New Partition Properties under AD

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We work in the theory $ZF + DC + AD$.

Recall the Erdős-Rado Partition notation.

**Definition**

$\kappa \rightarrow (\kappa)^\lambda$ if for every partition $\mathcal{P}: (\kappa)^\lambda \rightarrow \{0, 1\}$ of the increasing functions from $\lambda$ to $\kappa$ into two pieces, there is a homogeneous $H \subseteq \kappa$ of size $\kappa$.

We say $\kappa$ has the **strong** partition property if $\kappa \rightarrow (\kappa)^\kappa$, and say $\kappa$ has the **weak** partition property if $\kappa \rightarrow (\kappa)^\lambda$ for all $\lambda < \kappa$.

We usually use a reformulation of the partition property which uses c.u.b. homogeneous sets.

To state this, we need the notion of **type** of a function.
Definition
We say \( f: \alpha \rightarrow \text{On} \) has \textbf{uniform cofinality} \( \omega \) if there is an \( f': \alpha \times \omega \rightarrow \text{On} \) which is increasing in the second argument and
\[ f(\beta) = \sup_n f'(\beta, n) \quad \text{for all } \beta < \alpha. \]

Definition
\( f: \alpha \rightarrow \text{On} \) is of the \textbf{correct type} if \( f \) is increasing, everywhere discontinuous, and of uniform cofinality \( \omega \).

We can likewise define uniform cofinality \( \omega_1, \omega_2, \text{etc.} \).
More generally, for any \( g : \alpha \rightarrow \text{On} \) we can define \( f \) having uniform cofinality \( g \): there is an

\[
f' : \{ (\beta, \gamma) : \beta < \alpha, \gamma < g(\beta) \}
\]

with \( f(\beta) = \sup_{\gamma} f'(\beta, \gamma) \).

We use frequently the “almost everywhere” versions of these notions with respect to some measure \( \mu \) on \( \text{dom}(f) \).
Partition Relations

Definition

\( \kappa \overset{\text{c.u.b.}}{\rightarrow} (\kappa)^{\lambda} \) iff for every partition \( \mathcal{P} \) of the function \( f : \lambda \rightarrow \kappa \) of the correct type, there is a c.u.b. \( C \subseteq \kappa \) which is homogeneous for \( \mathcal{P} \).

The ordinary and c.u.b. version of the partition relation are essentially equivalent:

Fact

\[
\kappa \overset{\text{c.u.b.}}{\rightarrow} (\kappa)^{\lambda} \Rightarrow \kappa \rightarrow (\kappa)^{\lambda}
\]

\[
\kappa \rightarrow (\kappa)^{\omega \cdot \lambda} \Rightarrow \kappa \overset{\text{c.u.b.}}{\rightarrow} (\kappa)^{\lambda}
\]
Definition

A \subseteq \omega^\omega is \kappa-Suslin if there is a tree T \subseteq \omega \times \kappa such that

A = p[T] = \{x : \exists f \in \kappa^\omega \in (x, f) \in [T]\}. Let S(\kappa) = the \kappa-Suslin sets.

\kappa is a Suslin cardinal if S(\kappa) - \bigcup_{\lambda<\kappa} S(\lambda) \neq \emptyset.

Theorem (Steel-Woodin)

The Suslin cardinals are closed below their supremum.

Assuming V = L(\mathbb{R}), there is a largest Suslin cardinal \delta_1^2 (Martin-Steel), and the Suslin cardinals are c.u.b. in \delta_1^2.
The first $\omega$ many Suslin cardinals are:

$$\lambda_1 = \omega, \; \delta^1_1 = \omega_1, \; \lambda_3 = \omega_\omega, \; \delta^1_3 = \omega_\omega^{\omega+1}, \; \lambda_5 = \omega_\omega^\omega, \; \delta^1_5 = \omega_\omega^\omega+1,$$

$$\ldots, \lambda_{2n+1}, \; \delta^1_{2n+1}, \ldots$$

Here $\delta^1_{2n+1} = (\lambda_{2n+1})^+$, and $\lambda_{2n+1}$ is a cardinal of cofinality $\omega$ (Kechris).

