coding a function $F_{w^*}\colon \omega_1 \to \omega_1$ such that $[F_{w^*}]_{W^1_1} > \sup\{|x|\colon x \in W\}$. Thus, by third periodicity, II has a Δ^1_3 winning strategy τ.

Define $b\colon \omega_1 \to \omega_1$ inductively as follows. Let $b(\eta_0)$ be the maximum of $(\sup_{\eta<\eta_0} b(\eta)) + 1$ and

$$\sup\{F_{\tau(w_1,y)}(\eta_0)\colon \forall\eta < \eta_0\ [(w_1,y) \text{ is } \eta\text{-good } \wedge F_{w_1}(\eta) = b(\eta)]\}.$$

We show by induction on η_0 that:

4.45 Claim *a.) $b(\eta_0)$ is well-defined, that is, $b(\eta_0) < \omega_1$.*

b.) If (w_1, y) is $\leq \eta_0$-good and $\forall \eta \leq \eta_0\ F_{w_1}(\eta) = b(\eta)$, then $\forall \eta \leq \eta_0\ F_{w_2}(\eta) \leq F_{w_1}(\eta)$, where $w_2 = \tau(w_1, y)$.

Proof. Suppose the claim holds for all $\eta < \eta_0$. Note that if (w_1, y) is η-good for all $\eta < \eta_0$ and $\forall \eta < \eta_0\ F_{w_1}(\eta) = b(\eta)$, then by (b) and induction, $F_{w_2}(\eta_0)$ is defined, where $w_2 = \tau(w_1, y)$, as otherwise II would lose this run of the game. Now, $B_{\eta_0} \doteq \{(w_1, y) \colon \forall \eta < \eta_0\ [(w_1, y) \text{ is } \eta\text{-good} \wedge F_{w_1}(\eta) = b(\eta)]\}$ is $\boldsymbol{\Delta}^1_1$, as it is Δ^1_1 in any real coding η_0 and $b{\restriction}\eta_0$. The reasonableness of the coding $z \to F_z$ now gives (a) (c.f. (4) in definition 2.30). (b) at η_0 is now immediate from the definition of $b(\eta_0)$. ⊣

Next we claim that $[b]_{W^1_1} > |x|$ for all $x \in W$. If not, then by the invariance and initial segment properties of W', there is a w_1 in W' with $F_{w_1} = b$. Let y be such that $R(w_1, y)$, and have I play (w_1, y) against τ, producing $w_2 = \tau(w_1, y)$. Since $\forall \eta_0 < \omega_1\ (w_1, y)$ is η_0-good, a straightforward induction using (b) shows that $\forall \eta_0 < \omega_1\ F_{w_2}(\eta_0)$ is defined and $F_{w_2}(\eta_0) \leq F_{w_1}(\eta_0)$, a contradiction to II winning the game.

Finally, we show that the relation $F(z_1, z_2) \leftrightarrow z_1, z_2 \in \mathrm{WO} \wedge b(|z_1|) = |z_2|$ is Δ^1_3. We have $F(z_1, z_2)$ iff the following hold:

1. $z_1, z_2 \in \mathrm{WO}$
2. There is a $y \in \omega^\omega$ and a $z \in \mathrm{WO}$ with $|z| = |z_1| + 1$ and $|0|_{\prec_z} = |z_1|$ satisfying:
   (a) $\forall n\ y_n \in \mathrm{WO}$
   (b) The map $n \to |y_n|$ defines an order-preserving map from $\prec_z$ to ω_1.
   (c) For any $n \in \mathrm{dom}(\prec_z)$, $|y_n|$ is the maximum of $(\sup \{|y_m| \colon m \prec_z n\}) + 1$ and
   $$\sup \{F_{\tau(w_1,y)}(|n|_z) \colon \forall m \prec_z n\ (w_1, y) \text{ is } |m|_{\prec_z}\text{-good} \wedge F_{w_1}(|m|_{\prec_z}) = |y_m|\}.$$
   (d) $|y_0| = |z_2|$

It follows easily that $F \in \Sigma^1_2(\tau)$, so $F \in \Delta^1_3$. ⊣

We prove now the $m = 1$ case of the Kechris-Martin theorem. Let $R(x, w) \subseteq \omega^\omega \times \mathrm{WO}_2$ be Π^1_3 and invariant in the codes. Define $R'(x, w) \leftrightarrow w \in \mathrm{WO}_2 \wedge \exists w' \in \mathrm{WO}_2\ [|w'| \leq |w| \wedge R(x, w')]$. Clearly R' is invariant in the codes, and we claim that $R' \in \Pi^1_3$. To see this, note that

$$R'(x,w) \leftrightarrow w = \langle a,v\rangle \in \mathrm{WO}_2 \wedge \exists b \in \mathrm{WO}\ [\forall^* \beta < \omega_1\ |(T_v{\restriction}\beta)(|b|)| \leq |(T_v{\restriction}\beta)(|a|)| \wedge \forall z \in \mathrm{WO}_2(|z| = |\langle b,v\rangle| \to R(x,z))]$$

From lemma 4.40 and the $m = 0$ case of the theorem it follows that $R' \in \Pi^1_3$. Replacing now R with R', we may assume that $R(x,w)$ is also closed upwards in the codes w.

We employ a standard coding for the $\Delta^1_3(x)$ subsets of $\omega^\omega \times \omega^\omega$, uniformly in x. Thus, let $Q \subseteq (\omega^\omega)^3$ be Π^1_3 and such that for every $\Pi^1_3(x)$ set $A \subseteq (\omega^\omega)^2$, there is a real y recursive in x such that $A = Q_x$. Let $Q'_0(x,y,z) \leftrightarrow Q(x_0,y,z)$, and $Q'_1(x,y,z) \leftrightarrow Q(x_1,y,z)$. Let Q_0, Q_1 in Π^1_3 reduce Q'_0, Q'_1. We then then say x *codes a* Δ^1_3 *set* if $\forall y,z\ [Q_0(x,y,z) \vee Q_1(x,y,z)]$, in which case x codes the $\Delta^1_3(x)$ set $D_x = \{(y,z)\colon Q_0(x,y,z)\}$.

Returning to the proof, let $P(x) \leftrightarrow \exists w \in \mathrm{WO}_2\ R(x,w)$, where $R \in \Pi^1_3$ is invariant and closed upwards in the codes. From lemma 4.44 we have:

$$P(x) \leftrightarrow \exists y \in \Delta^1_3(x)\ [(y \text{ codes a } \Delta^1_3 \text{ relation } D_y \subseteq (\omega^\omega)^2) \wedge (D_y \subseteq \mathrm{WO}\times \mathrm{WO} \wedge D_y \text{ is invariant in the codes}) \wedge (D_y \text{ defines a total function from } \omega_1 \text{ to } \omega_1) \wedge \forall w \in \mathrm{WO}_2\ [(\forall^*_{W^1_1} \alpha < \omega_1\ (\alpha, f_w(\alpha)) \in D_y) \to R(x,w)]]$$

For "D_y defines a total function from ω_1 to ω_1" we use:

$$\forall x, z_1, z_2 \in \mathrm{WO}\ [D_y(x,z_1) \wedge D_y(x,z_2) \to |z_1| = |z_2|] \wedge \forall x \in \mathrm{WO}\ \exists z \in \mathrm{WO}\ [\forall z' \in \mathrm{WO}\ (|z'| = |z| \to D_y(x,z'))]$$

By the $m = 0$ case of the theorem, this expression defines a Π^1_3 set, so $P \in \Pi^1_3$, using lemma 4.40. This completes the $m = 1$ case of the theorem.

We prove now the general case $m > 1$ of the theorem. So let $P(x) \leftrightarrow \exists w \in \mathrm{WO}_{m+1}\ R(x,w)$, where $R \in \Pi^1_3$ is invariant in the codes w. Recall that for $w \in \mathrm{WO}_{m+1}$, f_w is the corresponding function from ω_1^m to ω_1 (defined W^m_1 almost everywhere) representing $|w|$. For any such f_w, there is a function $g : \omega_1 \to \omega_1$ such that $\forall^*_{W^m_1}(\alpha_1, \dots, \alpha_m)\ f_w(\vec{\alpha}) < g(\alpha_m)$. We may take $g = f_y$ for some $y = \langle a,u\rangle \in \mathrm{WO}_2$. Thus we have:

$$P(x) \leftrightarrow \exists y = \langle a,u\rangle \in \mathrm{WO}_2\ \exists z \in \mathrm{WO}_m\ [\forall^*_{W^{m-1}_1} \alpha_1, \dots, \alpha_{m-1}\ f_z(\vec{\alpha}) \prec_{T_u} |a| \wedge \forall w \in \mathrm{WO}_{m+1}\ [(\forall^*_{W^m_1} \alpha_1, \dots, \alpha_m\ f_w(\alpha_1, \dots, \alpha_m) = |(T_u{\restriction}\alpha_m)(f_z(\alpha_1, \dots, \alpha_{m-1}))|) \to R(x,w)]]$$

Note that the expression beginning with $\exists z \in \mathrm{WO}_m$ is invariant in the codes for y; it is equivalent to saying $\exists f\colon (\omega_1)^m \to \omega_1\ \forall^* \alpha_1, \dots, \alpha_m\ f(\alpha_1, \dots, \alpha_m) < f_y(\alpha_m)$ and $R(x, [f]_{W^m_1})$. By the $m = 1$ and $m - 1$ cases of the theorem, $P \in \Pi^1_3$. This completes the proof of theorem 4.38. □

5. Higher Descriptions

We assume AD throughout §5. We sketch in this section how the theory of §4 can be extended to higher levels. Indeed, the arguments here should extend to the general case of a successor Suslin cardinal in the hierarchy of $L(\mathbb{R})$. We will concentrate here, however, on the projective hierarchy, and in fact largely on the theory of $\boldsymbol{\delta}^1_5$, since all of the new ideas occur here. As we said in the introduction, our style here will be somewhat informal. We will concentrate on presenting the new ideas without getting lost in details; we will sometimes illustrate proofs by considering a representative example. The reader wishing to see the complete details for the next level of the analysis (the strong partition relation on $\boldsymbol{\delta}^1_3$, the computation of $\boldsymbol{\delta}^1_5$, and the weak partition relation on $\boldsymbol{\delta}^1_5$) can consult [11]. In §6 we will consider topics related to extending this theory further.

Reflecting on the arguments of the previous section, we see that there were two fundamental ingredients. First was the Kunen analysis, lemma 4.1, which provided an analysis of the equivalence classes of function $f\colon \omega_1 \to \omega_1$ with respect to the normal measure W^1_1 on ω_1. Second was the analysis, embodied in lemma 4.5, which showed how equivalence classes of functions with respect to the more general measures W^m_1 are generated from the normal measure analysis. The combinatorics of the process was described by the descriptions. Admittedly, the concept there was rather trivial (descriptions being just integers), and there was really no interesting combinatorics taking place. The situation changes as we move to the higher levels, though, and the concept of the description becomes a central point. Indeed, armed with the correct notion of description and proper generalization of the Kunen tree analysis (due to Martin, see below), the general step in the projective hierarchy analysis is quite similar to that of §4. Thus, we concentrate in this section on showing how these two key ingredients generalize.

5.1. Martin's Theorem on the Normal Measures

From the weak partition relation on $\boldsymbol{\delta}^1_3$, theorem 4.34, it follows that there are precisely three normal measures on $\boldsymbol{\delta}^1_3$, corresponding to the three regular cardinals $\omega, \omega_1, \omega_2$ below $\boldsymbol{\delta}^1_3$ [The weak partition relation shows that the c.u.b. filter restricted to points of one of these cofinalities is a normal measure. Conversely, any normal measure on $\boldsymbol{\delta}^1_3$ must give every c.u.b. set measure one, and by countable additivity must concentrate on one these cofinalities. Thus, it must coincide with one of these three normal measures.]

The ω-cofinal normal measure on $\boldsymbol{\delta}^1_3$ behaves just as the normal measure on ω_1, indeed the Kunen tree analysis is quite general and holds for all the $\boldsymbol{\delta}^1_{2n+1}$. Thus, there is a tree T on $\omega \times \boldsymbol{\delta}^1_{2n+1}$ such that for all $f\colon \boldsymbol{\delta}^1_{2n+1} \to \boldsymbol{\delta}^1_{2n+1}$, there is an $x \in \omega^\omega$ with T_x well-founded such that $\forall^* \alpha < \boldsymbol{\delta}^1_{2n+1}\ f(\alpha) < |T_x \restriction \alpha|$ (where $\forall^*$ refers to the ω-cofinal normal measure). The proof is a small variation of lemma 4.1. Instead of WF, one uses the set W defined by $W(x) \leftrightarrow \forall n\ P(x_n)$, where P is a $\boldsymbol{\Pi}^1_{2n+1}$-complete set. Let $\{\phi_n\}$ be a $\boldsymbol{\Pi}^1_3$ scale on P. Using $\{\phi_n\}$, define a tree U on $\omega \times \boldsymbol{\delta}^1_3$ with $p[U] = W$ (a branch $\vec{\alpha}$ through U_x gives subsequences $\vec{\alpha}_1$, $\vec{\alpha}_2$, *etc.*, such that for all m, $(x_m, \vec{\alpha}_m)$ is a branch through the tree of the scale $\{\phi_n\}$). We think of $x \in W$ as coding the ordinal $|x| \doteq \sup_n \phi_0(x_n)$. Note that for almost all $\alpha < \boldsymbol{\delta}^1_3$ with respect to the ω-cofinal normal measure, there is an $x \in W$ with $|x| = \alpha$ and $U_x \restriction \alpha$ is ill-founded. Let S be a complete $\boldsymbol{\Sigma}^1_{2n+1}$ set, which is $(\boldsymbol{\delta}^1_3)^-$-Suslin by theorem 2.14. Say $S = p[V]$, V a tree on $\omega \times (\boldsymbol{\delta}^1_3)^-$. Since $S \notin \boldsymbol{\Delta}^1_3$, it follows from theorem 2.15 that $\sup\{|V_x| : V_x \text{ is wellfounded }\} = \boldsymbol{\delta}^1_3$. The Kunen tree T is then constructed as in lemma 4.1.

For the other normal measures, however, the situation is different. To discuss this, we need to recall some facts from the homogeneous tree construction. A detailed account of this may be found in [18], we summarize the main points.

Recall that if T is a tree on $\omega \times \kappa$, and $s \in \omega^n$, then $T_s = \{\vec{\alpha} \in \kappa^n : (s, \vec{\alpha}) \in T\}$. If t extends s, let $\pi_{s,t}$ denote the natural map from T_t to T_s defined by $\pi_{s,t}(\vec{\alpha}) = \vec{\alpha} \restriction \mathrm{lh}(s)$. If μ is a measure on T_t, then $\pi_{s,t}(\mu)$ is a measure on T_s (recall $\pi_{s,t}(\mu)(A) = \mu(\pi^{-1}_{s,t}(A))$).

5.1 Definition *A tree T on $\omega \times \kappa$ is homogeneous if there are measures μ_s, for $s \in \omega^{<\omega}$, with $\mu_s(T_s) = 1$ such that if t extends s then $\pi_{s,t}(\mu_t) = \mu_s$, and having the following homogeneity property: for all $x \in \omega^\omega$, if T_x is ill-founded and for each n a set $A_n \subseteq T_{x \restriction n}$ is given with $\mu_{x\restriction n}(A_n) = 1$, then there is a sequence $\vec{\alpha} \in \kappa^\omega$ such that for all n, $\alpha \restriction n \in A_n$.*

The homogeneity property for T_x is equivalent to saying that the direct limit of the ultrapowers (of ON) by the measures $\mu_{x\restriction n}$ is well-founded.

We extend the definition in the obvious way to trees T on $\omega \times \omega \times \kappa$, *etc.* (in this case, the measures $\mu_{s,t}$ are indexed by pairs of sequences of the same length). We say T is a homogeneous tree for $P \subseteq \omega^\omega$ if T is homogeneous and $p[T] = P$.

If $P \subseteq 2^\omega$ is $\boldsymbol{\Pi}^1_1$, the standard Shoenfield construction gives a tree T_1 on $2 \times \omega_1$ with $P = p[T_1]$. T_1 may be defined so that for each $s \in 2^\omega$, there is a permutation π_s of $\{1, \ldots, n\}$, $n = \mathrm{lh}(s)$, with n occurring first, such that $(s, \vec{\alpha}) \in T_1$ iff $\vec{\alpha}$ is order-isomorphic to π_s. For π a permutation of $\{1, \ldots, n\}$, let W^π_1 be the natural measure on n-tuples $\vec{\alpha}$ which are order-isomorphic to π (*i.e.*, W^π_1 is equivalent to W^n_1 under the map which re-arranges $\vec{\alpha}$ into increasing order). Thus, the measures $W^{\pi_s}_1$ witness that T_1 is homogeneous.

If $S \subseteq 2^\omega$ is $\boldsymbol{\Sigma}^1_2$, then $S(x) \leftrightarrow \exists y\ P(x, y)$, where $P \in \boldsymbol{\Pi}^1_1$, so $P = p[T_1]$ for some homogeneous tree T_1 on $2 \times 2 \times \omega_1$ (if we identify T_1 with a tree T'_1 on $2 \times \omega_1$ by identifying the second and third coordinates of T_1 with the second coordinate of T_1', then T_1' is said to be *weakly homogeneous*). For $s, t \in 2^{<\omega}$ of the same length, let $\pi_{s,t}$ and $W_1^{\pi_{s,t}}$ be the permutation and measure associated to s, t, and T_1. We may assume without loss of generality that for $s, t \in 2^{<\omega}$ of the same length, that $\pi_{s,t}$ depends only on $s \restriction \mathrm{lh}(s) - 1$, $t \restriction \mathrm{lh}(t) - 1$. For convenience we also assume (without loss of generality) that for any s, t of the same length, $(T_1)_{s,t} \neq \emptyset$. If $Q = S^c$, then the homogeneous tree construction shows how, using the strong partition relation on ω_1, to get a homogeneous tree T_2 for Q. One way of doing this is as follows. For $s \in 2^{<\omega}$, let $\prec_s$ be the Kleene-Brouwer ordering on $(T_1)_s \subseteq (2 \times \omega_1)^{\leq \mathrm{lh}(s)}$. In specifying the Kleene-Brouwer ordering, we order pairs $(n, \alpha) \in 2 \times \omega_1$ first by α. It is convenient here to adopt a minor variation of the definition of $\mathcal{R}$ being a type-1 tree of uniform cofinalities, definition 4.25. First, we drop all sequences $\vec{p} = \langle p_1, i_1, \dots, p_m, i_m \rangle$ where $i_m = 0$ from the domain of $\mathcal{R}$. Thus, for any $\vec{p}$, $\mathcal{R}(\vec{p})$ is now either (ω) or a permutation p_{m+1} extending p_m. Secondly, we now allow either possibility for $\mathcal{R}(\vec{p})$ when $\vec{p}$ is maximal in $\mathrm{dom}(\mathcal{R})$. We define $<^{\mathcal{R}}$ exactly as before, and define $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ being of type $\mathcal{R}$ in the obvious way (for $\vec{p}$ maximal in $\mathrm{dom}(\mathcal{R})$ with $\mathcal{R}(\vec{p}) = p_{m+1}$, we require $f^{\vec{p}}(\alpha_1, \dots, \alpha_m)$ to almost everywhere have uniform cofinality $\{\beta : (\alpha_1, \dots, \alpha_m, \beta)$ is order-isomorphic to $p_{m+1}\}$). It is now easy to check that $\prec_s$ is of the form $<^{\mathcal{R}_s}$ for some type-1 tree of uniform cofinalities $\mathcal{R}_s$. In fact, we can define $\mathcal{R}_s$ as follows. Let $\vec{p} = \langle p_1, i_1, \dots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R}_s)$ iff $m \leq \mathrm{lh}(s)$, each $i_k = 1$ or 2, and $p_m = \pi_{s \restriction m, t}$, where $t = (i_1 - 1, \dots, i_m - 1)$. We set $\mathcal{R}_s(\vec{p}) = \pi_{s', t'}$, where s', t' are any immediate extensions of s, t (by our assumption, this only depends on s and t).

We define $(s, \vec{\beta}) \in T_2$ iff there is an $f \colon \mathrm{dom}(<^{\mathcal{R}_s}) \to \omega_1$ of type $\mathcal{R}_s$ with $\vec{\beta} = [f] \restriction \mathrm{lh}(s)$. To say $\vec{\beta} = [f] \restriction \mathrm{lh}(s)$ means that for all $i < \mathrm{lh}(s)$, $\beta_i = [f^{\langle p_1, i_1, \dots, p_k, i_k \rangle}]_{W_1^k}$, where $(i_1, \dots, i_k)$ is the i^{th} element of $2^{<\omega}$ in some reasonable enumeration, and $p_j = \pi_{(s \restriction j, (i_1, \dots, i_j))}$. From the strong partition relation on ω_1 it is not difficult to see that T_2 is homogeneous for Q, with measures $M^{\mathcal{R}_s}$. [For example, to show homogeneity, suppose $(T_2)_x$ is ill-founded, and each A_n has measure one with respect to $M^{\mathcal{R}_{x \restriction n}}$. Let C_n be a c.u.b. subset of ω_1 defining a $M^{\mathcal{R}_{x \restriction n}}$ measure one set contained in A_n. Let $C = \bigcap_n C_n$. Since $x \in Q = P^c$, $(T_1)_x$ is well-founded. Let f be order-preserving from the Kleene-Brouwer ordering on $(T_1)_x$ to C such that for all n, $f \restriction \mathrm{dom}(\prec_{x \restriction n})$ is of type $\mathcal{R}_{x \restriction n}$. Let $\beta_i = [f^{\langle p_1, i_1, \dots, p_k, i_k \rangle}]_{W_1^k}$, where again $(i_1, \dots, i_k)$ is the i^{th} element of $2^{<\omega}$ in our enumeration. Then $(\beta_0, \dots, \beta_n) \in A_n$ for all n.