Recall $\delta^1_{2n+2} = (\delta^1_{2n+1})^+$, and all the $\delta^1_n$ are regular (measurable).
Definition
Given an increasing, discontinuous sequence of cardinals $\{\kappa_i\}_{i<\theta}$ and a sequence of ordinals $\{\lambda_i\}_{i<\theta}$, we say a block function from $\vec{\lambda}$ to $\vec{\kappa}$ is a $\vec{f} = \{f_i\}_{i<\theta}$ where $f_i: \lambda_i \to \kappa_i - \sup_{j<i} \kappa_j$. A block c.u.b. set is a $\vec{C} = \{C_i\}_{i<\theta}$ where $C_i$ is c.u.b. in $\kappa_i - \sup_{j<i} \kappa_j$.

We next define the polarized partition property.

Definition
$\vec{\kappa} \rightarrow (\vec{\kappa})\vec{\lambda}$ if for every partition $\mathcal{P}$ of the block functions $\vec{f}: \vec{\lambda} \rightarrow \vec{\kappa}$ into two pieces, there is a block c.u.b. set $\vec{C}$ homogeneous for $\mathcal{P}$.

For sequences of length 3 we will write $(\kappa_0, \kappa_1, \kappa_2) \rightarrow (\kappa_0, \kappa_1, \kappa_2)^{\lambda_0, \lambda_1, \lambda_2}$. 
Main Results

Theorem (Apter, J, Löwe)

Let $\kappa$ be an inaccessible Suslin cardinal. Then
$$(\kappa, \kappa^+, \kappa^{++}) \rightarrow (\kappa, \kappa^+, \kappa^{++})^{\kappa, \kappa, \kappa}.$$ 

This extends a result of Kechris for the countable exponent case.

Theorem

For all regular $\kappa$ with $\delta^1_{2n+1} < \kappa < \delta^1_{2n+3}$ we have $\kappa \rightarrow (\kappa)^{\delta^1_{2n+1}}$ but $\kappa \nrightarrow (\kappa)^{\delta^1_{2n+2}}$.

Corollary

$\aleph_{\omega \cdot 2+1}$, $\aleph_{\omega^\omega+1}$ are regular cardinals without the weak partition property.
Application

In [Apter, J, Löwe] we used the first theorem to force over models of $\text{ZF} + \text{AD}$ to change the cofinalities of $\aleph_1$, $\aleph_2$, $\aleph_3$.

- There are 3 possibilities for $\aleph_1$. Namely, $\text{cof} = \omega$, $r$ (regular, non-measurable), or $m$ (measurable).
- There are 4 possibilities for $\aleph_2$, and 5 possibilities for $\aleph_3$.
- 13 of the 60 total possibilities are “trivially inconsistent.” For example, $\aleph_1$ regular, $\text{cof}(\aleph_2) = \aleph_1$, and $\text{cof}(\aleph_3) = \aleph_2$.

Theorem (Apter, J, Löwe)

Assuming suitable large cardinals, all of the remaining 47 cases are consistent with ZF.
Types of Suslin Cardinals

By a Lévy pointclass we mean a pointclass \( \Gamma \) closed under \( \exists^{\omega^\omega} \) or \( \forall^{\omega^\omega} \) (or both),

The Wadge hierarchy of Lévy pointclasses falls into projective hierarchies of 4 types.

We specialize to the Suslin pointclasses. The limit Suslin cardinals \( \kappa \) correspond to the bases of projective hierarchies.

Note that if \( \kappa \) is a limit Suslin cardinal then \( \Lambda = S(< \kappa) \) is a selfdual pointclass closed under quantifiers.

Also, \( o(\Lambda) = \delta(\Lambda) = \kappa \).
If $\text{cof}(\kappa) > \omega$, there is a nonselfdual pointclass $\Gamma_S$ (Steel pointclass) of Wadge degree $\kappa$ closed under $\forall^\omega$, $\land$, with $\text{sep}(\check{\Gamma}_S)$. Also, $\text{scale}(\Gamma_S)$.