Recall WO_2 is the $\mathbf{\Pi}^1_2$ set of codes for ordinals $< \omega_2$ (definition 4.36), and for $x \in \mathrm{WO}_2$, $|x| = [f_x]_{W_1^1} < \omega_2$ is the ordinal coded by x. The next lemma shows that we may get a homogeneous tree for WO_2 with an additional property.

5.2 Lemma *There is a homogeneous tree U on $\omega \times \omega_\omega$ with $WO_2 = p[U]$ and with the following property:*

$(\star)$: If $\{\psi_n\}$ is the scale on WO_2 corresponding to U (using left-most branches), then $\forall x \in WO_2\ |x| \leq \psi_0(x)$.

Proof. Let T be the Kunen tree of §4, where we use the linear-order version of theorem 4.2. Recall $x \in \mathrm{WO}_2$ if $x = \langle a, y\rangle$ where $a \in \mathrm{WO}$ and T_y is a well-ordering. Since T is Δ^1_1 in the codes, there is a $\mathbf{\Sigma}^1_1$ relation E such that for all $b \in \mathrm{WO}$:

$$E(y, b, x_1, x_2) \leftrightarrow [x_1, x_2 \in \mathrm{WO} \wedge |x_1|, |x_2| < |b| \wedge (|x_2|\ T_y\ |x_1|).$$

Let W_1 be a tree on ω^5 projecting to E. Let W_2 be a homogeneous tree on $\omega \times \omega_1$ for WO. We may assume that for all $(s, (\alpha_0, \ldots, \alpha_{n-1}) \in W_2$ that $\alpha_0 > \max\{\alpha_1, \ldots, \alpha_{n-1}\}$. Define the tree V on $\omega^2 \times \omega_1 \times \omega^2$ as follows. Let $(p, s, \vec{\alpha}, v, w) \in V$ iff $(s, \vec{\alpha}) \in W_2$ and (p, s, v, w) satisfies: for all $i < \mathrm{lh}(p) - 1$, if j is maximal so that $\langle i, j\rangle$, $\langle i+1, j\rangle < \mathrm{lh}(p)$, then

$$(p{\restriction}j, s{\restriction}j, (v_i(0), \ldots, v_i(j)), (v_{i+1}(0), \ldots, v_{i+1}(j)), (w_i(0), \ldots, w_i(j))) \in W_1,$$

where $v_i(k) = v(\langle i, k\rangle)$, and similarly for w. This last requirement is just building the Kunen-Martin tree for the relation E. In particular, V_y is well-founded iff for all $a \in \mathrm{WO}$, $T_y{\restriction}|a|$ is a well-order, that is, iff T_y is a well-order. Also, if T_y is well-founded, then for all $\beta < \omega_1$ the rank of $T_y{\restriction}\beta$ is less than or equal to the rank of $V^\beta_y \doteq$ those $(p, s, \vec{\alpha}, v, w) \in V_y$ with $\alpha_0 \leq \beta$ (as in the proof of the Kunen-Martin theorem).

Let W_3 be the homogeneous tree on $\omega \times \omega_\omega$ constructed from V, so $p[W_3] = \{y \colon T_y \text{ is well-founded }\}$. For each $p \in \omega^n$, the homogeneity measure μ_p will be of the form $M^{\mathcal{R}_p}$ for some type-1 tree of uniform cofinalities $\mathcal{R}_p$. Thus, $(p, \vec{\beta}) \in W_3$ iff there is an f of type $\mathcal{R}_p$ such that for all $j < n$, $\beta_j = [f^{\langle p_1, i_1, \ldots, p_k, i_k\rangle}]_{W^k_1}$, where $\langle p_1, i_1, \ldots, p_k, i_k\rangle$ is the j^{th} element of $\mathrm{dom}(\mathcal{R}_p)$ in some enumeration. We may assume that $\beta_0 = [f^{\langle p_1, i_1\rangle}]_{W^1_1}$, where i_1 is maximal so that $\langle p_1, i_1\rangle \in \mathrm{dom}(\mathcal{R}_p)$.

Finally, define U to be the tree which is the "conjunction" of W_2 and W_3:

$$((y(0), a(0), \ldots, y(n-1), a(n-1)), (\beta_0, \alpha_0, \ldots, \beta_{n-1}, \alpha_{n-1})) \in U \leftrightarrow (a{\restriction}n, (\alpha_0, \ldots, \alpha_{n-1})) \in W_2 \wedge (y{\restriction}n, (\beta_0, \ldots, \beta_{n-1})) \in W_3.$$

It is easy to see that U is homogeneous with measures of the form $W^\pi_1 \times M^{\mathcal{R}}$. Clearly $p[U] = \mathrm{WO}_2$. To verify $(\star)$, suppose $x = \langle a, y\rangle = (y(0), a(0), y(1), a(1), \ldots) \in \mathrm{WO}_2$. By definition $|x| \leq |T_y|$. It is enough to show that if $(y, \vec{\beta}) \in [W_3]$, then $\beta_0 \geq |T_y|$. For each n, let f_n be of type $\mathcal{R}_{y{\restriction}n}$ with $[f_n] = (\beta_0, \ldots, \beta_{n-1})$. For $C \subseteq \omega_1$, let $V_y{\restriction}C$ denote those $(s, \vec{\alpha}, v, w) \in V_y$ with all $\alpha_j \in C$. From the f_n we get a function f and a c.u.b. $C \subseteq \omega_1$ such that:

1. f is order-preserving from the Kleene-Brouwer ordering on $V_y \restriction C$ to ON.

2. For each n, f induces by restriction a function f'_n of type $\mathcal{R}_{y\restriction n}$ with $[f'_n] = [f_n] = (\beta_0, \dots, \beta_{n-1})$.

Thus, for all $\alpha \in C$ we have $f'_0(\alpha) \geq |V_y^\alpha \restriction C|$. However, for α a closure point of C (*i.e.*, α is the α^{th} element of C), we easily have $|V_y^\alpha \restriction C| = |V_y^\alpha|$. Hence, $\beta_0 = [f'_0]_{W_1^1} \geq [\alpha \to |V_y^\alpha|]_{W_1^1} \geq [\alpha \to |T_y \restriction \alpha|]_{W_1^1} = |T_y|$. ⊣

The above homogeneous tree construction, without the $(\star)$ argument, also shows that every $\boldsymbol{\Pi}^1_2$ set is the projection of a homogeneous tree on $\omega \times \omega_\omega$. This also shows that every $\boldsymbol{\Sigma}^1_3$ set is weakly homogeneous. Using the weak partition on $\boldsymbol{\delta}^1_3$ and the homogeneous tree construction again, one can then show that every $\boldsymbol{\Pi}^1_3$ set admits a homogeneous tree T on $\omega \times \boldsymbol{\delta}^1_3$. We wish, however, to modify this construction to obtain a homogeneous tree on a $\boldsymbol{\Pi}^1_3$-complete set with some additional properties. Our argument is really just Martin's analysis of functions with respect to the normal measures. The form we present it in here may be of use elsewhere. We sketch another version in theorem 5.6.

5.3 Theorem *There is a $\boldsymbol{\Pi}^1_3$ complete set P, a $\boldsymbol{\Pi}^1_3$-norm $z \to |z| < \boldsymbol{\delta}^1_3$ from P onto $\boldsymbol{\delta}^1_3$, and a homogeneous tree S on $\omega \times \boldsymbol{\delta}^1_3$ for P satisfying the following. There is a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$ there is a $z \in P$ with $|z| = \alpha$ and with $S_z \restriction (\sup_\nu j_\nu(\alpha))$ ill-founded, the supremum ranging over measures $\nu = M^{\mathcal{R}_s}$ occurring in the homogeneous tree U of lemma 5.2. Furthermore, for any z and $\vec{\beta} = (\beta_0, \beta_1, \dots)$, if $(z, \vec{\beta}) \in [S]$, then $|z| \leq \beta_0$.*

Proof. Recall according to theorem 4.33 our $\boldsymbol{\Delta}^1_3$ coding $z \to A_z$ of subsets of ω_ω (or $(\omega_\omega)^2$, etc.). We assume for this proof that for all z, $A_z \subseteq \omega_2 \times \omega_\omega \times \omega_\omega$. If $\alpha < \omega_2$, let $A_z^\alpha = \{(\beta, \gamma) \colon (\alpha, \beta, \gamma) \in A_z\}$. Define

$$P(z) \leftrightarrow \forall \alpha < \omega_2 \ A_z^\alpha \text{ is well-founded.}$$

Note that the relation

$$C(z, x_1, x_2, x_3) \leftrightarrow x_1 \in \mathrm{WO}_2 \wedge x_2, x_3 \in \mathrm{WO}_\omega \wedge (|x_1|, |x_2|, |x_3|) \in A_z$$

is $\boldsymbol{\Delta}^1_3$ by the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ unions (in fact, it is straightforward to show that C is Δ^1_3). Thus, $P \in \boldsymbol{\Pi}^1_3$. For $z \in P$, let $|z| = \sup_{\alpha < \omega_2} |A_z^\alpha|$. Using the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ unions and intersections, it is straightforward to check that this defines a $\boldsymbol{\Pi}^1_3$-norm onto $\boldsymbol{\delta}^1_3$.

Let $C^* \subseteq (\omega^\omega)^5$ be $\boldsymbol{\Pi}^1_2$ projecting to C. Let T^*_2 be a homogeneous tree on $\omega^5 \times \omega_\omega$ for C^*. Let U be a homogeneous tree on $\omega \times \omega_\omega$ for WO_2 satisfying

$(\star)$. Define a tree V on $\omega^2 \times \omega_\omega \times \omega \times \omega_\omega$ as follows. Set $(p, s, \vec{\alpha}, u, \vec{\beta}) \in V$ iff $(s, \vec{\alpha}) \in U$ and if $u = (u(0), \dots, u(n-1))$, then the $u(i)$ code more and more of reals $u_0, u_1, \dots$ and reals $w_0, w_1, \dots$. Each $u(j+1)$ extends the array coded by $u(j)$ by adding one extra pair $(u_i(k), u_{i+1}(k))$ for some i, k, and one extra value $w_i(k)$. For each i, k, if the pairs $(u_i(0), u_{i+1}(0))$, $\dots, (u_i(k-1), u_{i+1}(k-1))$ are added at stages $q_0, \dots, q_{k-1} < n$, then we require that

$$(p{\restriction}k, s{\restriction}k, u_i{\restriction}k, u_{i+1}{\restriction}k, w_i{\restriction}k, (\beta_{q_0}, \dots, \beta_{q_{k-1}})) \in T_2^*.$$

Again, V embodies the Kunen-Martin construction for the relation C. The tree V is also homogeneous, witnessed by measures $M_{p,s,u} = \nu_s \times \mu_{p,s,u}$ where ν_s are the homogeneity measures for U and $\mu_{p,s,u}$ are measures of the form $M^{\mathcal{R}_{p,s,u}}$ for some type-1 trees of uniform cofinalities $\mathcal{R}_{p,s,u}$ which are simply obtained from the type-1 trees giving the measures for T_2^* [Note that for any two type-1 trees $\mathcal{R}_1$ and $\mathcal{R}_2$, there is a type-1 tree $\mathcal{R}$ whose measure $M^{\mathcal{R}}$ projects naturally to both $M^{\mathcal{R}_1}$ and $M^{\mathcal{R}_2}$].

We now apply the homogeneous tree construction to V to produce a tree S on $\omega \times \boldsymbol{\delta}^1_3$ such that for all z, S_z is ill-founded iff V_z is well-founded. For any $p \in \omega^{<\omega}$, let $\prec_{V_p}$ denote the Kleene-Brouwer ordering on V_p. In specifying the Kleene-Brouwer order, we must say how the tuples $(s(n), \alpha_n, u(n), \beta_n)$ are ordered. In comparing two such tuples, it is important (at least for $n = 0$) that we order first by α_n (the remaining order is unimportant). We define $(p, \vec{\gamma}) \in S$ iff there exists an f which is order-preserving and of the correct type from V_p with the Kleene-Brouwer order to $\boldsymbol{\delta}^1_3$, and f represents $\vec{\gamma}$ in the following sense. First, we require $\gamma_0 = \sup(f)$. Let (s_i, u_i), $i \geq 1$, enumerate the pairs of finite sequences of the same length, with $\mathrm{lh}(s_i) \leq i$. Then γ_i, for $i \geq 1$, is represented with respect to $M_{p\restriction \mathrm{lh}(s_i), s_i, u_i}$ by the function $f^{s_i,u_i}(\vec{\alpha}, \vec{\beta}) = f(s_i, \vec{\alpha}, u_i, \vec{\beta})$. The weak partition relation on $\boldsymbol{\delta}^1_3$ shows that S is homogeneous (though we don't need this for the proof).

If $z \in P$, and hence V_z is well-founded, then easily S_z is ill-founded and in fact there is a $\vec{\gamma}$ with $(z, \vec{\gamma}) \in [S]$ such that $\gamma_0 \leq \omega \cdot |\prec_{V_z}|$, where $\prec_{V_z}$ is the Kleene-Brouwer ordering on V_z. That S_z being ill-founded implies $z \in P$ will be shown below.

We define now the c.u.b. set $C \subseteq \boldsymbol{\delta}^1_3$ as required. Fix for the moment $\alpha < \omega_2$, $\beta < \boldsymbol{\delta}^1_3$. Let

$$P_{\alpha,\beta} = \{z \colon \forall \alpha' \leq \alpha \ A_z^{\alpha'} \text{ is well-founded of rank } \leq \beta\}.$$

A standard computation, using the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ unions and intersections shows $P_{\alpha,\beta} \in \boldsymbol{\Delta}^1_3$. Note that if $z \in P_{\alpha,\beta}$ then V_z restricted to tuples $(s, \vec{\alpha}, u, \vec{\beta})$ such that $\alpha_0 \leq \alpha$ is well-founded; this uses property $(\star)$ of the tree U. Since $P_{\alpha,\beta} \in \boldsymbol{\Sigma}^1_3$ and is thus ω_ω-Suslin, an easy tree argument shows that $b(\alpha, \beta) \doteq \sup\{|\alpha|_{V_z} \colon z \in P_{\alpha,\beta}\} < \boldsymbol{\delta}^1_3$, where by $|\alpha|_{V_z}$

we mean the supremum of the ranks of $(s(0), \alpha, u(0), \gamma_0)$ in the Kleene-Brouwer ordering of V_z. This defines the function $b\colon \omega_2 \times \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$. Let then $C \subseteq \boldsymbol{\delta}^1_3$ be a c.u.b. set consisting of limit ordinals and closed under b.

Suppose now $\delta \in C$ and $\operatorname{cof}(\delta) = \omega_2$. Let $h\colon \omega_2 \to \delta$ be increasing and cofinal. Let $z \in \omega^\omega$ be such that $\forall \alpha < \omega_2\ A^\alpha_z$ is well-founded of rank $h(\alpha)$. Thus $P(z)$, and for all $\alpha < \omega_2$, $z \in P_{\alpha,h(\alpha)}$. For all $\alpha < \omega_2$ we have $|\alpha|_{V_z} \leq b(\alpha, h(\alpha)) < \delta$. Hence there is an order-preserving map f from the Kleene-Brouwer ordering of V_z to δ. If f represents $\vec{\gamma}$, then $\gamma_0 \leq \delta$ and for all n, $\gamma_n \leq \sup\limits_{p,s,u} j_{M_{p,s,u}}(\delta)$.

Suppose now $(z, (\beta_0, \beta_1, \ldots)) \in [S]$. We show that $z \in P$, and if we let $h(\alpha) = |A^\alpha_z|$ for $\alpha < \omega_2$, then $\sup(h) \leq \beta_0$. For each j, let $f_j\colon V_{z\restriction j} \to \boldsymbol{\delta}^1_3$ be order-preserving representing $(\beta_1, \ldots, \beta_j)$, and with $\beta_0 = \sup(f_j)$. Recall that for $i \leq j$ and $i > 0$, β_i is represented with respect to $M_{z\restriction \mathrm{lh}(s_i), s_i, u_i} = \nu_{s_i} \times \mu_{z\restriction \mathrm{lh}(s_i), s_i, u_i}$ by the function f^{s_i,u_i}_j. For each $i > 0$, let E_i be a $M_{z\restriction \mathrm{lh}(s_i), s_i, u_i}$ measure one set such that for all $j_1, j_2 \geq i$ and all $(\vec{\alpha}, \vec{\beta}) \in E_i$ we have $f^{s_i,u_i}_{j_1}(\vec{\alpha}, \vec{\beta}) = f^{s_i,u_i}_{j_2}(\vec{\alpha}, \vec{\beta})$. For $(\vec{\alpha}, \vec{\beta}) \in E_i$, let $f^{s_i,u_i}(\vec{\alpha}, \vec{\beta})$ denote the common value of $f^{s_i,u_i}_j(\vec{\alpha}, \vec{\beta})$ for $j \geq i$. Let A_i be a ν_{s_i} measure one set such that for all $\vec{\alpha} \in A_i$ we have that for $\mu_{z\restriction \mathrm{lh}(s_i), s_i, u_i}$ almost all $\vec{\beta}$ that $(\vec{\alpha}, \vec{\beta}) \in E_i$. Consider now $\alpha < \omega_2$, and fix $x \in \mathrm{WO}_2$ with $|x| = \alpha$. By homogeneity of U, fix $\vec{\alpha} = (\alpha_0, \alpha_1, \ldots)$ such that $(x, \vec{\alpha}) \in [U]$ and for all k, $\vec{\alpha}\restriction k \in \bigcap_i A_i$, where the intersection runs over the i such that $s_i = x\restriction k$.

Consider the tree

$$V_{z,x,\vec{\alpha}} \cap \vec{B} = \{(u, \vec{\beta})\colon (z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), \vec{\alpha}\restriction \mathrm{lh}(u), u, \vec{\beta}) \in V \\ \wedge\ \vec{\beta} \in B_{z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), u}\}$$

where $B_{z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), u}$ is a $\mu_{z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), u}$ measure one set such that $\{\vec{\alpha}\restriction \mathrm{lh}(u)\} \times B_{z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), u} \subseteq E_i$ where i is such that $(s_i, u_i) = (x\restriction \mathrm{lh}(u), u)$. The map $(u, \vec{\beta}) \to f^{x\restriction \mathrm{lh}(u), u}(\vec{\alpha}\restriction \mathrm{lh}(u), \vec{\beta})$ is order-preserving from the Kleene-Brouwer order of $V_{z,x,\vec{\alpha}} \cap \vec{B}$ to β_0. In particular, the tree $V_{z,x,\vec{\alpha}} \cap \vec{B}$ has rank at most β_0. On the other hand, the proof of the Kunen-Martin theorem shows that the tree of finite sequences $(y_0, \ldots, y_n)$ of reals such that for all $i < n$, $C(z, x, y_{i+1}, y_i)$, embeds into $V_{z,x,\vec{\alpha}} \cap \vec{B}$ (we use here the homogeneity of T^*_2, which allows the "witness sequences" from the Kunen-Martin proof to be chosen in the B sets). Thus, A^α_z is well-founded of rank at most $|V_{z,x,\vec{\alpha}} \cap \vec{B}| \leq \beta_0$. Since $\alpha < \omega_2$ was arbitrary, we have $\sup(h) \leq \beta_0$.

We have proved the theorem for points of cofinality ω_2. A similar (but slightly easier) construction works for points of cofinality ω_1, using WO in place of WO_2. The cofinality ω case, we already observed, is the Kunen result. ⊣

5.4 Remark *If $\{\phi_n\}_{n\in\omega}$ is the semi-scale on P corresponding to S (using left-most branches), then one can show that $\{\phi_n\}$ is a (not necessarily regular) $\mathbf{\Pi}^1_3$-scale.*

From theorem 5.3, Martin's theorem now follows quickly.

5.5 Theorem (Martin) *There is a tree T on $\omega \times \boldsymbol{\delta}^1_3$ such that for all $f\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ there is a $z \in \omega^\omega$ with T_z well-founded, and a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$, $f(\alpha) < |T_z \restriction \sup_\nu j_\nu(\alpha)|$, the supremum ranging over measures ν occurring in the homogeneous trees on $\mathbf{\Pi}^1_1$, $\mathbf{\Pi}^1_2$-complete sets.*

Proof. The argument is now almost identical to the Kunen case. Fix P, S as in theorem 5.3. Let W be a tree on $\omega \times \omega_\omega$ such that $p[W]$ is $\mathbf{\Sigma}^1_3$-complete, and thus $\sup\{|W_w|\colon W_w \text{ is well-founded }\} = \boldsymbol{\delta}^1_3$. Let U be the tree on $\omega^2 \times \boldsymbol{\delta}^1_3 \times \omega \times \omega_\omega$ defined by:

$$(p, s, \vec{\alpha}, u, \vec{\beta}) \in U \leftrightarrow (s, \vec{\alpha}) \in S \wedge (u, \vec{\beta}) \in W$$
$$\wedge\ \exists \sigma(\sigma \text{ extends } p \wedge \sigma(s) = u).$$

If $f\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$, then play the game where I plays out z, II plays out y, and II wins iff $(z \in P) \to (W_y$ is well-founded and $|W_y| > f(|z|))$. II wins by boundedness (a $\mathbf{\Sigma}^1_1$ subset of P codes a bounded below $\boldsymbol{\delta}^1_3$ set of ordinals). If σ is a winning strategy for II, and C is as in theorem 5.3, then for all $\alpha \in C$, $|U_\sigma \restriction \sup_\nu j_\nu(\alpha)| > f(\alpha)$. Weaving the second, third, fourth, and fifth coordinates of U into a single coordinate produces a tree T as desired. ⊣

We refer to the tree T of theorem 5.5 as the *Martin tree*.

In [11] a version of Martin's theorem is presented which does not get the extra information about the homogeneous scale, but which refines slightly the inequality. Actually, the refined inequality can also be obtained by examining the proof of theorem 5.3. Nevertheless, this second variation of the proof has, we feel, enough advantages to warrant presenting. Specifically, we will use this second variation of the proof in theorem 5.7.

5.6 Theorem (Martin) *There is a tree T on $\omega \times \boldsymbol{\delta}^1_3$ such that for all $f\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$, there is a z with T_z well-founded and a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$, $f(\alpha) < |T_z \restriction \sup_\nu j_\nu(\alpha)|$, where if $\mathrm{cof}(\alpha) = \omega_1$ the supremum ranges over the measures W^m_1 occurring in the homogeneous tree on a $\mathbf{\Pi}^1_1$-complete set, and if $\mathrm{cof}(\alpha) = \omega_2$, the supremum ranges over the measures occurring in the homogeneous tree on a $\mathbf{\Pi}^1_2$-complete set (if $\mathrm{cof}(\alpha) = \omega$, we use $|T_z \restriction \alpha|$).*

Proof. The proof of this version is similar to that of the version presented above, so we give a sketch. Let P, U be as in theorem 5.3, and let ν_s be the homogeneity measures for U. Define (where A^α_z is as in the proof of theorem 5.3)

$$P'(z,x) \leftrightarrow x \in \mathrm{WO}_2 \ \wedge \ A_z^{|x|} \text{ is well-founded} .$$

Thus $P' \in \boldsymbol{\Pi}^1_3$ and $P(z) \leftrightarrow \forall x\ [x \in \mathrm{WO}_2 \rightarrow P'(z,x)]$. Let V be a homogeneous tree on $\omega \times \omega \times \boldsymbol{\delta}^1_3$ for P' with $\boldsymbol{\delta}^1_3$-complete measures $\mu_{s,t}$ (which the usual homogeneous tree construction gives). Define now the tree W on $\omega \times \boldsymbol{\delta}^1_3$ as follows. Let $\{s_n\}_{n\in\omega}$ be an enumeration of $\omega^{<\omega}$ with any sequence preceding its extensions. Define $((z(0),\dots,z(n-1)),(\beta_0,\dots,\beta_{n-1})) \in W$ iff for all $j<n$, if $s_j = (s(0),\dots,s(k-1))$ and $q_0 < q_1 < \cdots < q_{k-1} = j$ are such that $s_{q_l} = (s(0),\dots,s(l))$, and $f_0,\dots,f_{k-1}$ represent $\beta_{q_0},\dots,\beta_{q_{k-1}}$ with respect to $\nu_{s(0)},\dots,\nu_{(s(0),\dots,s(k-1))}$, then

$$\forall^*_{\nu_{s_j}} \gamma_0,\dots,\gamma_{k-1}\ (z{\restriction}k, s_j, (f_0(\gamma_0),\dots,f_{k-1}(\gamma_0,\dots,\gamma_{k-1}))) \in V.$$

We claim first that $P = p[W]$. $p[W] \subseteq P$ is easily checked, and uses only the homogeneity of U. If $z \in P$, consider the ordinal game G_z where I plays out $x \in \omega^\omega$, $\vec{\alpha} = (\alpha_0,\alpha_1,\dots) \in (\omega_\omega)^\omega$ with $\alpha_0 < \omega_2$, II plays out $\vec{\delta} \in (\boldsymbol{\delta}^1_3)^\omega$, and II wins iff $\forall n\ [(x{\restriction}n, \vec{\alpha}{\restriction}n) \in U \rightarrow (z{\restriction}n, x{\restriction}n, \vec{\delta}{\restriction}n) \in V]$. The game is closed for II, hence determined, and the usual homogeneity argument, using the $\boldsymbol{\delta}^1_3$ completeness of the measures for V, shows that I cannot have a winning strategy [otherwise, by $\boldsymbol{\delta}^1_3$-completeness of the measures $\mu_{s,t}$, we could stabilize I's moves on measure one sets, and then by the homogeneity of those measures, get a play for II which wins]. A winning strategy τ for II then gives functions f_s, and thus ordinals β_i, such that $(z,\vec{\beta}) \in [W]$. Namely, if $\mathrm{lh}(s_i) = k$, then β_i is represented with respect to ν_{s_i} by the function $f_{s_i}(\gamma_0,\dots,\gamma_{k-1}) = \tau(s_i,(\gamma_0,\dots,\gamma_{k-1}))$.

Fix for the moment $\alpha < \omega_2$, $\beta < \boldsymbol{\delta}^1_3$. Let $G_{z,\alpha}$ be the game defined just as G_z, except I's first ordinal move α_0 satisfies $\alpha_0 \leq \alpha$. Let $P_{\alpha,\beta}$ be as in theorem 5.3. For $z \in P_{\alpha,\beta}$, let $b(z,\alpha) < \boldsymbol{\delta}^1_3$ be least so that II has a winning strategy in $G_{z,\alpha}$ playing ordinals $< b(z,\alpha)$ (this exists since $\boldsymbol{\delta}^1_3$ is regular). We claim that $b(\alpha,\beta) \doteq \sup\{b(z,\alpha) \colon z \in P_{\alpha,\beta}\} < \boldsymbol{\delta}^1_3$. This follows from the fact that $P_{\alpha,\beta}$ is ω_ω-Suslin. To see this, consider the auxiliary game where I plays out $z, x \in \omega^\omega$, $\vec{\gamma} \in (\omega_\omega)^\omega$, $\vec{\alpha} \in (\omega_\omega)^\omega$ with $\alpha_0 \leq \alpha$, and II plays out $\vec{\delta} \in (\boldsymbol{\delta}^1_3)^\omega$, and II wins iff

$$\forall n\ [((z{\restriction}n, \vec{\gamma}{\restriction}n) \in U_2 \wedge (x{\restriction}n, \vec{\alpha}{\restriction}n) \in U) \rightarrow (z{\restriction}n, x{\restriction}n, \vec{\delta}{\restriction}n) \in V],$$

where U_2 is a tree on $\omega \times \omega_\omega$ with $p[U_2] = P_{\alpha,\beta}$. II again wins, and $b(\alpha,\beta) \leq$ any ordinal η large enough so that II can win this auxiliary game playing ordinals $< \eta$.

Let now $C \subseteq \boldsymbol{\delta}^1_3$ be a c.u.b. set consisting of limit ordinals and closed under b. Let $\delta \in C$ with $\mathrm{cof}(\delta) = \omega_2$. Let $h \colon \omega_2 \rightarrow \delta$ be increasing and cofinal, and $z \in P$ be such that $\forall \alpha < \omega_2\ |A^z_\alpha| = h(\alpha)$. Then we claim that

$z \in p[W{\restriction}\sup_\nu j_\nu(\delta)]$, the supremum ranging over the measures ν_s for U. We must show that II can win G_z playing only ordinals $< \delta$. If I first moves $(x(0), \alpha_0)$, II picks the least ordinal $\eta \leq b(\alpha_0, h(\alpha_0)) < \delta$ such that II can win $G_{z,\alpha}$ starting from that position playing only ordinals $< \eta$. II then follows the canonical winning strategy for the closed game $G_{z,\alpha}{\restriction}\eta$.

Thus, we have produced a tree W and a c.u.b. C such that $P = p[W]$ and for all $\alpha \in C$ of cofinality ω_2 there is a $z \in p[W{\restriction}\sup_{\nu_s} j_{\nu_s}(\alpha)]$ such that $|z| = \alpha$, that is, $\alpha = \sup_{\beta<\omega_2} |A_z^\beta|$. Again, the argument for cofinality ω_1 points is similar, using WO instead of WO_2. The argument now finishes exactly as before (i.e., theorem 5.5). ⊣

Theorems 5.5, 5.6 are the correct extension of the Kunen analysis on ω_1 to $\boldsymbol{\delta}^1_3$. The fact that the identity function of the Kunen analysis is replaced by $\alpha \to \sup_\nu j_\nu(\alpha)$ is, at bottom, the source of the combinatorial complications at the higher levels.

The proofs of theorems 5.3, 5.5, and 5.6 are quite general. For example, the proof of theorem 5.6 generalizes to the following (the proof is identical to that of theorem 5.6 so we omit it).

5.7 Theorem *Let $\lambda < \kappa$ be regular cardinals, and $\boldsymbol{\Gamma}$ a non-selfdual pointclass closed under $\forall^{\omega^\omega}, \wedge, \vee$. Assume:*

1. *There is a $\boldsymbol{\Delta}$ coding of the ordinals less than λ. That is, there is a $\boldsymbol{\Delta}$ set $C \subseteq \omega^\omega$ and a map $x \to |x|_C < \lambda$ for $x \in C$ such that the relations $(x_1, x_2 \in C \wedge |x_1|_C \leq |x_2|_C)$ and $(x_1, x_2 \in C \wedge |x_1|_C < |x_2|_C)$ are both in $\boldsymbol{\Delta}$.*

2. *There is a homogeneous tree U on C such that for all $x \in C$, $|x| \leq \psi_0(x) < \lambda$, where $\{\psi_n\}$ is the semi-scale corresponding to U.*

3. *There is a map $z \to A_z \subseteq \lambda \times \kappa$, for $z \in \omega^\omega$, satisfying:*

   (a) $\forall f\colon \lambda \to \kappa\ \exists z\ A_z = f$.

   (b) The relation $P'(z, x) \leftrightarrow [x \in C \wedge \exists! \beta\ A_z(|x|_C, \beta)]$ is in $\boldsymbol{\Gamma}$.

   (c) For all $\alpha < \lambda$, $\beta < \kappa$, $P_{\alpha,\beta} \doteq \{z\colon \forall \alpha' \leq \alpha\ \exists \beta' \leq \beta\ [A_z(\alpha', \beta') \wedge \forall \beta''(A_z(\alpha', \beta'') \to \beta' = \beta'')]\}$ is in $\boldsymbol{\Delta}$.

4. *Every $\boldsymbol{\Gamma}$ set admits a homogeneous tree on κ with κ-complete measures.*

5. *Every $\boldsymbol{\Delta}$ set is α-Suslin for some $\alpha < \kappa$. Also, if $A \subseteq P \doteq \{z\colon \forall x \in C\ P'(z, x)\}$ is in $\exists^{\omega^\omega}\boldsymbol{\Delta}$, then $\sup\{|z|\colon z \in A\} < \kappa$, where for $z \in P$, $|z|$ is the supremum of the range of the function $A_z\colon \lambda \to \kappa$.*

Then there is a tree W on $\omega \times \kappa$ with $p[W] = P$ and a c.u.b. $D \subseteq \kappa$ such that for all $\alpha \in D$ with $\mathrm{cof}(\alpha) = \lambda$, there is a $z \in P$ with $|z| = \alpha$ and $W_z \restriction (\sup_\nu j_\nu(\alpha))$ is ill-founded, the supremum ranging over the measures ν for the homogeneous tree U.

5.8 Remark *The hypotheses imply that the prewellordering property falls on the $\boldsymbol{\Gamma}$ side. For if not, then $\check{\boldsymbol{\Gamma}}$ is closed under well-ordered unions. From (3c) it follows that there is a κ increasing sequence of $\boldsymbol{\Delta}$ sets. Thus, there is a $\check{\boldsymbol{\Gamma}}$ prewellordering of length κ. If $\boldsymbol{\Delta}$ is not closed under $\exists^{\omega^\omega}$, then by (5) this prewellordering is α-Suslin for some $\alpha < \kappa$, a contradiction. If $\boldsymbol{\Delta}$ is closed under real quantification, then $\kappa = o(\boldsymbol{\Delta})$, and $\boldsymbol{\Delta}$ is closed under $< \kappa$ unions. Then P is a union of $\boldsymbol{\Delta}$ sets, so is in $\check{\boldsymbol{\Gamma}}$, and hence in $\boldsymbol{\Delta}$. This contradicts (5). Thus,* $\mathrm{pwo}(\boldsymbol{\Gamma})$*. Similarly, it can be shown that $\boldsymbol{\Delta}$ is closed under $< \kappa$ length unions and intersections.*

This general form of Martin's theorem is particularly useful when combined with another result of Martin and Steel (c.f. [28]) which provides the existence of homogeneous trees in a general setting. We recall this theorem. Let $\{s_i\}_{i \in \omega}$ be an enumeration of $\omega^{<\omega}$ with each sequence preceding its proper extensions, and such that if $s_i = t^\frown a$, $s_j = t^\frown b$, then $a < b \to i < j$. View each real σ as a strategy for II via $\sigma(s_{i+1}) = \sigma(i)$ (note $s_0 = \emptyset$).

Let $\boldsymbol{\Gamma}$ be a Steel pointclass (i.e., $\boldsymbol{\Gamma}$ is non-selfdual, closed under $\forall^{\omega^\omega}$, $\mathrm{pwo}(\boldsymbol{\Gamma})$, and $\boldsymbol{\Delta}$ is closed under real quantification). By $\mathrm{red}(\boldsymbol{\Gamma})$, let $U, V \subseteq (\omega^\omega)^3$ be disjoint $\boldsymbol{\Gamma}$ sets such that for every disjoint $\boldsymbol{\Gamma}$ sets $A, B \subseteq (\omega^\omega)^2$ there is an $x \in \omega^\omega$ with $A = U_x$, $B = V_x$. We say $\boldsymbol{\Delta}$ is uniformly closed under $\exists^{\omega^\omega}$ if the relations

$$R(x,z) \leftrightarrow \forall z, w\ [U_x(z,w) \vee V_x(z,w)] \wedge \exists w\ U_x(z,w)$$
$$S(x,z) \leftrightarrow \forall z, w\ [U_x(z,w) \vee V_x(z,w)] \wedge \forall w\ U_x(z,w)$$

are in $\boldsymbol{\Gamma}$.

5.9 Theorem (AD) (Martin, Steel) *Let $\boldsymbol{\Gamma}$ be a non-selfdual pointclass, $A \in \boldsymbol{\Gamma} - \check{\boldsymbol{\Gamma}}$, and assume A and A^c are both Suslin. Let $B = \{\sigma \colon \forall y\ \sigma(y) \in A\}$. Then B is $\forall^{\omega^\omega}\boldsymbol{\Gamma}$-complete and B admits a scale $\{\psi_n\}$ whose corresponding tree T_ψ is homogeneous. If $\{\phi_n\}$ is a $\boldsymbol{\Gamma}$ very good scale on A and either $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$ or $\boldsymbol{\Delta}$ is uniformly closed under $\exists^{\omega^\omega}$, then $\{\psi_n\}$ is a $\forall^{\omega^\omega}\boldsymbol{\Gamma}$ scale. If $\boldsymbol{\Gamma}$ is closed under $\forall^\omega$, countable unions and intersections, then the measures in T_ψ will be κ-complete, where $\kappa =$ the supremum of the lengths of the $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ prewellorderings of the reals.*

Combining this with theorem 5.7 we have the following.

5.10 Theorem $(\mathrm{AD} + V = L(\mathbb{R}))$ *Let $\kappa < \boldsymbol{\delta}^2_1$ be a regular limit Suslin cardinal, and $\lambda < \kappa$ be regular. Let $\boldsymbol{\Gamma}$ be the pointclass closed under $\forall^{\omega^\omega}$ such* *that $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$. Assume there is a $\mathbf{\Delta}$ coding of λ with homogeneous tree U as in (1), (2) of theorem 5.7. Then there is a tree W on $\omega \times \kappa$ with $p[W] = P$, a $\mathbf{\Gamma}$-complete set, and a map $z \to |z| < \kappa$ for $z \in P$ satisfying:*

1. *If $S \subseteq P$ is in $\mathbf{\Delta}$, then* $\sup\{|z| : z \in S\} < \kappa$.
2. *There is a c.u.b. $C \subseteq \kappa$ such that for all $\alpha \in C$ of cofinality λ, there is a $z \in P$ such that $|z| = \alpha$ and $W_z \restriction (\sup_\nu j_\nu(\alpha))$ is ill-founded, the supremum ranging over the homogeneity measures ν for U.*

Proof(sketch). $\mathbf{\Gamma}$ is closed under countable unions and intersections from [35], and also from [37] $\mathbf{\Gamma}$ has the scale property. The proof of theorem 3.3 of [35] also shows that $\mathbf{\Delta}$ is uniformly closed under $\exists^{\omega^\omega}$. Note also that κ is the supremum of the lengths of the $\mathbf{\Delta}$ prewellorderings, and $\mathbf{\Delta}$ is closed under $< \kappa$ unions and intersections. By theorem 5.9, some $\mathbf{\Gamma}$-complete set admits a homogeneous tree with κ-complete measures, and thus so does every $\mathbf{\Gamma}$ set. The coding $z \to A_z \subseteq \lambda \times \kappa$ is given simply from the coding lemma (relative to some $\mathbf{\Delta}$ prewellordering of length λ and some $\mathbf{\Gamma}$-norm on a $\mathbf{\Gamma}$-complete set). We define P, W as in theorem 5.6, using a homogeneous tree V with κ-complete measures for P'. (1)-(5) of theorem 5.7 are easily checked. ⊣

We note that if λ is a regular Suslin cardinal, the hypotheses of the previous theorem are automatically satisfied. For there is a non-selfdual pointclass $\mathbf{\Lambda}$ closed under $\forall^{\omega^\omega}$ with the scale property such that λ is the supremum of the lengths of the $\mathbf{\Delta} = \mathbf{\Delta}(\mathbf{\Lambda})$ prewellorderings. If λ is inaccessible, $\mathbf{\Delta}$ is uniformly closed under $\exists^{\omega^\omega}$. If λ is a successor Suslin, then from theorem 4.3 of [37], $\mathbf{\Lambda} = \forall^{\omega^\omega}\mathbf{\Lambda}^-$ with $\mathbf{\Lambda}^-$ closed under $\exists^{\omega^\omega}$ with scale($\mathbf{\Lambda}^-$). In either case, by theorem, 5.9, there is a $\mathbf{\Lambda}$-scale $\{\psi_n\}$ on a $\mathbf{\Lambda}$-complete set C whose tree U is homogeneous. We must have ψ_0 onto λ in this case (assuming without loss of generality that the scale is regular, i.e., all norms map onto an initial segment on the ordinals). Letting our coding of λ be given by $|x| = \psi_0(x)$ for $x \in C$, this verifies (1), (2) of theorem 5.7.

We make some comments on the possible significance of theorem 5.10 to extending the inductive analysis of the projective sets to higher levels of $L(\mathbb{R})$. In analyzing the measures on κ, the first step of the proof of theorem 4.8 shows it is necessary to have at hand an analysis of the functions on κ with respect to the semi-normal measures (the measures that give every c.u.b. set measure one). At successor Suslin cardinals, theorem 5.7 and the analysis below κ should give the analog of theorem 5.6. If κ is singular, there should be no analog of Martin's theorem required, though other methods become necessary (we will discuss some of these in the next section). For κ inaccessible Suslin, theorem 5.10 is a step towards providing the necessary result, but is not complete as it handles only the normal measures on κ corresponding to fixed cofinalities $\lambda < \kappa$.

5.2. Some Canonical Measures

For the remainder of this section we return to the projective hierarchy, and discuss the other main ingredient; the theory of descriptions. As we said in the introduction, our purpose here is not to present complete details, but rather to exposit the main ideas. We will frequently illustrate a proof by considering a case which shows the central idea.

According to theorem 5.6, in analyzing functions $f\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to the ω_1 cofinal normal measure, we need to consider ultrapowers by the measures W^m_1. There is nothing more to say in this case. With respect to the ω_2-cofinal normal measure, we need consider ultrapowers by the measures $M^{\mathcal{R}}$ occurring in the homogeneous tree on a $\boldsymbol{\Pi}^1_2$ set. One suspects that there is a combinatorially simpler family which "dominates" these measures. Indeed, it simplifies considerably the resulting theory to have such a family at hand.

Let $<_m$ be the ordering on $(\omega_1)^m$ corresponding to the permutation $\pi = (m, 1, 2, \dots, m-1)$. Thus,

$$(\alpha_1, \dots, \alpha_m) <_m (\beta_1, \dots, \beta_m) \leftrightarrow \\ (\alpha_m, \alpha_1, \dots, \alpha_{m-1}) <_{lex} (\beta_m, \beta_1, \dots, \beta_{m-1}).$$

Recall that a function f from the domain of a well-ordering $\prec$ to the ordinals is of the correct type if it order-preserving with respect to $\prec$, of uniform cofinality ω, and everywhere discontinuous.

5.11 Definition *S^m_1 is the measure on ω_{m+1} induced by the strong partition relation on ω_1 and functions $h\colon \mathrm{dom}(<_m) \to \omega_1$ of the correct type. That is, A has measure one if there is a c.u.b. $C \subseteq \omega_1$ such that $[f]_{W^m_1} \in A$ for all $f : \mathrm{dom}(<_m) \to C$ of the correct type.*

Thus, S^1_1 is the ω cofinal normal measure on ω_2. We let W_1 denote the collection of measures W^m_1, and S_1 the collection S^m_1.

According to the next theorem, we need only consider ultrapowers by the measures S^m_1 in theorem 5.6.

5.12 Theorem *There is a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\theta \in C$ of cofinality ω_2, $\sup_\nu j_\nu(\theta) = \sup_m j_{S^m_1}(\theta)$, the first supremum ranging over measures ν of the form $M^{\mathcal{R}}$ for some type-1 tree of uniform cofinalities $\mathcal{R}$.*

We let C be the set of θ closed under the embeddings j_ν, for $\nu = M^{\mathcal{R}}$ (recall theorem 4.35). The theorem follows easily from the following lemma:

5.13 Lemma *Let $\nu = M^{\mathcal{R}}$. There is an $m \in \omega$, a measure μ on ω_ω, and a function $h\colon \omega_{m+1} \to (\omega_\omega)^{<\omega}$ satisfying the following:*

1. *$\forall^*_{S_1^m}\alpha\ \exists\vec{\gamma}\in\mathrm{dom}(\nu)\ \forall^*_\mu\beta\ h(\alpha)(\beta)<\vec{\gamma}$. By $h(\alpha)(\beta)<\vec{\gamma}$ we mean that if $\vec{\gamma}$ is represented by $f\colon \mathrm{dom}(<^{\mathcal{R}})\to\omega_1$ of type $\mathcal{R}$, and $h(\alpha)(\beta)$ by $g\colon \mathrm{dom}(<^{\mathcal{R}})\to\omega_1$ of type $\mathcal{R}$, then $[g^{\langle p_1,i_1\rangle}]_{W_1^1}<[f^{\langle p_1,i_1\rangle}]_{W_1^1}$ for all $\langle p_1,i_1\rangle\in\mathrm{dom}\,\mathcal{R}$.*

2. *If $A\subseteq\mathrm{dom}(\nu)$ has ν measure one, then $\forall^*_{S_1^m}\alpha\ \forall^*_\mu\beta\ h(\alpha)(\beta)\in A$.*

To see this proves the theorem, fix $\delta=[F]_\nu<j_\nu(\theta)$, where $\theta\in C$. Define ϵ by: $\forall^*_{S_1^m}\alpha\ \forall^*_\mu\beta\ \epsilon(\alpha)(\beta)=F(h(\alpha)(\beta))$. From (2) this depends only on $[F]_\nu$, and from (1) it follows easily that $\epsilon<j_{S_1^m}(\theta)$. [We use here the fact that if $F\colon\mathrm{dom}(\nu)\to\mathrm{ON}$, then there is a ν measure one set A such that if $f,g\colon\mathrm{dom}(<^{\mathcal{R}})\to\omega_1$ of type $\mathcal{R}$ represent $[f],[g]\in A$ and $[g^{\langle p_1,i_1\rangle}]<[f^{\langle p_1,i_1\rangle}]$ for all $\langle p_1,i_1\rangle\in\mathrm{dom}(\mathcal{R})$, then $F([g])\le F([f])$. This follows by an easy partition argument.] From (2), the map $\pi(\delta)=\epsilon$ is an embedding of $j_\nu(\theta)$ into $j_{S_1^m}(\theta)$.

The lemma is proved by a direct construction of the measure μ. We illustrate with a case. Suppose $<^{\mathcal{R}}$ is lexicographic ordering on tuples $\langle\gamma_1,i_1,\gamma_2,\gamma_3\rangle$ where $\gamma_3<\gamma_2<\gamma_1<\omega_1$, and $i_1\in\{0,1\}$ (we have removed irrelevant indices from our notation now). Also, ν is induced by functions $f\colon\mathrm{dom}(<^{\mathcal{R}})\to\omega_1$ of the correct type (ν is actually the two-fold product of the measure on ω_4 corresponding to the permutation $(3,2,1)$). In this case, take $m=4$ and $\mu=W_1^2\times S_1^2$. We define the function h as required. Let $\alpha<\omega_5$ be represented by $f_\alpha\colon\mathrm{dom}(<_4)\to\omega_1$ of the correct type. Define $h(\alpha)$ so that for almost all $\vec{\beta}=(\beta_1,\beta_2,\beta_3)\in\mathrm{dom}\,\mu$, so $\beta_1<\beta_2<\omega_1$ and $\beta_3<\omega_3$ is represented by $f_{\beta_3}\colon\mathrm{dom}(<_2)\to\omega_1$ of the correct type, $h(\alpha)(\vec{\beta})\in\mathrm{dom}(\nu)$ is represented by $h(\alpha)(\vec{\beta})(\langle\gamma_1,i_1,\gamma_2,\gamma_3\rangle)=f_\alpha(\beta_{i_1},\gamma_2,f_{\beta_3}(\gamma_3,\gamma_2),\gamma_1)$. It is easy to check that this is well-defined and satisfies (1), (2). □

We state one more embedding theorem which helps to simplify the analysis. By theorem 2.14, $\boldsymbol{\delta}^1_5=\lambda_5^+$, where λ_5 is least such that every $\boldsymbol{\Pi}^1_4$ set is λ_5-Suslin. The weak partition relation on $\boldsymbol{\delta}^1_3$ and the homogeneous tree construction shows every $\boldsymbol{\Pi}^1_3$ set admits a homogeneous tree with measures ν_s on $\boldsymbol{\delta}^1_3$. Thus, every $\boldsymbol{\Sigma}^1_4$ set is weakly homogeneous with the same family of measures. The homogeneous tree construction again shows every $\boldsymbol{\Pi}^1_4$ set is λ-Suslin, where $\lambda=\sup_\nu j_\nu(\boldsymbol{\delta}^1_3)$, ν ranging over the measures in a homogeneous tree on a $\boldsymbol{\Pi}^1_3$-complete set (granting the strong partition relation on $\boldsymbol{\delta}^1_3$, the resulting tree on $\omega\times\lambda$ is also homogeneous). Thus, $\lambda_5\le\lambda$. Computing the upper bound for $\boldsymbol{\delta}^1_5$ is thus reduced to bounding the ultrapowers $j_\nu(\boldsymbol{\delta}^1_3)$.

Again, one suspects that a simpler family of measures will suffice here.

5.14 Definition *W_3^m is the measure on $\boldsymbol{\delta}^1_3$ induced by the weak partition relation on $\boldsymbol{\delta}^1_3$, functions $f:\omega_{m+1}\to\boldsymbol{\delta}^1_3$ of the correct type, and the measure S_1^m on ω_{m+1}. That is, $W_3^m(A)=1$ iff there is a c.u.b. $C\subseteq\boldsymbol{\delta}^1_3$ such that for all $f:\omega_{m+1}\to C$ of the correct type, $[f]_{S_1^m}\in A$.*

5.15 Theorem *Let ν be a measure on $\boldsymbol{\delta}^1_3$ occurring in the homogeneous tree on a $\boldsymbol{\Pi}^1_3$-complete set. Then for some $m \in \omega$, $j_\nu(\boldsymbol{\delta}^1_3) \leq j_{W^m_3}(\boldsymbol{\delta}^1_3)$.*

5.16 Remark *The theorem actually holds for any measure ν on $\boldsymbol{\delta}^1_3$, although this requires the analysis of measures on $\boldsymbol{\delta}^1_3$ to show. The proof of theorem 5.15 is similar to that of theorem 5.12. The reader can find the details in [11].*

5.3. Higher Descriptions

In view of theorem 5.15, the basic problem in computing the upper bound for $\boldsymbol{\delta}^1_5$ is to analyze equivalence classes of functions $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to the measures W^m_3. This leads us to the notion of a level 1 description. It is helpful to consider some examples first.

Let us construct first an equivalence class of a function $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to W^1_3. We must define $F([f]_{S^1_1})$, for $f\colon \omega_2 \to \boldsymbol{\delta}^1_3$ of the correct type. We will define $F(f)$ for any such f, and note that our definition only depends on $[f]_{S^1_1}$. $F(f)$ is defined to be the ordinal represented with respect to $K_1 = S^1_1$ by the function which assigns to $[h_1]_{W^1_1}$ (here $h_1\colon \omega_1 \to \omega_1$ is of the correct type) the value $F(f, [h_1])$. We will define $F(f, h_1)$ for any such h_1, and note that this depends only on $[h_1]_{W^1_1}$, so we set $F(f, [h_1]) = F(f, h_1)$. Finally, $F(f, h_1)$ is the ordinal represented with respect to $K_2 = S^1_1$ by the function which assigns to $[h_2]_{W^1_1}$ the value $F(f, h_1, h_2) \doteq f([h_1 \circ h_2]_{W^1_1})$. Extending our earlier notational convention, we abbreviate this definition by saying $\forall^* f\ \forall^* h_1\ \forall^* h_2\ F(f, h_1, h_2) = f([h_1 \circ h_2])$. We could also write $\forall^* f\ \forall^* h_1\ \forall^* h_2\ F(f, h_1, h_2) = f([h])$, where $\forall^*_{W^1_1} \alpha\ h(\alpha) = h_1(h_2(\alpha))$.

It is easy to see that this definition is well-defined. Note, however, the map $(h_1, h_2) \to f([h_1 \circ h_2]_{W^1_1})$ is not well-defined with respect to $S^1_1 \times S^1_1$, that is, it does not just depend on $[h_1]_{W^1_1}, [h_2]_{W^1_1}$. Note also that it is important that we compose the functions h_1, h_2 in the order shown; the other way does not lead to a well-defined definition.

This simple example shows that the basic operation of composition leads to well-defined definitions of equivalence classes of functions $F\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$. In the general definition of a level 1 description, we generalize by allowing finitely many functions $h_1, \dots, h_t$, where each h_i is either a function $h_i\colon \mathrm{dom}(<_m) \to \omega_1$ of the correct type, or a finite tuple of ordinals $\beta_1 < \cdots < \beta_m < \omega_1$. That is, instead of S^1_1 in the example above, we allow measures S^m_1, W^m_1.

We first consider one more example. We define now the equivalence class of three functions F_1, F_2, F_3 with respect to the measure W^2_3. Thus, we must define $F_i([f]_{S^2_1}) = F_i(f)$ where $f\colon \omega_3 \to \boldsymbol{\delta}^1_3$ is of the correct type. In all three case we will use the sequence of measures $K_1 = K_2 = S^2_1$,

$K_3 = W_1^2$. We define F_1 by:

$$\forall^* f\ \forall^* h_1\ \forall^* h_2 \forall^* h_3 = (\beta_1, \beta_2)\ F_1(f, h_1, h_2, h_3) = \sup \{f(\delta) \colon \delta < [h]_{W_1^2}\},$$

where $\forall^*_{W_1^2} \alpha_1, \alpha_2\ h(\alpha_1, \alpha_2) = h_1(1)(\alpha_2)$ (recall $h_1(1)(\alpha_2) = \sup_{\gamma < \alpha_2} h_1(\gamma, \alpha_2)$). We define $F_2(f, h_1, h_2, h_3) = f([h])$, where

$$h(\alpha_1, \alpha_2) = h_1(h_2(1)(\alpha_1), \alpha_2).$$

We define $F_3(f, h_1, h_2, h_3) = f([h])$, where

$$h(\alpha_1, \alpha_2) = h_1(h_2(\beta_2, \alpha_1), \alpha_2).$$

It is easy to check that all three functions are well-defined, and that $\forall^*_{W_3^2}[f]$ $F_3(f) < F_2(f) < F_1(f)$. We will return to this example in a moment.

In general, a description d will be a finitary object defined relative to a sequence $K_1, \dots, K_t$ of measures, each of the form W_1^r or S_1^r. It will describe how, given the functions $h_1, \dots, h_t$, to generate the function h as in the above examples. In the first example, $h \colon \omega_1 \to \omega_1$, and in the second example, we had $h \colon (\omega_1)^2 \to \omega_1$ in all three cases. In general we will have $h \colon (\omega_1)^m \to \omega_1$ for some $m \in \omega$. The set $\mathcal{D}$ of descriptions will be partitioned accordingly as $\mathcal{D} = \bigcup_m \mathcal{D}_m$. The $d \in \mathcal{D}_m$, which we call the m-descriptions, will thus generate $h \colon (\omega_1)^m \to \omega_1$ given the functions $h_1, \dots, h_t$. We write also $\mathcal{D}_m(K_1, \dots, K_t)$ to denote those m-descriptions defined relative to $K_1, \dots, K_t$.

Thus, in the first example the underlying description (which we haven't defined yet) lies in $\mathcal{D}_1(S_1^1, S_1^1)$, and in the second, the descriptions lie in $\mathcal{D}_2(S_1^2, S_1^2, W_1^2)$.

Given a description $d \in \mathcal{D}_m(K_1, \dots, K_t)$, we will write $h(d; h_1, \dots, h_t)$ to denote the function $h \colon (\omega_1)^m \to \omega_1$ generated according to d from the $\vec{h}$.

Fix now $m \in \omega$ and the sequence $K_1, \dots, K_t$. The primitive descriptions in $\mathcal{D}_m(K_1, \dots, K_t)$ are those which do not involve composing functions. We refer to these as the *basic* descriptions, and the others as non-basic. As we define the descriptions, we simultaneously define $k(d) \in \{1, \dots, t\} \cup \{\infty\}$ for each description which gives the least k so that the function h_k is involved in the definition of $h(d; \vec{h})$. As we define the descriptions, we also define how to interpret them, that is, we define $h(d; \vec{h})(\alpha_1, \dots, \alpha_m)$. For $h_k : \mathrm{dom}(<_r) \to \omega_1$ order-preserving, and $l \leq r$, recall the definition of $h_k(l) : (\omega_1)^l \to \omega_1$ from definition 4.22. In this case it reduces to (for $l = r$, $h_k(l) = h_k$):

$$h_k(l)(\alpha_1, \dots, \alpha_l) = \sup \{h_k(\alpha_1, \dots, \alpha_{l-1}, \beta_1, \dots, \beta_{r-l}, \alpha_l) : \\ \alpha_{l-1} < \beta_1 < \cdots < \beta_{r-l} < \alpha_l\}.$$

5.17 Definition (Descriptions)

1. *(Basic) We allow:*

   (a) *$d = (p)$ where p is an integer $1 \leq p \leq m$ (which we put in parentheses to distinguish from a level -1 description). We define $h(d;\vec{h})(\alpha_1,\dots,\alpha_m) = \alpha_p$. We set $k(d) = \infty$.*

   (b) *$d = (k;p)$ where $1 \leq k \leq t$, K_k is of the form $K_k = W_1^r$, and $1 \leq p \leq r$. In this case, we define $h(d;\vec{h})(\alpha_1,\dots,\alpha_m) = \beta_p$, where $h_k = (\beta_1,\dots,\beta_r)$. We set $k(d) = k$.*

2. *(Non-Basic) Suppose $1 \leq k \leq t$ and $K_k = S_1^r$. We allow:*

   (a) *$d = (k;d_r,d_1,\dots,d_l)$, where d_1,…,d_l,$d_r \in \mathcal{D}_m(K_1,\dots,K_t)$, $l < r$, and $k(d_1)$,…,$k(d_l)$,$k(d_r) > k$. We set $k(d) = k$ and define:*

   $$h(d;\vec{h})(\vec{\alpha}) = h_k(l+1)(h(d_1;\vec{h})(\vec{\alpha}),\dots,h(d_l;\vec{h})(\vec{\alpha}),h(d_r;\vec{h})(\vec{\alpha}))$$

   (b) *$d = (k;d_r,d_1,\dots,d_l)^s$, where d_1,…,d_l,$d_r \in \mathcal{D}_m(K_1,\dots,K_t)$, $l < r$, $k(d_1)$,…,$k(d_l)$,$k(d_r) > k$, and s is a formal symbol (which stands for "sup"). We require in this case that $r \geq 2$, $l \geq 1$. We set $k(d) = k$ and define:*

   $$h(d;\vec{h})(\vec{\alpha}) = \sup\{h_k(l+1)(h(d_1;\vec{h})(\vec{\alpha}),\dots,h(d_{l-1};\vec{h}),\beta, h(d_r;\vec{h})(\vec{\alpha}))\colon \beta < h(d_l;\vec{h})\}.$$

We write $(k;d_r,d_1,\dots,d_l)^{(s)}$ to indicate the symbol s may or may not appear.

This completes the definition of $\mathcal{D}$ and the "interpretation function" h. We will write d^m when we wish to emphasize $d \in \mathcal{D}_m$. In the first example above, the description is given by $d^1 = (1;(2;(1)))$. In the second example, the three descriptions are given by $d_a^2 = (1;(2))$, $d_b^2 = (1;(2);(2;(1)))$, and $d_c^2 = (1;(2);(2;(1),(3;2)))$.

If $d_1, d_2 \in \mathcal{D}_m(K_1,\dots,K_t)$, define $d_1 < d_2$ iff $\forall^* h_1 \dots \forall^* h_t\ h(d_1;\vec{h}) < h(d_2;\vec{h})$ almost everywhere. Here $\forall^*$ refers to the measure on functions of the correct type induced by the strong partition relation. Note that in a non-basic description $d = (k;d_r,d_1,\dots,d_l)$ the component descriptions are listed in their order of significance in determining $h(d;\vec{h})$. It is not difficult to reformulate the $<$ relation on $\mathcal{D}_m(K_1,\dots,K_t)$ in a purely "syntactical" manner, we leave this to the reader.

The requirement that $k(d_r), k(d_1),\dots,k(d_l) > k = k(d)$ for non-basic descriptions d is one of two necessary to ensure the equivalence class of $h(d;\vec{h})$ is well-defined with respect to the measures $K_1,\dots,K_t$. This guarantees that for almost all $h_{k+1},\dots,h_t$, the values $h(d_r;\vec{h})(\vec{\alpha})$, etc, that we are putting into h_k will lie in a c.u.b. set on which two functions h_k, h'_k (with $[h_k] = [h'_k]$) agree. The other requirement is that these values be in the correct order. We can now state this requirement, which is referred to as "condition C".

5.18 Definition (Definition of C) *We say $d \in \mathcal{D}$ satisfies condition C if either d is basic or else d is non-basic, say of the form $d = (k; d_r, d_1, \ldots, d_l)^{(s)}$, all $d_r, d_1, \ldots, d_l$ satisfy C, and $d_1 < d_2 < \ldots < d_l < d_r$.*

It is now easy to check that if d satisfies this condition, then the equivalence class of $h(d; \vec{h})$ is well-defined in the following precise sense:

5.19 Lemma *Suppose $d \in \mathcal{D}_m(K_1, \ldots, K_t)$ satisfies condition C. Then for almost all h_1, if $[h'_1] = [h_1]$, then for almost all h_2, if $[h'_2] = [h_2]$, …, for almost all h_t if $[h'_t] = [h_t]$, then $[h(d; h_1, \ldots, h_t)]_{W_1^m} = [h(d; h'_1, \ldots, h'_t)]_{W_1^m}$.*

□

All of the descriptions in the examples above satisfy C.

From now on, we officially let $h(d; \vec{h})$ be the ordinal $< \omega_{m+1}$ (where $d \in \mathcal{D}_m$) represented by the function h.

We expand a little our notational convention mentioned at the end of the introduction. Suppose $K_1, \ldots, K_t$ is a sequence of measures, $d \in \mathcal{D}(K_1, \ldots, K_t)$ satisfies C, and $P \subseteq \mathrm{ON}$. When we write

$$\forall^* h_1, \ldots, h_t \ P(h(d; h_1, \ldots, h_t)),$$

we mean: $\forall^*_{K_1} \eta_1$ if $[h_1] = \eta_1$, then $\forall^*_{K_2} \eta_2$ if $[h_2] = \eta_2$, …, $\forall^*_{K_t} \eta_t$ if $[h_t] = \eta_t$, then $P(h(d; h_1, \ldots, h_t))$. If $\theta \in \mathrm{ON}$, we write

$$\forall^* h_1, \ldots, h_t \ P(\theta(h_1, \ldots, h_t))$$

to mean: if we fix a representing function $\eta_1 \to \theta(\eta_1)$ for θ with respect to K_1 then $\forall^*_{K_1} \eta_1$ if $[h_1] = \eta_1$, then if we fix a representing function $\eta_2 \to \theta([h_1], \eta_2)$ for $\theta([h_1])$ with respect to K_2, then $\forall^*_{K_2} \eta_2$ if $[h_2] = \eta_2$, …, if we fix a representing function $\eta_t \to \theta([h_1], \ldots, [h_{t-1}], \eta_t)$ for $\theta([h_1], \ldots, [h_{t-1}])$ with respect to K_t, then $\forall^*_{K_t} \eta_t$ if $[h_t] = \eta_t$, then $P(\theta([h_1], \ldots, [h_t]))$.

We can use these conventions simultaneously. For example, for d satisfying C we may write

$$\forall^* h_1, \ldots, h_t \ \mathrm{cof}(h(d; h_1, \ldots, h_t)) < \theta(h_1, \ldots, h_t).$$

This abbreviates: $\forall^*_{K_1} \eta_1$ if $[h_1] = \eta_1$, then $\forall^*_{K_2} \eta_2$ if $[h_2] = \eta_2$, …, $\forall^*_{K_t} \eta_t$ if $[h_t] = \eta_t$, then $\mathrm{cof}(h(d; h_1, \ldots, h_t)) < \theta([h_1], \ldots, [h_t])$. Written out in full, this becomes: if $\eta_1 \to \theta(\eta_1)$ represents θ with respect to K_1, then $\forall^*_{K_1} \eta_1$, if $[h_1] = \eta_1$ and $\eta_2 \to \theta([h_1], \eta_2)$ represents $\theta([h_1])$ with respect to K_2, then $\forall^*_{K_2} \eta_2$, if $[h_2] = \eta_2$ and $\eta_3 \to \theta([h_1], [h_2], \eta_3)$ represents $\theta([h_1], [h_2])$

with respect to K_3, ..., $\forall^*_{K_t}\eta_t$, if $[h_t] = \eta_t$, then $\mathrm{cof}(h(d; h_1, \dots, h_t)) < \theta([h_1], \dots, [h_t])$. Such a statement is well-defined by lemma 5.19.

Recall the purpose of a description $d \in \mathcal{D}_m$ is to generate an ordinal $\alpha < \omega_{m+1}$ which we can plug into a function $f\colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ in our attempt to generate an equivalence class $[F]_{W^m_3}$. By "plug in" we mean either take $f(\alpha)$ or $\sup_{\alpha'<\alpha} f(\alpha')$. Condition C guarantees the ordinal α is well-defined in an appropriate sense. However, it does not guarantee α will be representable by a function $h\colon \mathrm{dom}(<_m) \to \omega_1$ of the correct type, or that it is a limit of such ordinals. Thus we introduce another condition, condition D, below.

First we extend slightly the notion of a description. Let $\bar{\mathcal{D}}_m$ be the set of objects ("extended descriptions") of the form (d) or $(d)^s$ where $d \in \mathcal{D}_m$ satisfies C, together with one new object $()^s$. We write $(d)^{(s)}$ to denote either (d) or $(d)^s$.

5.20 Definition (Definition of D) *Suppose $d \in \bar{\mathcal{D}}_m(K_1, \dots, K_t)$. Then:*

1. *(d) satisfies condition D if $\forall^* h_1, \dots, h_t\ h(d; \vec{h})\colon (\omega_1)^m \to \omega_1$ is of the correct type almost everywhere (i.e. restricted to a measure one set with respect to W^m_1).*

2. *$(d)^s$ satisfies condition D if $\forall h_1, \dots, h_t\ h(d; \vec{h})$ is the supremum of ordinals representable by functions $h\colon (\omega_1)^m \to \omega_1$ of the correct type almost everywhere. We also define $()^s$ to satisfy D.*

We can now describe our generation of equivalence classes.

5.21 Definition *Let $m \in \omega$, $K_1, \dots, K_t \in W_1 \cup S_1$, let $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \dots, K_t)$ satisfy condition D, and let $g\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$. We define an ordinal which we denote by $(g; (d)^{(s)}; K_1, \dots, K_t)$. We represent this ordinal with respect to W^m_3 by the function which assigns to $[f]_{S^m_1}$ the ordinal $(g; f; (d)^{(s)}; K_1, \dots, K_t)$, for $f : \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type. We represent $(g; f; (d)^{(s)}; K_1, \dots, K_t)$ with respect to K_1 by the function which assigns to $[h_1]$ the ordinal $(g; f; (d)^{(s)}; h_1, K_2, \dots, K_t)$, and in general, represent $(g; f; (d)^{(s)}; h_1, \dots, h_{i-1}, K_i, \dots, K_t)$ with respect to K_i by the function which assigns to $[h_i]$ the ordinal $(g; f; (d)^{(s)}; h_1, \dots, h_i, K_{i+1}, \dots, K_t)$. Finally, we define $(g; f; (d)^{(s)}, h_1, \dots, h_t)$ by cases as follows:*

1. *If s does not appear, then $(g; f; (d); h_1, \dots, h_t) = g(f(h(d; \bar{h})))$.*

2. *If s appears, then $(g; f; (d)^s; h_1, \dots, h_t) = g(\sup_{\beta < h(d;\vec{h})} f(\beta))$.*

3. *For the object $()^s$, we define $(g; f; ()^s; h_1, \dots, h_t) = g(\sup_{\beta<\omega_{m+1}} f(\beta))$.*

Of particular importance is the case $g = \text{id}$, the identity function. In the first example considered previously, the equivalence class $[F]_{W_3^1}$ constructed was $(\text{id}; (d); K_1, K_2)$, where $d = (1; (2; (1)))$, and $K_1 = K_2 = S_1^1$. The three functions of the second example represent the ordinals $(\text{id}; (d_a)^s; K_1, K_2, K_3$, $(\text{id}; (d_b); K_1, K_2, K_3)$, and $(\text{id}; (d_c); K_1, K_2, K_3)$ respectively.

It turns out, though we will not prove this fully here, that the cardinals $\boldsymbol{\delta}_3^1 < \kappa < (\boldsymbol{\delta}_5^1)^-$ are precisely the ordinals of the form $(\text{id}; (d)^{(s)}; K_1, \dots, K_t)$.

We have thus seen how descriptions generate equivalence classes of functions $F\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ with respect to the measures W_3^m, just as the trivial descriptions did for functions $F\colon \omega_1 \to \omega_1$ with respect to the measures W_1^m, definition 4.3. The main task remaining is to formulate and prove a result analogous to lemma 4.5, the "main lemma" in the theory of trivial descriptions. To do that we need the correct analog of the lowering operator $\mathcal{L}$ (which we will still call $\mathcal{L}$).

If we fix $m \in \omega$ and measures $K_1, \dots, K_t \in W_1 \cup S_1$, then the relation $<$ on $\mathcal{D}_m(K_1, \dots, K_t)$ is a linear ordering. For $d \in \mathcal{D}_m(K_1, \dots, K_t)$, $\mathcal{L}(d)$ will again give the description preceding d in this ordering, except for a unique minimal description. For the sake of completeness we will give the complete definition of $\mathcal{L}$, though will be content to illustrate the proof of the main lemma through one of our examples. The $\mathcal{L}$ operation is defined by defining a series of approximations $\mathcal{L}^k$ to it. $\mathcal{L}^k(d)$ will only be defined for d with $k(d) \geq k$. Roughly speaking, $\mathcal{L}^k$ is the result of holding $h_1, \dots, h_{k-1}$ constant and lowering with respect to $h_k, \dots, h_t$ only. We will thus take $\mathcal{L}(d) = \mathcal{L}^1(d)$. Following [11], we define $\mathcal{L}^k$ as follows.

5.22 Definition (Definition of $\mathcal{L}^k$) *Let $m \in \omega$ and $K_1, \dots, K_t \in W_1 \cup S_1$. Let $k \in \{1, \dots, t\} \cup \{\infty\}$, and assume $d \in \mathcal{D}_m(K_1, \dots, K_t)$ with $k(d) \geq k$. Then $\mathcal{L}^k(d)$ is defined by reverse induction on k through the following cases:*

I *$k = \infty$. So, $d = (i)$ where $1 \leq i \leq m$. If $i > 1$, then $\mathcal{L}^\infty(d) = (i-1)$. For $i = 1$, d is minimal with respect to $\mathcal{L}^\infty$.*

II *$1 \leq k \leq t$.*

1. *$k = k(d)$.*
   (a) *$d = (k; p)$ is basic. For $p > 1$, we set $\mathcal{L}^k(d) = (k; p-1)$, and for $p = 1$ we define d to be minimal.*
   (b) *$d = (k; d_r, d_1, \dots, d_l)$, where $K_k = S_1^r$ and $l = r - 1$. We set $\mathcal{L}^k(d) = (k; d_r, d_1, \dots, d_l)^s$ if $l \geq 1$, and if $r = 1$ and $d = (k; d_r)$, we set $\mathcal{L}^k(d) = d_r$.*
   (c) *$d = (k; d_r, d_1, \dots, d_l)$, where $K_k = S_1^r$ and $l < r-1$. First assume $l \geq 1$. If $\mathcal{L}^{k+1}(d_r)$ is defined and $> d_l$, we set $\mathcal{L}^k(d) = (k; d_r, d_1, \dots, d_l, \mathcal{L}^{k+1}(d_r))$. If $\mathcal{L}^{k+1}(d_r)$ is not defined or is $\leq d_l$, we set $\mathcal{L}^k(d) = (k; d_r, d_1, \dots, d_l)^s$. If $l = 0$ (so*

$d = (k; d_r)$), we set $\mathcal{L}^k(d) = (d_r; \mathcal{L}^{k+1}(d_r))$ if $\mathcal{L}^{k+1}(d_r)$ is defined, and otherwise $\mathcal{L}^k(d) = d_r$.

(d) *$d = (k; d_r, d_1, \ldots, d_l)^s$, where $K_k = S_1^r$ (so $l \geq 1$). We set $\mathcal{L}^k(d) = (k; d_r, d_1, \ldots, d_{l-1}, \mathcal{L}^{k+1}(d_l))$ if $\mathcal{L}^{k+1}(d_l)$ is defined and if $l \geq 2$ also satisfies $\mathcal{L}^{k+1}(d_l) > d_{l-1}$. Otherwise, we set $\mathcal{L}^k(d) = (k; d_r, d_1, \ldots, d_{l-1})^s$ if $l \geq 2$ and for $l = 1$, $\mathcal{L}^k(d) = d_r$.*

2. *$k < k(d)$, $K_k = W_1^r$.*

   (a) *d not minimal with respect to $\mathcal{L}^{k+1}$. We set $\mathcal{L}^k(d) = \mathcal{L}^{k+1}(d)$.*

   (b) *d is minimal with respect to $\mathcal{L}^{k+1}(d)$. We set $\mathcal{L}^k(d) = (k; r)$.*

3. *$k < k(d)$, $K_k = S_1^r$.*

   (a) *d not minimal with respect to $\mathcal{L}^{k+1}$. We then set $\mathcal{L}^k(d) = (k; \mathcal{L}^{k+1}(d))$.*

   (b) *d minimal with respect to $\mathcal{L}^{k+1}$. Then d is minimal with respect to $\mathcal{L}^k$.*

Recall our previous example where $m = 2$, $K_1 = K_2 = S_1^2$, $K_3 = W_1^2$, and we had the three descriptions $d_a = (1; (2))$, $d_b = (1; (2); (2; (1)))$, and $d_c = (1; (2); (2; (1), (3; 2)))$. The reader can check now that $\mathcal{L}(d_a) = d_b$, $\mathcal{L}(d_b) = (1; (2); (2; (1)))^s$, and $\mathcal{L}((1; (2); (2; (1)))^s) = d_c$.

We also extend the $\mathcal{L}$ operation to $\bar{\mathcal{D}}_m(K_1, \ldots, K_t)$ as follows. We set $\mathcal{L}((d)) = ((d)^s)$, and $\mathcal{L}((d)^s) = (\mathcal{L}^{(p)}(d))$, where $\mathcal{L}^{(p)}(d)$ denotes the p^{th} iterate of $\mathcal{L}$, and p is least so that $\mathcal{L}^{(p)}(d)$ satisfies condition D. If such a p does not exist, we say $(d)^s$ is minimal with respect to $\mathcal{L}$. Finally, for the distinguished object $()^s$, we define $\mathcal{L}(()^s) = (\tilde{d})^{(s)}$, where $\tilde{d}$ is the maximal description in $\mathcal{D}_m(K_1, \ldots, K_t)$ such that $(\tilde{d})$ or $(\tilde{d})^s$ satisfies D (in the first case, s does not appear, and in the second it does). If there are no descriptions satisfying D, then $()^s$ is declared minimal.

To illustrate, let $m = 2$, and consider the sequence of measures K_1, K_2, K_3 where $K_1 = K_2 = S_1^2$, and $K_3 = W_1^2$. Applying the $\mathcal{L}$ operation repeatedly to $()^s \in \bar{\mathcal{D}}_2(K_1, K_2, K_3)$ results in a sequence whose first few terms are: $(d_0)^s$, (d_1), $(d_1)^s$, (d_2), $(d_2)^s$, (d_3), $(d_3)^s$, $(d_4)^s$, (d_5), $(d_5)^s$, (d_6), $(d_6)^s$,

(d_7), $(d_7)^s$, (d_8), $(d_8)^s$, (d_9), where:

$$
\begin{aligned}
d_0 &= (1; (2; (2))) \\
d_1 &= (1; (2; (2)); (2; (2); (1))) \\
d_2 &= (1; (2; (2)); (2; (2); (1)))^s \\
d_3 &= (1; (2; (2)); (2; (2); (1))^s) \\
d_4 &= (1; (2; (2)); (2; (2); (1))^s)^s \\
d_5 &= (1; (2; (2)); (2; (2); (3; 2))) \\
d_6 &= (1; (2; (2)); (2; (2); (3; 2)))^s \\
d_7 &= (1; (2; (2)); (2; (2); (3; 2))^s) \\
d_8 &= (1; (2; (2)); (2; (2); (3; 2))^s)^s \\
d_9 &= (1; (2; (2)); (2; (2); (3; 1)))
\end{aligned}
$$

Note that (d_0), and (d_4) do not satisfy D.

We now state our "main lemma", the analog of lemma 4.5.

5.23 Theorem (Main Lemma) *Let $(d)^{(s)} \in \bar{\mathcal{D}}(K_1,\ldots,K_t)$ satisfy D. Suppose $\theta < (\mathrm{id};(d)^{(s)};K_1,\ldots,K_t)$. Then:*

1. *If $(d)^{(s)}$ is not minimal with respect to $\bar{\mathcal{L}}$, then there is a $g\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ such that $\theta < (g; \mathcal{L}((d)^{(s)});\, K_1,\, \ldots,\, K_t)$.*
2. *If $(d)^{(s)}$ is minimal with respect to $\mathcal{L}$, then $\theta < \boldsymbol{\delta}^1_3$.*

We will illustrate the proof of the main lemma by considering the example $(d_a)^s$ above, where $d_a = (1; (2))$. So, fix $\theta < (\mathrm{id}; (d_a)^s; K_1, K_2, K_3)$. Thus,

$$\forall^*_{W^2_3} f\ \forall^* h_1, h_2, h_3\ [\theta(f, \vec{h}) < (\mathrm{id}; f; (d_a)^s; \vec{h}) = \sup\{f(\gamma)\colon \gamma < h(d_a; \vec{h})].$$

Hence,

$$\forall^* f\ \exists \delta\ \forall^* h_1, h_2, h_3\ [\delta(\vec{h}) < h(d_a; \vec{h}) \wedge \theta(f, \vec{h}) < f(\delta(\vec{h}))].$$

Suppose now $\delta \in \mathrm{ON}$ is such that $\forall^* h_1, h_2, h_3\ \delta(\vec{h}) < h(d_a; \vec{h})$. In other words,

$$\forall^* h_1, h_2, h_3\ \forall^*_{W^2_1} \alpha_1, \alpha_2\ \delta(\vec{h})(\alpha_1, \alpha_2) < h(d_a; \vec{h})(\alpha_1, \alpha_2).$$

Recall that $h(d_a; \vec{h})(\alpha_1, \alpha_2) = h_1(1)(\alpha_2) = \sup_{\eta < \alpha_2} h_1(\eta, \alpha_2)$. Thus,

$$\forall^* h_1, h_2, h_3\ \forall^* \alpha_1, \alpha_2\ \exists \eta < \alpha_2\ [\delta(\vec{h})(\alpha_1, \alpha_2) < h_1(\eta, \alpha_2)].$$

It follows that

$$\forall^* h_1, h_2, h_3\ \exists g\colon \omega_1 \to \omega_1\ \forall^* \alpha_1, \alpha_2\ [\delta(\vec{h})(\alpha_1, \alpha_2) < h_1(g(\alpha_1), \alpha_2)].$$

In this expression, only the equivalence class of g with respect to W_1^1 matters, and we may assume the g is of the correct type. Using the ω_2-additivity of the measure S_1^1, it follows that

$$\forall^* h_1, h_2\ \exists g\colon \omega_1 \to \omega_1\ \forall^* h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(\vec{h})(\alpha_1, \alpha_2) < h_1(g(\alpha_1), \alpha_2)].$$

For fixed h_1 of the correct type, and fixed $\delta(h_1) \in \mathrm{ON}$ such that

$$\forall^* h_2\ \exists g\colon \omega_1 \to \omega_1\ \forall^* h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(h_1)(h_2, h_3)(\alpha_1, \alpha_2) < h_1(g(\alpha_1), \alpha_2)],$$

we consider the partition $\mathcal{P}$: we partition $h_2\colon \mathrm{dom}(<_2) \to \omega_1$ of the correct type with the extra value $g(\alpha)$ inserted between $h_2(1)(\alpha)$ and $N_{h_2}(h_2(1)(\alpha))$ (with $g(\alpha)$ of uniform cofinality ω) according to whether

$$\forall^* h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(h_1)(h_2, h_3)(\alpha_1, \alpha_2) < \ h_1(g(\alpha_1), \alpha_2)].$$

From lemma 4.24 (with $m = 2$, $n = 1$, $r = 1$) it follows that a c.u.b. set cannot be homogeneous for the contrary side of the partition. Let C be homogeneous for $\mathcal{P}$, and $g(\alpha) =$ the ω^{th} element of C greater that α. We then have that for any $h_2\colon \mathrm{dom}(<_2) \to C'$ of the correct type

$$\forall^* h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(h_1)(h_2, h_3)(\alpha_1, \alpha_2) < h_1(g(h_2(1)(\alpha_1)), \alpha_2)].$$

Since for almost all h_1, $\delta(h_1)$ satisfies the hypothesis of the partition, we have:

$$\forall^* h_1\ \exists g\colon \omega_1 \to \omega_1\ \forall^* h_2, h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(h_1, h_2, h_3)(\alpha_1, \alpha_2) < \\ h_1(g(h_2(1)(\alpha_1)), \alpha_2)].$$

Fix a representing function $h_1 \to \delta(h_1)$ for δ, and consider the partition $\mathcal{P}$: we partition $h_1\colon \mathrm{dom}(<_2) \to \omega_1$ of the correct type with the extra values $g(\gamma_1, \gamma_1)$ (of uniform cofinality ω) inserted between $h_1(\gamma_1, \gamma_2)$ and $N_{h_1}(h_1(\gamma_1, \gamma_2))$ according to whether

$$\forall^* h_2, h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(h_1, h_2, h_3)(\alpha_1, \alpha_2) < N_g(h_1(h_2(1)(\alpha_1), \alpha_2)) \\ = g(h_2(1)(\alpha_1), \alpha_2).$$

It follows from lemma 4.24 that we cannot have a c.u.b. set C homogeneous for the contrary side of the partition. For if so, fix a representing function $h_1 \to \delta(h_1)$ for δ, and fix then a $h_1\colon \mathrm{dom}(<_2) \to C$ of the correct type such that for some $\bar{g}\colon \omega_1 \to \omega_1$ of the correct type, which we fix, we have:

$$\forall^* h_2, h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(\vec{h})(\alpha_1, \alpha_2) < h_1(\bar{g}(h_2(1)(\alpha_1)), \alpha_2)].$$

Define $\tilde{g}(\gamma_1, \gamma_2) = h_1(\bar{g}(\gamma_1), \gamma_2)$. Apply then lemma 4.24 to $f = h_1$ and $\tilde{g}$ (and $r = 2$). This produces h_1', g' which are ordered as in $\mathcal{P}$, have range in C, and for which the property stated in $\mathcal{P}$ holds, a contradiction. If we fix now a c.u.b. C homogeneous for $\mathcal{P}$, and define $g(\alpha) =$ the ω^{th} element of C greater than α, then we have:

$$\forall^* h_1, h_2, h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(\vec{h})(\alpha_1, \alpha_2) < g(h_1(h_2(1)(\alpha_1), \alpha_2))].$$

We now have that for almost all $f\colon \omega_3 \to \boldsymbol{\delta}^1_3$ of the correct type, there is a $g\colon \omega_1 \to \omega_1$ such that

$$\forall^* h_1, h_2, h_3\ \theta(f, \vec{h}) < f(\gamma) \tag{I.1}$$

where:

$$\forall^* \alpha_1, \alpha_2\ \gamma(\alpha_1, \alpha_2) = g(h_1(h_2(1)(\alpha_1), \alpha_2)) = g(h(d_b; \vec{h})(\alpha_1, \alpha_2)), \tag{I.2}$$

and $d_b = (1; (2); (2; (1)))$ as before.

Fix a representing function $f \to \theta(f)$ for θ with respect to W^2_3, and consider the partition $\mathcal{P}$: we partition $f\colon \omega_3 \to \boldsymbol{\delta}^1_3$ of the correct type with the extra values $f_2(\alpha)$ of uniform cofinality ω inserted between $f(\alpha)$ and $f(\alpha + 1)$ according to whether $\forall^* h_1, h_2, h_3\ \theta(f)(\vec{h}) < f_2(\gamma)$, where $\gamma = h(d_b; \vec{h})$. There cannot be a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ homogeneous for the contrary side, for if so, fix $f\colon \omega_3 \to C$ of the correct type such that there is a $g\colon \omega_1 \to \omega_1$ as in equations I.1, I.2, and fix such a g. Define $f_2\colon \omega_3 \to C$ by: $f_2(\gamma) = f(\delta)$, where $\forall^* \alpha_1, \alpha_2\ \delta(\alpha_1, \alpha_2) = g(\gamma(\alpha_1, \alpha_2))$. A variation of lemma 4.24 shows that there are $f', f_2'\colon \omega_3 \to C$ of the correct type and ordered as in $\mathcal{P}$ such that $[f']_{S^2_1} = [f]_{S^2_1}$, $[f_2']_{S^2_1} = [f_2]_{S^2_1}$. This contradicts the homogeneity of C for the contrary side. Let $C \subseteq \boldsymbol{\delta}^1_3$ now be a c.u.b. set homogeneous for $\mathcal{P}$. Define $g\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ by $g(\alpha) =$ the ω^{th} element of C greater than α. We then have that $\forall^* f\ \forall^* h_1, h_2, h_3\ \theta(f, \vec{h}) < g(f(\gamma))$, where $\gamma = h(d_b; \vec{h})$. In other words, $\theta < (g; (d_b); K_1, K_2, K_3)$. □

From lemma 5.23 and theorem 5.5 our main result analyzing equivalence classes with respect to the measures W^m_3 on $\boldsymbol{\delta}^1_3$ now follows.

5.24 Theorem *Suppose* $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \ldots, K_t)$ *satisfies D. If* $(d)^{(s)}$ *is not minimal with respect to* $\mathcal{L}$ *then*

$$(\mathrm{id}; (d)^{(s)}; K_1, \ldots, K_t) \leq [\sup_{K_{t+1} \in W_1 \cup S_1} (\mathrm{id}; \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t, K_{t+1})]^+.$$

Here $\mathcal{L}((d)^{(s)})$ *is computed relative to the sequence* $K_1, \ldots, K_t$*. If* $(d)^{(s)}$ *is minimal with respect to* $\mathcal{L}$*, then* $(\mathrm{id}; (d)^{(s)}; K_1, \ldots, K_t) = \boldsymbol{\delta}^1_3$.

Proof. We consider the first case, and suppose $\theta < (\mathrm{id}; (d)^{(s)}; K_1, \ldots, K_t)$. By lemma 5.23, there is a $g\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ such that

$$\theta < (g; \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t).$$

Let T be as in theorem 5.5, and fix a real x and a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$, $g(\alpha) < |T_x \restriction \sup_{K_{t+1}} j_{K_{t+1}}(\alpha)|$, the supremum ranging over measures $K_{t+1} \in W_1 \cup S_1$. For $\alpha < \boldsymbol{\delta}^1_3$, let

$$l(\alpha) = \sup_{K_{t+1}} j_{K_{t+1}}(\alpha) \text{ and } l'(\alpha) = |T_x \restriction l(\alpha)|.$$

We define a well-founded relation $\prec$ on $\lambda \doteq (l; \mathcal{L}((d)^{(s)}); K_1, \dots, K_t)$ by: $\rho_1 \prec \rho_2$ iff

$$\forall^*_{W^m_3} f \ \forall^* h_1, \dots, h_t \ |T_x \restriction \lambda(f, \vec{h})(\rho_1(f, \vec{h}))| < |T_x \restriction \lambda(f, \vec{h})(\rho_2(f, \vec{h}))|.$$

Easily, $|\prec| \geq (l'; \mathcal{L}((d)^{(s)}); K_1, \dots, K_t) \geq \theta$. It follows that

$$(\mathrm{id}; (d)^{(s)}; K_1, \dots, K_t) \leq [(\sup_{K_{t+1}} j_{K_{t+1}}; \mathcal{L}((d)^{(s)}); K_1, \dots, K_t)]^+.$$

By countable additivity of the measures $W^m_3, K_1, \dots, K_t$, we have that if $\alpha < (\sup_{K_{t+1}} j_{K_{t+1}}; \mathcal{L}((d)^{(s)}); K_1, \dots, K_t)$ then there is a K_{t+1} such that $\alpha < (j_{K_{t+1}}; \mathcal{L}((d)^{(s)}); K_1, \dots, K_t)$. Also, from the definitions of these ordinals it is immediate that

$$(j_{K_{t+1}}; \mathcal{L}((d)^{(s)}); K_1, \dots, K_t) = (\mathrm{id}; \mathcal{L}((d)^{(s)}); K_1, \dots, K_t, K_{t+1}).$$

The result now follows. ⊣

To compute the upper bound for $\boldsymbol{\delta}^1_5$, it suffices to compute the rank of the $\mathcal{L}$ operation, in a suitable sense. Namely, fix m and consider the set of all tuples $((d)^{(s)}; K_1, \dots, K_t)$ where $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \dots, K_t)$ satisfies D relative to $K_1, \dots, K_t$. Let $\prec_m$ be the transitive relation on this set generated by the relation $((d)^{(s)}; K_1, \dots, K_t) \prec_m (\mathcal{L}((d)^{(s)}); K_1, \dots, K_t, K_{t+1})$ for all K_{t+1}, where $\mathcal{L}((d)^{(s)})$ is again computed relative to the sequence $K_1, \dots, K_t$. The relation $\prec_m$ is easily well-founded. Let $|\vec{s}|_m$ denote the rank of the tuple $\vec{s}$, computed in the slightly non-standard manner by: $|\vec{s}|_m = (\sup_{\vec{t} \prec_m \vec{s}} |\vec{t}|_m) + 1$; by convention if $\vec{s}$ is minimal, then $|\vec{s}| = 0$ (thus, at limit ranks, this is one more than the usual rank).

An immediate induction on the $\prec_m$ rank using theorem 5.24 then shows:

5.25 Theorem *For all $m \in \omega$, $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \dots, K_t)$ satisfying D, we have* $(\mathrm{id}; (d)^{(s)}; K_1, \dots, K_t) \leq \aleph_{\omega+1+|((d)^{(s)}; K_1, \dots, K_t)|_m}$. □

Let θ_m be the supremum of the $|\ |_m$ ranks of the tuples $((d)^{(s)}; K_1, \dots, K_t)$ where $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \dots, K_t)$. From theorem 5.25 and the homogeneous tree analysis (c.f. two paragraphs before definition 5.14) we thus have $\boldsymbol{\delta}^1_5 \leq [\aleph_{\sup_m \theta_m}]^+$.

The computation of the θ_m is a purely combinatorial problem, and is relatively straightforward. We omit the proof, and simply state the result that $\theta_m = \omega^{\omega^m}$ (ordinal exponentiation). As an immediate corollary we have:

5.26 Corollary $\boldsymbol{\delta}^1_5 \leq \aleph_{\omega^{\omega^\omega}+1}$.

At this point we have extended the basic ingredients in the theory, Martin's theorem and the description analysis, from the $\boldsymbol{\delta}^1_1$ level to the $\boldsymbol{\delta}^1_3$ level, and we have used this to do one step in the next level of the inductive analysis, namely the upper bound for $\boldsymbol{\delta}^1_5$. To finish the next level analysis, it remains to prove the strong partition relation on $\boldsymbol{\delta}^1_3$, the lower bound for $\boldsymbol{\delta}^1_5$, and the weak partition relation for $\boldsymbol{\delta}^1_5$. The proofs in all cases follow in outline those of §4, using the description analysis, theorem 5.24. Since we have now illustrated all of the ideas which go into these arguments, we will content ourselves with this. The complete details of these arguments can be found in [11]. We mention only that the analysis of measures on $\boldsymbol{\delta}^1_3$ and on λ_5 require the notions of type-2 and type-3 trees of uniform cofinalities respectively (roughly corresponding to the measures occurring in homogeneous trees on $\boldsymbol{\Pi}^1_3, \boldsymbol{\Pi}^1_4$-complete sets).

5.4. Some Further Results

We close this section with some remarks on generalizations and refinements of the results discussed. One can show that all of the ordinals $(\mathrm{id}; (d)^{(s)}; K_1, \ldots, K_t)$ are actually cardinals (theorem 5.24 shows that all cardinals between $\boldsymbol{\delta}^1_3$ and λ_5 must be of this form). This is proved in [13]. It is also not difficult to show that if μ is a semi-normal measure on a cardinal κ having the strong partition relation, then $j_\mu(\kappa)$ is a regular cardinal. For the three normal measures μ_ω, μ_{ω_1}, μ_{ω_2} on $\boldsymbol{\delta}^1_3$, these ultrapowers are computed to be $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$, $\aleph_{\omega\cdot 2+1}$, and $\aleph_{\omega^\omega+1}$ respectively. These three regular cardinals κ all satisfy $\kappa \rightarrow (\kappa)^\lambda$ for all $\lambda < \boldsymbol{\delta}^1_4$, but $\kappa \not\rightarrow (\kappa)^{\boldsymbol{\delta}^1_4}$. One can also compute the cofinalities of all successor cardinals between $\boldsymbol{\delta}^1_4$ and λ_5. The result, from [13] is:

5.27 Theorem *Suppose* $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1} < \aleph_{\alpha+1} < \aleph_{\omega^{\omega^\omega}+1} = \boldsymbol{\delta}^1_5$. *Let* $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, *where* $\omega^\omega > \beta_1 \geq \cdots \geq \beta_n$ *be the normal form for* α. *Then:*

- *If* $\beta_n = 0$, *then* $\mathrm{cof}(\kappa) = \boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$.
- *If* $\beta_n > 0$, *and is a successor ordinal, then* $\mathrm{cof}(\kappa) = \aleph_{\omega\cdot 2+1}$.
- *If* $\beta_n > 0$ *and is a limit ordinal, then* $\mathrm{cof}(\kappa) = \aleph_{\omega^\omega+1}$.

Finally, one can extend the results of this section to all levels of the projective hierarchy. One first defines the measures W^m_{2n+1}, $S^{l,m}_{2n+1}$, for $m \in \omega$, $1 \leq l \leq 2^{n+1} - 1$, assuming the weak and strong partition relations on $\boldsymbol{\delta}^1_{2n+1}$ respectively (for $n = 0$, we set $S^m_1 = S^{1,m}_1$ to agree with our previous notation). Order these (to be defined) families of measures as: W^m_1, $S^{1,m}_1$, W^m_3, $S^{1,m}_3$, $S^{2,m}_3$, $S^{3,m}_3$, W^m_5, etc. W^m_{2n+1} is defined to be the measure on $\boldsymbol{\delta}^1_{2n+1}$ induced from the weak partition relation on $\boldsymbol{\delta}^1_{2n+1}$, functions $f\colon \mathrm{dom}(S^{l_0,m}_{2n-1}) \to \boldsymbol{\delta}^1_{2n+1}$, where $l_0 = 2^n - 1$, and the measure $S^{l_0,m}_{2n-1}$ on $\mathrm{dom}(S^{l_0,m}_{2n-1}) < \lambda_{2n+1}$. $S^{1,m}_{2n+1}$ is the measure on $(\boldsymbol{\delta}^1_{2n+1})^{+m}$ defined just as S^m_1 was, using $\boldsymbol{\delta}^1_{2n+1}$ in place of ω_1. For $l > 1$, $S^{l,m}_{2n+1}$ is the measure on $\mathrm{dom}(S^{l,m}_{2n+1}) < \lambda_{2n+3}$ induced by the strong partition relation on $\boldsymbol{\delta}^1_{2n+1}$, functions $f\colon \boldsymbol{\delta}^1_{2n+1} \to \boldsymbol{\delta}^1_{2n+1}$ of the correct type, and the measure ν on $\boldsymbol{\delta}^1_{2n+1}$. Here ν is the measure on $\boldsymbol{\delta}^1_{2n+1}$ induced by the weak partition relation on $\boldsymbol{\delta}^1_{2n+1}$, functions $h\colon \mathrm{dom}(\mu) \to \boldsymbol{\delta}^1_{2n+1}$, and the measure μ, where μ is the m^{th} measure in the $l-1^{st}$ family. In [8] it is shown that these measures dominate the general measures in the homogeneous trees on $\boldsymbol{\Pi}^1_{2n+1}, \boldsymbol{\Pi}^1_{2n+2}$-complete sets, in the sense of theorems 5.12, 5.15. The notion of a level n description is introduced there, and the analog of the main theorem, theorem 5.24 is proved (with a suitable generalization of the $\mathcal{L}$ operator). The ranks of these generalized $\mathcal{L}$ operations are also computed, giving the upper bounds for the $\boldsymbol{\delta}^1_{2n+1}$. As mentioned in the introduction, the result is $\boldsymbol{\delta}^1_{2n+1} \leq \aleph_{\omega(2n-1)+1}$, where $\omega(0) = 0$ and $\omega(n+1) = \omega^{\omega(n)}$. With the main theorem, the inductive step is then similar to that of §4 or [11] (see the forthcoming [7]). In particular, $\sup_n \boldsymbol{\delta}^1_n = \aleph_{\epsilon(0)}$, where $\epsilon(0) = \sup_m \omega(m)$. Also, the regular cardinals between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$ are given by the ultrapowers of $\boldsymbol{\delta}^1_{2n+1}$ by the normal measures on $\boldsymbol{\delta}^1_{2n+1}$, corresponding to the regular cardinals below $\boldsymbol{\delta}^1_{2n+1}$. Thus, there are $2^{n+1} - 1$ regular cardinals between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$.

In fact, these arguments extend with little modification up to $\boldsymbol{\delta}^1_{\omega_1} = \aleph_{\omega_1}$. Here $\boldsymbol{\delta}^1_\alpha$ is the supremum of the lengths of the $\boldsymbol{\Delta}^1_\alpha$ prewellorderings, where $\boldsymbol{\Sigma}^1_\alpha$ is the α^{th} pointclass closed under $\exists^{\omega^\omega}$ (so $\boldsymbol{\Sigma}^1_0 = \boldsymbol{\Sigma}^0_1$, and for limit α, $\boldsymbol{\delta}^1_\alpha = \sup_{\beta<\alpha} \boldsymbol{\delta}^1_\beta$). For there are no new measures on $\boldsymbol{\delta}^1_\alpha$ for α limit $< \omega_1$, and a coding of $\mathcal{P}(\boldsymbol{\delta}^1_\alpha)$ may be constructed trivially from codings of $\mathcal{P}(\boldsymbol{\delta}^1_\beta)$, $\beta < \alpha$. Also, the only normal measures on the $\boldsymbol{\delta}^1_{\alpha+2n+1}$ for limit α correspond to the fixed cofinalities below $\boldsymbol{\delta}^1_{\alpha+2n+1}$ (since there will be only countably many regular cardinals below $\boldsymbol{\delta}^1_{\alpha+2n+1}$). Again, the ultrapowers of the $\boldsymbol{\delta}^1_{\alpha+2n+1}$ by these normal measures, together with the $\boldsymbol{\delta}^1_{\alpha+2n+1}$ precisely constitute the regular cardinals below $\boldsymbol{\delta}^1_{\omega_1}$.

At $\aleph_{\omega_1}$, or at any limit Suslin cardinal δ of cofinality $> \omega$, the situation changes as there are, of course, new measures on δ. One can show directly here that the next Suslin cardinal after $\boldsymbol{\delta}^1_{\omega_1}$ has cardinality the supremum of the ultrapowers of $\boldsymbol{\delta}^1_{\omega_1}$ by the measures on $\boldsymbol{\delta}^1_{\omega_1}$. Actually, Martin has proved a general result which shows the same fact for any limit Suslin cardinal of cofinality $> \omega$ (or any successor Suslin cardinal as well). In unpublished work, the author has analyzed these measures and shown that the supremum of their ultrapowers is $\aleph_{\aleph_\omega}$. We will consider some of the problems associated with further extending the theory in the next section.

6. Global Results

We consider in this section some results and problems of a "global" nature, that is, related to the attempt to push the structural theory of $L(\mathbb{R})$ up through Θ. The problems we consider here (and in some cases solve) seem to be necessary for further extensions of the theory, but are almost certainly not sufficient. Identifying the remaining obstructions remains a central goal of this subject. Nevertheless, some of the results we mention in this section are of independent interest. We will not require any of the results of §§4, 5 for this section. We assume AD throughout this section, occasionally assuming $V = L(\mathbb{R})$ as well.

6.1. Generic Codes

Kechris and Woodin [22] have developed a theory of generic codes for uncountable ordinals which we will use in several of the arguments of this section. We will only need, however, the most basic lemma of their theory, the one asserting the existence of a generic coding function. For the sake of completeness, we give their proof of this result.

We say an ordinal α is *reliable* if there is a $P \subseteq \omega^\omega$ and a scale $\{\phi_n\}_{n\in\omega}$ with $\phi_n \colon P \to \alpha$ with ϕ_0 onto α. Every Suslin cardinal is easily reliable (c.f. lemma 4.6 of [37]), and in [37] it is shown from $\mathrm{AD} + V = L(\mathbb{R})$ that every reliable cardinal is a Suslin cardinal. Actually, using some additional arguments this can be shown to follow from just AD. There are, however, many reliable ordinals which are not cardinals, as, for example, the set of reliable ordinals is c.u.b. in every $\boldsymbol{\delta}^1_{2n+1}$. For the purposes of generic codes, it is convenient to slightly strengthen the definition of reliable to include the requirement that the scale relations $\leq^*_n$, $<^*_n$ are both Suslin and co-Suslin. This only has the effect of removing the largest Suslin cardinal, if there is one, from consideration. We henceforth officially adopt this stronger form of the definition.

If α is reliable, $S \in \mathcal{P}_{\omega_1}(\alpha)$, and $\beta \in S$, we say S is *β-honest* if there is an $x \in P$ with $\phi_0(x) = \beta$ and $\forall n\ \phi_n(x) \in S$ (this notion is defined relative to the choice of P and $\{\phi_n\}$). We say S is honest if it is β-honest for all $\beta \in S$. For $x \in P$ we frequently write $|x|$ for $\phi_0(x)$. If S is a countable set of ordinals, then S^ω (S having the discrete topology) is homeomorphic to ω^ω and so carries a natural notion of category. When we speak of meager or comeager, we are always referring to this topology. If $p \in S^{<\omega}$, we write $\forall^*_p s \in S^\omega$ to mean for comeager many s in the neighborhood $N_p = \{s \in S^\omega \colon s \restriction \mathrm{lh}(p) = p\}$. Throughout, we let $S(\kappa)$ denote the pointclass of κ-Suslin sets.

Recall that from AD, every set $A \subseteq S^\omega$ has the Baire property. In particular, A is either meager or else comeager on some neighborhood N_p. Also from AD, a well-ordered union of meager sets is meager ("additivity of category").

6.1 Theorem (Kechris-Woodin) *Let α be reliable, witnessed by P, $\{\phi_n\}$. There is Lipschitz continuous function $G \colon \alpha^\omega \to \omega^\omega$ satisfying:*

1. $\forall \vec{s} = (\alpha_0, \alpha_1, \ldots) \in \alpha^\omega \ \forall n \in \omega\ [G(\vec{s})_n \in P \wedge \phi_0(G(\vec{s})_n) \leq \alpha_n]$.
2. *If $\vec{s} = (\alpha_0, \alpha_1, \ldots) \in \alpha^\omega$ enumerates an honest set S, then for all $n \in \omega$, $\phi_0(G(\vec{s})_n) = \alpha_n$.*

Proof. Let T be the tree of the scale $\{\phi_n\}$, so T is a tree on $\omega \times \alpha$. For $\gamma < \alpha$, let

$$T_\gamma = \{(s, \vec{\alpha}) \in T : \alpha_0 \leq \gamma\}.$$

Consider the following ordinal game:

I	α_0		α_1		α_2		α_3		$\ldots$	
II		β_0		β_1		β_2		β_3		$\ldots$
		$x(0)$		$x(1)$		$x(2)$		$x(3)$		$\ldots$

Here $\alpha_i, \beta_i < \alpha$, $x(i) \in \omega$, so I plays out $\vec{\alpha} \in \alpha^\omega$, and II plays out $\vec{\beta} \in \alpha^\omega$ and $x \in \omega^\omega$. Let $S = \{\alpha_i \colon i \in \omega\}$. x codes reals $x_0, x_1, \ldots, x_i$ and $\vec{\beta}$ codes sequences $(\vec{\beta})_i \in \alpha^\omega$ in the usual manner, so $x_i(j) = x(\langle i, j \rangle)$, and $(\vec{\beta})_i(j) = \beta_{\langle i,j \rangle}$ (we assume $\langle i, j \rangle \geq i$ for all j, so II does not have to play any of the $x_i(j)$ or $(\vec{\beta})_i(j)$ until I has played α_i). II wins the run of the game iff

$$\forall i\ (x_i, \alpha_i{}^\frown(\vec{\beta})_i) \in [T] \wedge \forall i \forall y\ [y \in p[T_{\alpha_i} \restriction S] \to \phi_0(y) \leq \phi_0(x_i)].$$

Now I cannot have a winning strategy, for as soon as I plays α_i, II can pick some $x_i \in P$ with $\phi_0(x_i) = \alpha_i$, and pick $(\vec{\beta})_i$ with $(x_i, \alpha_i{}^\frown(\vec{\beta})_i) \in [T]$ and proceed to play these to defeat I's strategy. Thus, if the game is determined, then II has a winning strategy τ. Ignoring the ordinal moves of τ gives the function G as desired.

To show the game is determined, it is enough to observe that it is Suslin, co-Suslin by theorem 2.20. The first conjunct in the payoff definition trivially defines a Suslin, co-Suslin set (in fact, a closed set). For the second conjunct, note that

$$y \in p[T_{\alpha_i} \restriction S] \leftrightarrow \exists \pi \in \omega^\omega \; \forall n \; (y \restriction n, (\alpha_i, \alpha_{\pi(1)}, \dots, \alpha_{\pi(n-1)})) \in T.$$

From the closure of the Suslin sets under $\vee^\omega, \wedge^\omega, \exists^{\omega^\omega}$, and $\forall^{\omega^\omega}$ (the latter by the second periodicity theorem; see remark 2.10), it follows that this relation, and thus the second conjunct, is Suslin, co-Suslin.

⊣

If there is a largest Suslin cardinal Ξ, then theorem 6.1 does not immediately give a generic coding function $G\colon \Xi^\omega \to \omega^\omega$ at Ξ. In this case $\boldsymbol{\Gamma} = S(\Xi)$ will be a non-selfdual pointclass closed under real quantification, $\wedge$, $\vee$, and scale($\boldsymbol{\Gamma}$). The scale relations $\leq_n^*$, $<_n^*$ will not be co-Suslin, however. Nevertheless, we can argue that a generic coding function G still exists. To see this, fix a $\boldsymbol{\Gamma}$-scale $\{\phi_n\}_{n\in\omega}$ on a $\boldsymbol{\Gamma}$-complete set A. Without loss of generality, all of the norms ϕ_n are onto Ξ. Recall that Ξ is regular and a limit of Suslin cardinals. Also, the Suslin cardinals are c.u.b. in Ξ. By boundedness, there is a c.u.b. $C \subseteq \Xi$ such that for all $\alpha \in C$ and $\beta < \alpha$, if $B^i_\beta = \{x \in A\colon \phi_i(x) < \beta\}$, then $\sup\{\phi_j(x)\colon j \in \omega \wedge x \in B^i_\beta\} < \alpha$. Note that every Suslin cardinal in C is reliable with respect $\{\phi_n\}$. For every Suslin cardinal $\alpha \in C$, there is a generic coding function at α with respect to $A_\alpha = \{x \in A\colon \forall i \; \phi_i(x) < \alpha\}$ and the norms $\phi_i \restriction A_\alpha$ (from theorem 6.1). It suffices to show that we can get a function which to each such α assigns a generic coding function G_α with respect to A_α and the $\phi_i \restriction A_\alpha$. For then we can define $G\colon \Xi^\omega \to \omega^\omega$ by: $G(\alpha_0, \alpha_1, \dots) = z$ where for all j we have $(z)_j = (G_{\alpha'_j}(\beta_0, \beta_1, \dots))_j$ where α'_j is the least Suslin cardinal in C greater than α_j and $\beta_k = \alpha_k$ if $\alpha_k < \alpha'_j$ and otherwise $\beta_k = \alpha_j$. Using the definition of C it is not hard to check that G is a generic coding function (the point is that if S is α_j-honest then $S \cap \alpha'_j$ is α_j-honest). It remains to show that we can uniformly define the G_α for $\alpha \in C$ a Suslin cardinal. Let $\mathcal{G}_\alpha$ denote the generic coding game as in theorem 6.1 using A_α and the $\phi_i \restriction A_\alpha$. The payoff set for II is Suslin and co-Suslin, and is uniformly Suslin (but not uniformly co-Suslin). For all Suslin cardinals α in C, II has a winning strategy in $\mathcal{G}_\alpha$. From a Suslin representation for II's payoff set and the fact that II has a winning strategy, we can uniformly in α get a winning strategy for II in $\mathcal{G}_\alpha$, which then gives us the generic coding function G_α. In fact, either of the two proofs that Suslin, co-Suslin ordinal games are determined shows this (c.f. theorem 2.2 of [31] or theorem 2.5 of [20], also theorem 2 of [29]).

We are frequently only concerned with getting a real which codes the ordinal α_0. Thus, let $G_0\colon \alpha^\omega \to \omega^\omega$ be a Lipschitz continuous function so that $\forall \vec{s} \in \alpha^\omega \; G_0(\vec{s}) = G(\vec{s})_0$. The functions G_0, G are referred to as generic coding functions, and we fix them for the remainder of this section. Of course, these functions depend on the choice of the set P and the scale $\{\phi_n\}$, but we suppress writing this. Frequently, α will be a Suslin cardinal.

Recall theorem 2.22, according to which any game on α whose payoff depends only on $G(\vec{s})$ is determined (where $\vec{s}$ is the sequence I and II build, assuming now $V = L(\mathbb{R})$). Thus, for any game on α with payoff set $R \subseteq \alpha^\omega$, there is a determined game $R' \subseteq \alpha^\omega$ approximating R. Namely, define $R'(\vec{s}) \leftrightarrow R(|G(\vec{s})_0|, |G(\vec{s})_1|, \ldots)$. R' is always determined, and if $\vec{s}$ enumerates an honest set, then $R'(\vec{s}) \leftrightarrow R(\vec{s})$.

The generic coding functions are particularly useful when combined with the existence of supercompactness measures. Recall that from $\mathrm{AD} + V = L(\mathbb{R})$ there is a supercompactness measure (i.e., a fine, normal, countably additive ultrafilter) ν on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^2_1)$, which in turn induces one on $\mathcal{P}_{\omega_1}(\delta)$ for any $\delta \leq \boldsymbol{\delta}^2_1$. Woodin has shown ([40]) that the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\delta)$, for any $\delta \leq \boldsymbol{\delta}^2_1$ is unique. Woodin has also shown that here is a supercompactness measure on $\mathcal{P}_{\omega_1}(\lambda)$ for any $\lambda < \Theta$ assuming $\mathrm{AD} + V = L(\mathbb{R})$.

In fact, the existence of generic coding functions can be used to give a quick proof of the existence of the supercompactness measure on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^2_1)$, assuming $\mathrm{AD} + V = L(\mathbb{R})$. Recall $\boldsymbol{\delta}^2_1$ is a Suslin cardinal, and $S(\boldsymbol{\delta}^2_1) = \boldsymbol{\Sigma}^2_1$ has the scale property. Let $G\colon (\boldsymbol{\delta}^2_1)^\omega \to \omega^\omega$ be a generic coding function at $\boldsymbol{\delta}^2_1$ (see the remarks after the proof of theorem 6.1). If $A \subseteq \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^2_1)$, define $\nu(A) = 1$ iff II has a winning strategy in the game G_A: I and II alternate playing $\alpha_0, \alpha_1, \ldots$ building $\vec{s} \in \lambda^\omega$, and II wins iff $S = \{|G(\vec{s})_0|, |G(\vec{s})_1|, \ldots\} \in A$. It is not hard to check that this defines a fine, normal measure on $\mathcal{P}_{\omega_1}(\lambda)$, using standard dovetailing arguments and the fact that either player can play to ensure S is honest. Alternatively, one can argue just using theorem 6.1 as follows. Let μ be a normal measure on $\boldsymbol{\delta}^2_1$ (the proof that the $\boldsymbol{\delta}^1_{2n+1}$ are measurable works for $\boldsymbol{\delta}^2_1$). For every reliable $\lambda < \boldsymbol{\delta}^2_1$ there is a generic coding function $G\colon \lambda^\omega \to \omega^\omega$ from 6.1, and this gives a supercompactness measure on $\mathcal{P}_{\omega_1}(\lambda)$ as above. By Woodin's theorem, this supercompactness measure on $\mathcal{P}_{\omega_1}(\lambda)$ is unique, call it ν_λ. A supercompactness measure on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^2_1)$ can then be defined by $\nu(A) = 1$ iff $\forall^*_\mu \lambda\ \forall^*_{\nu_\lambda} S \in \mathcal{P}_{\omega_1}(\lambda)\ S \in A$.

As an example of using generic coding arguments, we prove, following [12], the following theorem.

6.2 Theorem *Let κ be a regular Suslin cardinal less than the supremum of the Suslin cardinals. Let $\lambda < \Theta$ be a cardinal with $j_{\nu_\alpha}(\lambda) = \lambda$ for all $\alpha < \kappa$, where ν_α is the supercompactness measure on $\mathcal{P}_{\omega_1}(\alpha)$. Then* $\mathrm{cof}(\lambda^+) > \kappa$.

6.3 Corollary *If $\omega_1 < \lambda^+ < \Theta$, then* $\mathrm{cof}(\lambda^+) > \omega_1$.

Proof. Fix κ, λ as above, and assume $f\colon \kappa \to \lambda^+$ is cofinal. For $\alpha < \kappa$, let $\alpha' < \kappa$ denote the least reliable ordinal $> \alpha$ relative to the scale used in constructing the generic coding function G_0 for κ (it is not hard to see that $\alpha' < \kappa$ using the regularity of κ). Consider the game where I, II play $\alpha_0, \alpha_1, \ldots$ building $\vec{s}$, and II plays also $x(0), x(1), \ldots \in \omega$ building $x \in \omega^\omega$. II wins iff x codes a well-ordering of λ of length $\geq f(\phi_0(G_0(\vec{s})))$. Here we code subsets of λ by reals in some manner which is not important, say by the coding lemma. I cannot have a winning strategy, for as soon as I plays α_0, II can enumerate an honest set containing α_0 and closed under I's winning strategy, and play some x coding a well-ordering of λ of length $\geq f(\alpha_0)$.

A winning strategy τ for II gives (ignoring II's ordinal moves) a Lipschitz continuous $\mathcal{F} \colon \kappa^\omega \to \omega^\omega$ such that

1. For all $\vec{s} \in \kappa^\omega$, $\mathcal{F}(\vec{s})$ codes a well-ordering of λ.

2. For all $\alpha < \kappa$ and all $\vec{s}$ enumerating an honest S containing α, $\mathcal{F}(\alpha^\frown \vec{s})$ codes a well-ordering of λ of length $\geq f(\alpha)$.

Let $|\beta|_x$ denote the rank of β in the well-ordering coded by x.

Fix for the moment $\alpha < \kappa$ and an honest set $S \in \mathcal{P}_{\omega_1}(\alpha')$ containing α. Define a tree T on $S^{<\omega} \times \lambda$ as follows. Put $((p_0, \beta_0), \ldots, (p_n, \beta_n))$ in T iff

1. For all $i < n$, p_{i+1} extends p_i.

2. $\forall^*_{p_{i+1}} \vec{s} \in S^\omega \ |\beta_{i+1}|_{\mathcal{F}(\alpha^\frown \vec{s})} < |\beta_i|_{\mathcal{F}(\alpha^\frown \vec{s})}$.

3. $\forall i \leq n \ \exists \eta_i \in \mathrm{ON} \ \forall^*_{p_i} \vec{s} \in S^\omega \ |\beta_i|_{\mathcal{F}(\alpha^\frown \vec{s})} = \eta_i$.

The last clause guarantees that T is well-founded, as the η_i are decreasing along any branch. We define an order-preserving map π from the tree of the ϵ relation on $f(\alpha)$ into T. Suppose

$$\pi(\gamma_0, \gamma_1, \ldots, \gamma_n) = ((p_0, \beta_0), \ldots, (p_n, \beta_n))$$

has been defined, and assume inductively that

$$\forall^*_{p_n} \vec{s} \in S^\omega \ |\beta_n|_{\mathcal{F}(\alpha^\frown \vec{s})} = \eta_n \geq \gamma_n.$$

If $\gamma_{n+1} < \gamma_n$, then let $\pi(\gamma_0, \ldots, \gamma_{n+1})$ be the least sequence

$$((p_0, \beta_0), \ldots, (p_{n+1}, \beta_{n+1}))$$

extending $((p_0, \beta_0), \ldots, (p_n, \beta_n))$ such that

$$\forall^*_{p_{n+1}} \vec{s} \in S^\omega \ |\beta_{n+1}|_{\mathcal{F}(\alpha^\frown \vec{s})} = \eta_{n+1} \geq \gamma_{n+1} \text{ and } \eta_{n+1} < \eta_n.$$

The additivity of category shows that $p_{n+1}, \beta_{n+1}, \eta_{n+1}$ exist.

The definition of the tree $T = T^\alpha_S$ is uniform is α, S. Let now $F(\alpha) = [S \to T^\alpha_S]_{\nu_{\alpha'}}$. $F(\alpha)$ may be viewed as a well-ordering of $j_{\nu_{\alpha'}}(\lambda) = \lambda$, and clearly $|F(\alpha)| \geq f(\alpha)$. This gives a well-ordering of λ of length λ^+, a contradiction. ⊣

The analog of corollary 6.3 with $\kappa = \boldsymbol{\delta}^1_{2n+1}$ replacing ω_1 was shown by Kechris and Woodin to hold for λ below the supremum of the projective ordinals, i.e., $\lambda < \aleph_{\epsilon(0)}$ (this also follows from the projective hierarchy analysis, c.f. theorem 5.27). Along with 6.3, this suggests the following conjecture:

6.4 Conjecture *If κ is a regular Suslin cardinal, and $\kappa < \lambda^+ < \Theta$, then* $\mathrm{cof}(\lambda^+) > \kappa$.

6.2. Weak Square and Uniform Cofinalities

One of the important ingredients in the projective hierarchy analysis is the analysis of uniform cofinalities. In the general step, one has the notion of a type-n tree of uniform cofinalities $\mathcal{R}$ with associated measure $M^{\mathcal{R}}$, and it is necessary to analyze the possible uniform cofinalities with respect to these measures. Consider then the general question: given a measure ν on κ, what are the possible uniform cofinalities of a function $f\colon \kappa \to \lambda \in \mathrm{ON}$ with respect to ν? One possibility is that for some function g from κ to the regular cardinals we have $\forall^*_\nu \alpha\ f(\alpha)$ has uniform cofinality $g(\alpha)$. We call these the trivial uniform cofinalities. What are the possible non-trivial uniform cofinalities? Suppose, for example, $\kappa = \omega_2$, $\nu = S^1_1$, and $\lambda \in \mathrm{ON}$. Are the possible non-trivial uniform cofinalities for $f\colon \omega_2 \to \lambda$ the same as for $f\colon \omega_2 \to \boldsymbol{\delta}^1_3$? Intuitively, it seems as though the possible non-trivial uniform cofinalities should depend only on κ (and ν), and not on λ (a phenomenon reminiscent of the coding lemma). For small λ, one can extend the projective hierarchy arguments directly to answer this question, but for large λ, such an inductive approach does not seem to help.

From these considerations the author formulated a combinatorial principle called $\boxminus_{\kappa,\lambda}$. In fact, this principle also arose independently from attempts to extend some joint work with Becker ([2], [6]). The statement of the principle follows.

6.5 Definition *Let κ, $\lambda < \Theta$. $\boxminus_{\kappa,\lambda}$ is the assertion that for all $f\colon \kappa \to \lambda$ such that* $\mathrm{cof}(f(\alpha)) \leq \kappa$ *for all $\alpha < \kappa$, there is an $A \subseteq \lambda$ of size $\leq \kappa$ such that for all $\alpha < \kappa$, $A \cap f(\alpha)$ is cofinal in $f(\alpha)$.*

Thus, $\boxminus_{\kappa,\lambda}$ can be viewed as a choice principle. The principle can be stated for any cardinal κ, but for non-Suslin κ can fail. For example, it is not difficult to see that $\boxminus_{\omega_2,\omega_3}$ fails (using just that $\mathrm{cof}(\omega_3) = \omega_2$). In [12], the following theorem is proved.

6.6 Theorem $(\mathrm{AD} + V = L(\mathbb{R}))$ *$\boxminus_{\kappa,\lambda}$ holds for any Suslin cardinal κ and any $\lambda < \Theta$.*

This theorem provides a positive answer to the question on uniform cofinalities asked above. Specifically, we have the following.

6.7 Theorem ($\mathrm{AD} + V = L(\mathbb{R})$) *Let κ be a Suslin cardinal, μ a measure on κ, $\lambda < \Theta$, and $f\colon \kappa \to \lambda$. Then one of the following holds.*

1. $\forall^*_\mu \alpha \ \operatorname{cof}(f(\alpha)) \leq \kappa$. *Then the uniform cofinality of f with respect to μ is realized by a function $f'\colon \kappa \to \kappa$.*
2. $\forall^*_\mu \alpha \ \operatorname{cof}(f(\alpha)) > \kappa$. *Let $g(\alpha) = \operatorname{cof}(f(\alpha))$. Then there is an h with domain $\{(\alpha, \beta)\colon \alpha < \kappa \wedge \beta < g(\alpha)\}$ such that $h(\alpha, \beta) < f(\alpha)$ and $\forall^*_\mu \alpha \ f(\alpha) = \sup\{h(\alpha, \beta)\colon \beta < g(\alpha)\}$.*

We require a preliminary lemma. Throughout, ν denotes the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$.

6.8 Lemma *Let κ be a Suslin cardinal and $\lambda \in ON$ with $\operatorname{cof}(\lambda) > \kappa$. Suppose $F\colon \mathcal{P}_{\omega_1}(\kappa) \to \lambda$. Then $\exists \delta < \lambda \ \forall^*_\nu S \ F(S) < \delta$.*

Proof. Fix an $S(\kappa)$-bounded prewellordering (C, ψ) of length λ according to theorem 2.25. Play the game where I plays $\alpha_0, \alpha_2, \dots$, II plays $\alpha_1, \alpha_3, \dots$ and $x(0), x(1), \dots \in \omega$, and II wins iff $x \in C$ and $\psi(x) \geq F(S)$, where $S = \{|G(\vec{s})_0|, |G(\vec{s})_1|, \dots\}$ and $\vec{s} = (\alpha_0, \alpha_1, \dots)$. The game is determined and easily I cannot win. A winning strategy for II gives a Lipschitz continuous function $\mathcal{F}\colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that $\forall \vec{s} \in \kappa^\omega \ \mathcal{F}(\vec{s}) \in C$ (ignoring II's ordinal moves in computing $\mathcal{F}(\vec{s})$), and for all $\vec{s}$ enumerating an honest set S closed under $\mathcal{F}$, $\psi(\mathcal{F}(\vec{s})) \geq F(S)$. Let $w \in B \leftrightarrow \exists \vec{s} \in \kappa^\omega \ w = \mathcal{F}(\vec{s})$. $B \subseteq C$ and is κ-Suslin, so $\delta = \sup\{\phi(w)\colon w \in B\} < \lambda$ (to see B is κ-Suslin, note that the Lipschitz continuous $\mathcal{F}$ can be coded by the coding lemma with the pointclass $S(\kappa)$). Thus, $\forall^*_\nu S \ F(S) < \delta$. $\dashv$

Proof of theorem 6.7. Assume first $\forall^*_\mu \alpha \ \operatorname{cof}(f(\alpha)) \leq \kappa$. By $\boxminus_{\kappa,\lambda}$, let $A \subseteq \lambda$ have size κ such that $\forall^*_\mu \alpha$ ($A \cap f(\alpha)$ is cofinal in $f(\alpha)$). Taking the transitive collapse of A, we may assume that $\lambda < \kappa^+$. Let $\prec$ be a well-ordering of κ of length $> \lambda$. For $\alpha < \kappa$, let $R(\alpha) \leq \kappa$ be least such that $\sup\{|\beta|_\prec\colon \beta < R(\alpha) \wedge |\beta|_\prec < f(\alpha)\} = f(\alpha)$. For $\beta < R(\alpha)$, let $l(\alpha, \beta) = |\beta|_\prec$ if $|\beta|_\prec < f(\alpha)$, and 0 otherwise. R, l provide a liftup to f, as in the proof of lemma 4.19. The uniform cofinality of f with respect to μ is the same as that of R.

Assume now $\forall^*_\mu \alpha \ \operatorname{cof}(f(\alpha)) > \kappa$. Let $g(\alpha) = \operatorname{cof}(f(\alpha))$. The game argument above produces a Lipschitz continuous $\mathcal{F}\colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that for all $\alpha < \kappa$, and all $\vec{s} \in \kappa^\omega$ enumerating an honest set containing α and closed under $\mathcal{F}$, $\mathcal{F}(\alpha^\frown s)$ codes (ignoring II's ordinal moves) an increasing $g(\alpha)$ sequence cofinal in $f(\alpha)$. The exact manner in which reals code $g(\alpha)$ sequences below λ is not important, say by the coding lemma with respect to a suitably large pointclass.

Fix for the moment $\alpha < \kappa$ and an honest S containing α and closed under $\mathcal{F}$. For $p \in S^{<\omega}$ and $\beta < g(\alpha)$ define $h(\alpha, \beta, S, p)$ to be the least $\gamma < f(\alpha)$ such that $\forall^*_p s \in S^\omega\ \mathcal{F}(\alpha^\frown s)(\beta) < \gamma$ if one exists, and 0 otherwise. Define $h(\alpha, \beta, S) = \sup_{p \in S^{<\omega}} h(\alpha, \beta, S, p)$. Clearly $h(\alpha, \beta, S) < f(\alpha)$. If $\gamma < f(\alpha)$, then by additivity of category there is a $p \in S^{<\omega}$, a $\beta < g(\alpha)$, and a $\eta < f(\alpha)$ such that $\eta > \gamma$ and

$$\forall^*_p s \in S^\omega\ \mathcal{F}(\alpha^\frown s)(\beta) = \eta.$$

Thus, $h(\alpha, \beta, S, p) > \gamma$. Hence, $f(\alpha) = \sup_{\beta < g(\alpha)} h(\alpha, \beta, S)$. Also, an easy argument shows that $h(\alpha, \beta, S)$ is monotonically increasing in β. Define

$$h(\alpha, \beta) = \text{ the least } \delta < f(\alpha) \text{ such that } \forall^*_\nu S \in \mathcal{P}_{\omega_1}(\kappa)\ h(\alpha, \beta, S) < \delta.$$

By lemma 6.8, this is well-defined. Fix now $\alpha < \kappa$, and suppose towards a contradiction that $\rho \doteq \sup_{\beta < g(\alpha)} h(\alpha, \beta) < f(\alpha)$. We have

$$\forall^*_\nu S\ \exists \beta < g(\alpha)\ [h(\alpha, \beta, S) > \rho].$$

By lemma 6.8 and monotonicity, $\exists \beta_0 < g(\alpha)\ \forall^*_\nu S\ [h(\alpha, \beta_0, S) > \rho]$. However, $\forall^*_\nu S\ h(\alpha, \beta_0, S) \leq h(\alpha, \beta_0) \leq \rho$. ⊣

Theorem 6.6 has other applications as well. For example, in [12] it is used to show the following.

6.9 Theorem ($\mathrm{AD} + V = L(\mathbb{R})$) *Let κ be a regular cardinal which is either a Suslin cardinal or the successor of a Suslin cardinal. Then κ is $\boldsymbol{\delta}^2_1$-supercompact.*

6.10 Corollary ($\mathrm{AD} + V = L(\mathbb{R})$) *All the projective ordinals $\boldsymbol{\delta}^1_n$ are $\boldsymbol{\delta}^2_1$-supercompact.*

Solovay [34] first showed, assuming $\mathrm{AD}^{\mathbb{R}}$, that $\boldsymbol{\delta}^1_1 = \omega_1$ is λ-supercompact for all $\lambda < \Theta$. The work of Martin-Steel [29] and Harrington-Kechris [5] showed that $\boldsymbol{\delta}^1_1$ is $\boldsymbol{\delta}^2_1$-supercompact from just AD. The $\kappa = \boldsymbol{\delta}^1_2$ case of corollary 6.10 is due to Becker [1]. As with ω_1, it is open whether such κ are λ-supercompact for all $\lambda < \Theta$.

One of the main ideas in the proof of theorem 6.6 involves combining certain category methods with the generic coding arguments. We will not prove theorem 6.6 in detail here. Rather, we present a result whose proof uses the same idea.

We fix some notation for the remainder of this section. We assume $\mathrm{AD} + V = L(\mathbb{R})$. κ will henceforth denote a Suslin cardinal, and ν the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$. From the scale analysis in $L(\mathbb{R})$ (see

[37]) there is a κ-Suslin set P and a κ-Suslin scale $\{\phi_n\}$ on P with ϕ_0 onto κ (for κ below the supremum of the projective ordinals, that is, $\kappa = \boldsymbol{\delta}^1_{2n+1}$ or $\kappa = (\boldsymbol{\delta}^1_{2n+1})^-$, this is immediate). We write $|x|$ for $\phi_0(x)$, for $x \in P$. The generic coding functions G_0, G are defined relative to $P, \{\phi_n\}$, and are henceforth fixed. The pointclass $S(\kappa)$ of κ-Suslin sets is closed under $\exists^{\omega^\omega}$, so by the coding lemma we may code subsets of κ within the pointclass $S(\kappa)$. In particular, strategies (Lipschitz continuous functions) may be coded within $S(\kappa)$. If $\tau \in \omega^\omega$ codes a strategy, we also write τ for the strategy it codes. Thus, if τ codes a strategy $\tau\colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$, the relation $\tau(\alpha_1, \dots, \alpha_n) = ((\beta_0, \dots, \beta_n), (a_0, \dots, a_n))$ is κ-Suslin in the codes (with respect to ϕ_0). For any other object we need to code by reals, the exact manner in which we do so is not important, say by using the coding lemma with respect to some sufficiently large pointclass.

Suppose $\tau\colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$ is a strategy, and $\vec{s} = (\alpha_0, \alpha_2, \dots) \in \kappa^\omega$. Let $(\alpha_1, \alpha_3, \dots)$ be the ordinal part of τ's response, and $x = (x(0), x(1), \dots)$ the integer part. Let $S = \{\alpha_0, \alpha_1, \dots\} \in \mathcal{P}_{\omega_1}(\kappa)$. We say x codes a comeager set $A \subseteq S^\omega$ and a continuous function $f\colon A \to \omega^\omega$ provided x_0 codes the comeager set A, and x_1 the continuous function f as follows. x_0 codes A by having each $(x_0)_n$ code a dense open $D_n \subseteq S^\omega$ such that $A = \bigcap_n D_n$. To say y codes the dense open set $D \subseteq S^\omega$ means each $y(k)$ codes a sequence $u_k \in \omega^{<\omega}$, and $D = \bigcup_k N_{u_k^*}$, where if $u_k = (a_0, \dots, a_l)$, then $N_{u_k^*}$ is the basic open set in S^ω determined by the sequence $u_k^* = (\alpha_{a_0}, \dots, \alpha_{a_l})$. Likewise, x_1 codes f by coding a sequence of tuples of integers $(a_0, \dots, a_l, b_0, \dots, b_m)$, where for $u = (a_0, \dots, a_l)$ coding a basic open set in D_k we have $m \geq k$ and for $s \in A \cap N_{u^*}$, $f(s)$ extends $(b_0, \dots, b_m)$. It is easy to see that for a fixed enumeration $\alpha_0, \alpha_1, \dots$ of a set S, the set of x coding a comeager set and a continuous function on S^ω is $\boldsymbol{\Pi}^0_2$.

If $\tau\colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$ is a strategy and $\vec{s} \in \kappa^\omega$, we usually write $\tau(\vec{s})$ to denote the real obtained as the integer moves of τ against $\vec{\alpha}$.

6.11 Theorem ($\mathrm{AD} + V = L(\mathbb{R})$) *Let κ be a Suslin cardinal, and ν the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$. Suppose $F\colon \mathcal{P}_{\omega_1}(\kappa) \to \lambda < \Theta$. Then* $\mathrm{cof}([F]_\nu) > \kappa$ *iff* $\forall^*_\nu S \in \mathcal{P}_{\omega_1}(\kappa)\ [\mathrm{cof}(F(S)) > \omega]$.

The full proof of this theorem can be found in [12]. We will prove here a somewhat weaker version which still suffices to illustrate the main idea used in the proof of 6.6. Specifically, we show here that

1. If $\forall^*_\nu S\ [\mathrm{cof}(F(S)) \leq \omega]$ then $\mathrm{cof}([F]_\nu) \leq \kappa$.
2. If $\forall^*_\nu S\ [\mathrm{cof}(F(S)) > \kappa]$, then $\mathrm{cof}([F]_\nu) > \kappa$.

The proof of the full theorem in [12] uses these ideas plus also the Becker-Kechris method used in proving the invariance of $L[T_{2n+1}]$ (see [3]).

Proof. Suppose first that $\forall^*_\nu S\ \operatorname{cof}(F(S)) \leq \omega$. Play the game where I plays $\alpha_0, \alpha_2, \ldots$, II plays $\alpha_1, \alpha_3, \ldots$ and $x(0), x(1), \in \omega$, and II wins iff x codes an ω sequence of ordinals cofinal in $F(S')$, where $S' = \{|G(\vec{s})_0|, |G(\vec{s})_1|, \ldots\}$, and $\vec{s} = (\alpha_0, \alpha_1, \ldots)$. II has a winning strategy, since II can defeat any strategy for I by enumerating an honest set S closed under I's strategy, and playing an x coding an ω sequence cofinal in $F(S)$. A winning strategy for II gives a Lipschitz continuous function $\mathcal{F}\colon \kappa^\omega \to \omega^\omega$ (ignoring the ordinal moves) such that for all $\vec{s} \in \kappa^\omega$ $\mathcal{F}(\vec{s})$ codes an ω sequence of ordinals $|\mathcal{F}(\vec{s})_0|, |\mathcal{F}(\vec{s})_1|, \ldots$, and for all $\vec{s}$ enumerating an honest set S closed under $\mathcal{F}$, $\sup_i |\mathcal{F}(\vec{s})_i| = F(S)$.

For $S \in \mathcal{P}_{\omega_1}(\kappa)$ honest and closed under $\mathcal{F}$, define $G(S) \subseteq F(S)$ by: $G(S) = \{G_{n,p}(S)\colon n \in \omega, p \in S^{<\omega}\}$, where $G_{n,p}(S) \doteq$ the least ordinal $\beta < F(S)$ such that $\forall^*_p \vec{s} \in S^\omega\ |\mathcal{F}(\vec{s})_n| = \beta$ if such an ordinal exist, and $G_{n,p}(S) = 0$ otherwise. The additivity of category shows that $G(S)$ is cofinal in $F(S)$. Thus, $[G]_\nu$ is cofinal in $[F]$, and by normality, $[G]_\nu$ is a set of size $\leq \kappa$. This shows the first claim.

Suppose now $\forall^*_\nu S\ \operatorname{cof}(F(S)) > \kappa$. Suppose towards a contradiction that $\operatorname{cof}([F]_\nu) \leq \kappa$, and let $B \subseteq [F]_\nu$ be cofinal with $|B| \leq \kappa$. By the usual game argument as above and theorem 2.25, there is a Lipschitz continuous $\mathcal{F}\colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that for all $\vec{s} \in \kappa^\omega$, $u \doteq \mathcal{F}(\vec{s})$ codes a $S(\kappa)$-bounded prewellordering (C_u, ψ_u) of some limit length which we denote by $|u|$, and for all $\vec{s}$ enumerating an honest set S closed under $\mathcal{F}$, $|u| = F(S)$.

If $\beta < [F]_\nu$, we say a real z is β-good if:

1. z codes a Lipschitz continuous function $z\colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that if $s_0 \in \kappa^\omega$ enumerates an honest set S closed under z and $\mathcal{F}$, then $z(s_0)$ codes a comeager set $A_{z(s_0)} \subseteq S^\omega$, and a continuous function $z(s_0, -)\colon A_{z(s_0)} \to \omega^\omega$ such that for all $s_1 \in A_{z(s_0)}$, $w \doteq z(s_0, s_1)$ is in the $S(\kappa)$-bounded union (C_u, ψ_u) coded by $u \doteq \mathcal{F}(s_1)$.

2. $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ \forall^* s_0\ \forall^* s_1 \in S^\omega$ the rank of $w = z(s_0, s_1)$ in the $S(\kappa)$-bounded union coded by $u = \mathcal{F}(s_1)$ is greater than $\beta(S)$. (Recall $S \to \beta(S)$ represents β).

We first claim that for all $\beta < [F]_\nu$ there is a $z \in \omega^\omega$ such that z is β-good. To see this, fix a function $S \to \beta(S)$ representing β with respect to ν, and play the game where I plays $\alpha_0, \alpha_2, \ldots$, II plays $\alpha_1, \alpha_3, \ldots$, and $x(0), x(1), \ldots$, and II wins iff x codes a comeager set $A_x \subseteq S^\omega$, where $S = \{|G(\vec{s})_0|, |G(\vec{s})_1|, \ldots\}$, $\vec{s} = (\alpha_0, \alpha_1, \ldots)$, and a continuous function $x(-)\colon A_x \to \omega^\omega$ such that for all $s_1 \in A_x$, $x(s_1) = \overline{0}$ if $\mathcal{F}(s_1) = u$ does not code $F(S)$, and otherwise the rank of $x(s_1)$ in the $S(\kappa)$-bounded union coded by u is $> \beta(S)$. This game is again determined. Suppose I won by σ. Let S be an honest set closed under σ and $\mathcal{F}$, and such that $\operatorname{cof}(F(S)) > \kappa$, and $F(S) > \beta(S)$. II will enumerate S in the $\alpha_1, \alpha_3, \ldots$. Let $R \subseteq S^\omega \times \omega^\omega$

be defined by: $R(s_1, w) \leftrightarrow [s_1$ enumerates $S \wedge w \in C_u$ where $u = \mathcal{F}(s_1) \wedge \psi_u(w) \geq \beta(S)]$. From AD, we may uniformize R by R' on a comeager set. Also, every function defined on a comeager subset of S^ω is continuous restricted to a comeager set. Let $x \in \omega^\omega$ code such a comeager set $A_x \subseteq S^\omega$ (coding neighborhoods using only the $\alpha_1, \alpha_3, \ldots$) and continuous function $x(-)\colon A_x \to \omega^\omega$. If II plays this x, then II defeats I. A winning strategy τ for II then gives a β-good real.

By the coding lemma, there is a $S(\kappa)$ set $C \subseteq \omega^\omega$ such that $\forall \tau \in C$ $\exists \beta < [F]_\nu$ τ is β-good, and $\forall \beta \in B$ $\exists \tau \in C$ τ is β-good. We define now $G\colon \mathcal{P}_{\omega_1}(\kappa) \to \mathrm{ON}$ such that $[G]_\nu < [F]_\nu$ but $\forall \beta \in B$ $\forall^*_\nu S$ $G(S) > \beta(S)$, a contradiction.

Let S be honest and closed under $\mathcal{F}$. Let $G(S)$ be the least $\alpha \in \mathrm{ON}$ such that $\forall^* s_1 \in S^\omega$ $G(S, s_1) < \alpha$, where $G(S, s_1)$ is defined as follows. Let $u = \mathcal{F}(s_1)$, so (C_u, ψ_u) is a $S(\kappa)$-bounded prewellordering of length $F(S)$. Set $G(S, s_1) = \sup\{\psi_u(w)\colon w \in B_{s_1}\}$, where

$$w \in B_{s_1} \leftrightarrow \exists s_0 \text{ enumerating } S\ \exists \tau \in C\ [(S \text{ is closed under } \tau) \wedge (s_1 \in A_{\tau(s_0)}) \wedge (w = \tau(s_0, s_1))].$$

Easily, $B_{s_1} \in S(\kappa)$. Also, $B_{s_1} \subseteq C_u$, and so by boundedness, $G(S, s_1) < F(S)$ for all $s_1 \in S^\omega$. By additivity of category, $G(S) < F(S)$. Thus, $[G]_\nu < [F]_\nu$.

Fix now $\beta \in B$ and a function $S \to \beta(S)$ representing β, and let $\tau \in C$ be β-good. Let S be honest, closed under $\mathcal{F}$ and τ, and such that (2) in the definition of β-good above holds for S for this τ. We show that $G(S) > \beta(S)$, a contradiction. It is enough to show that $\forall^* s_1 \in S^\omega$ $G(S, s_1) > \beta(S)$. Fix s_0 enumerating S so that the remaining clause in (2) of β-good is satisfied. If $s_1 \in A_{\tau(s_0)}$ and $w = \tau(s_0, s_1)$, then $w \in B_{s_1}$ (using τ and s_0 as witnesses) and so $G(S, s_1) \geq \psi_u(w)$, where $u = \mathcal{F}(s_1)$. On the other hand, from the choice of S, s_0 and (2) of β-good we have $\forall^* s_1$ $\psi_u(w) > \beta(S)$. Thus, $\forall^* s_1$ $G(S, s_1) > \beta(S)$. ⊣

6.3. Some Final Remarks

We close this paper with some final (somewhat tentative) thoughts on extending the structural theory throughout $L(\mathbb{R})$. The analysis, of course, is inductive, and proceeds by induction on the Suslin cardinals. As we remarked earlier, the arguments of §§4, 5 should provide the necessary ingredients at successor Suslin cardinals. At singular Suslin cardinals δ, theorems 6.6, 6.7, and similar results should provide a basis for the analysis. Aside from providing an analysis of the uniform cofinalities (theorem 6.7), these techniques should provide a method for "gluing together" the description analyses at the lower Suslin cardinals to obtain one at δ (theorem 6.7

may be viewed as a simple case of this; it shows how to glue together the pointwise cofinalities below δ to obtain a uniform cofinality).

At inaccessible Suslin cardinals δ, one gets some facts for "free", such as the strong partition relation on δ (see [20]). However, it still seems necessary to analyze the measures on δ to permit analysis at the next Suslin cardinal. One of the main problems here, as we mentioned earlier, is analyzing the semi-normal measures on δ (which is where the first step in the "pressing down" analysis of the measures leaves one; see the proof of theorem 4.8). The normal measures on δ corresponding to fixed cofinalities $\kappa < \delta$ seem well-behaved (for example, we get theorem 5.10), but there are, in general, many more semi-normal measures on δ.

Let $S \subseteq \delta$ be a "thin" stationary set. By thin we mean for all $\alpha \in S$, $S \cap \alpha$ is not stationary in α. Then the c.u.b. filter restricted to S defines a normal measure μ_S on δ. We refer to this as the atomic normal measure corresponding to S. This is shown using the strong partition relation on δ. For example, to see this defines an ultrafilter, for $A \subseteq \delta$ consider the partition of $f\colon \delta \to \delta$ of the correct type according to whether $\alpha(f, S) \in A$, where $\alpha(f, S) =$ the least limit point of $\mathrm{ran}(f)$ in S.

Given thin stationary sets S_1, S_2, define $S_1 \prec S_2$ iff there is a c.u.b. $C \subseteq \delta$ such that for all $f\colon \delta \to C$ of the correct type, $\alpha(f, S_1) < \alpha(f, S_2)$. The strong partition relation on δ shows that $\prec$ is a well-ordering on equivalence classes $[S]$, where $S \sim T$ iff there is a c.u.b. $C \subseteq \delta$ such that $S \cap C = T \cap C$. Equivalently, $S_1 \prec S_2$ iff there is a c.u.b. $C \subseteq \delta$ such that for all $\alpha \in C \cap S_2$, S_1 is stationary in α. Let $o(\delta)$ denote the rank of this prewellordering, and $o(S)$ the rank of S in $\prec$ (this forms a generalized notion of Mahlo rank; δ is inaccessible if $o(\delta) \geq \delta$, Mahlo if $o(\delta) \geq \delta + 1$, etc.). Note that the atomic normal measure corresponding to S depends only on $[S]$.

If $o(\delta)$ is fairly small compared with δ, we can transfer a semi-normal measure μ on δ onto a smaller ordinal, and thereby begin to analyze μ. Suppose, for example, $o(\delta) = \delta + \omega_1$, and μ concentrates on inaccessible cardinals. A generic coding argument, which we omit, shows that we may pick thin stationary sets S_α, $\alpha < \omega_1$ which are pairwise disjoint and $o(S_\alpha) = \delta + \alpha$. For μ almost all $\alpha < \delta$, let $\alpha \in S_{f(\alpha)}$. Then $\nu = f(\mu)$ is a measure on ω_1. Let μ' be the measure obtained by integrating the μ_S with respect to ν, that is, $\mu'(A) = 1$ iff $\forall^*_\nu \alpha < \omega_1\ \forall^*_{\mu_{S_\alpha}} \beta\ (\beta \in A)$. It is then not hard to see that $\mu = \mu'$. Thus we have analyzed the semi-normal measures on δ.

One can attempt to extend these arguments to larger values of $o(\delta)$. Suppose, for example, that $o(\delta) = j_{\nu_\omega}(\delta)$, where ν_ω denotes the ω-cofinal normal measure on δ. Using the strong partition relation on δ, ν_ω induces a measure $\mathcal{V}$ on $j_{\nu_\omega}(\delta)$ (using functions of the correct type, say). Integrating the $\nu_{[S_\alpha]}$ using $\mathcal{V}$ produces a semi-normal measure ν on δ (there is a problem now in trying to pick representatives S_α for the equivalence classes; the measure $\nu_{[S_\alpha]}$, however, is still well-defined). In fact, if $o(\delta)$ is large enough we may lift an arbitrary measure μ on δ to a new measure ν on δ in this manner (using μ in place of ν_ω). It seems reasonable (but not clear) that one might reverse the above process, and thus reduce the semi-normal measure ν on δ to a "smaller" measure μ on δ and proceed inductively. The measure is smaller in the following sense.

6.12 Lemma *Let μ be a measure on δ, where δ is an inaccessible Suslin cardinal (and so has the strong partition property). Let $\mathcal{V}$ be the measure on $j_\mu(\delta)$ induced by the strong partition property, functions $F\colon \delta \to \delta$ of the correct type, and the measure μ on δ. Assume $o(\delta) \geq j_\mu(\delta)$, and let ν be the measure on δ defined by: $\nu(A) = 1$ iff $\forall^*_{\mathcal{V}}\beta < j_\mu(\delta)\ \forall^*_{[S_\beta]}\alpha < \delta\ \alpha \in A$ (here $[S_\beta]$ is the β^{th} equivalence class in the stationary set ordering). Then $j_\mu(\delta) < j_\nu(\delta)$.*

The proof of the lemma is not difficult, we omit it (the basic fact is that if $S_1 \prec S_2$ then $j_{S_1}(\delta)$ embeds into $[f]_{S_2}$, where $f(\alpha) =$ the next Suslin cardinal after α).

Unfortunately, the main definite result along these lines at the moment is a negative one; it asserts that for Suslin cardinals δ where $S(\delta)$ is closed under real quantification, $o(\delta)$ is closed under the above ultrapower operation in the following precise sense (c.f. [10]).

6.13 Theorem *Let δ be a Suslin cardinal with $S(\delta)$ closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$. Let $\beta < o(\delta)$, and ν_β the corresponding atomic normal measure. Then $j_{\nu_\beta}(\delta) < o(\delta)$. Furthermore, $\mathrm{cof}(o(\delta)) > \delta$, and $\mathrm{cof}(o(\delta)) \neq j_{\nu_\beta}(\delta)$ for all $\beta < o(\kappa)$.*

It is conjectured in [10] that $o(\delta)$ is regular for such δ; theorem 6.13 seems a step towards showing that. A recent theorem of Steel [38] shows that in $L(\mathbb{R})$ every regular cardinal below Θ is measurable (using techniques of inner model theory). Thus, granting the above conjecture, new (normal) measures appear on δ which seem not to be approachable "from below" in the previous sense. Undoubtedly, new techniques will be necessary.

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