**Fact (Steel)**

*If $\kappa$ is regular, then $\Gamma_S$ is closed under $\lor$, $\land$.*

Wren $\kappa$ is regular, we have boundedness of $\Delta_S$ sets with respect to $\Gamma_S$-norms on $\Gamma_S$-complete sets.
Type I $\text{cof}(\kappa) = \omega$. Let $\Sigma_0^\kappa = \bigcup_\omega (S(< \kappa))$. Then $\text{scale}(\Sigma_0^\kappa)$, $\text{scale}(\Pi_1^\kappa)$, $\text{scale}(\Sigma_2^\kappa)$, etc. The Suslin cardinals are $\kappa = \lambda_1^\kappa$, $\kappa^+ = \delta_1^\kappa$, $\lambda_3^\kappa$, $\delta_3^\kappa$, etc.

Type II $\text{cof}(\kappa) > \omega$, $\Gamma_S$ not closed under $\lor$.

Type III $\text{cof}(\kappa) > \omega$, $\Gamma_S$ closed under $\lor$ but not $\exists^\omega \forall^\omega \kappa$ necessarily regular).

Type IV $\text{cof}(\kappa) > \omega$, $\Gamma$ closed under $\exists^\omega \forall^\omega$.

For $\kappa$ an inaccessible Suslin cardinal we are in Type III or Type IV.

The Type I hierarchies will play an important role in the proof.
We fix the inaccessible Suslin cardinal \( \kappa \). Let \( C \subseteq \kappa \) be the c.u.b. set of limit Suslin cardinals. Let \( C_\omega \subseteq C \) be the points of cofinality \( \omega \).

\( \kappa \) has the strong partition property by Kechris-Kleinberg-Moschovakis-Woodin.

Let \( \mu \) be the \( \omega \)-cofinal, normal measure on \( \kappa \).

For \( \alpha \in C_\omega \), let \( \mu_\alpha \) be the \( \omega \)-cofinal, normal measure on \( \alpha^+ \). We have \( j_{\mu_\alpha}(\alpha^+) = \alpha^{++} \).
We have the following picture.

\[ \Sigma_0^\alpha = \bigcup_{\lambda < \alpha} S(\lambda) \]

\[ S(\alpha) = \Sigma_1^\alpha, \quad S(\alpha^+) = \Sigma_2^\alpha \]

\[ \text{scale}(\Sigma_0^\alpha) \quad \text{scale}(\Pi_1^\alpha) \]
Plan of the proof

1. We show $[\alpha \mapsto \alpha^+]_\mu = \kappa^+$. 
2. We show $\delta = [\alpha \mapsto \alpha^{+++}]_\mu \leq \kappa^{++}$. 
3. We show for all $\theta < \omega_1$ that $(\kappa, \kappa^+, \delta) \rightarrow (\kappa, \kappa^+, \delta)^\theta$. 
   It follows that $\delta$ is regular, so $\delta = \kappa^{++}$. 
4. Finally, we show $(\kappa, \kappa^+, \delta) \rightarrow (\kappa, \kappa^+, \delta)^\kappa$
The Trees $T^+$ and $T^{++}$

We define two trees $T^+$ and $T^{++}$ on $\omega \times \kappa$.

**Lemma**

For any $f: \kappa \to \kappa$ with $f(\alpha) < \alpha^+$ there is a $x \in \omega^\omega$ with $T^+_x$ wellfounded and such that $\forall^* \alpha \; f(\alpha) < |T^+_x \upharpoonright \alpha|$.

**Lemma**

For any $f: \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$ there is a $x \in \omega^\omega$ with $T^{++}_x$ wellfounded such that $\forall^* \alpha \; (f(\alpha) < [\beta \mapsto |T^{++}_x \upharpoonright \beta|]_{\mu(\alpha)})$.

The first lemma shows $[\alpha \mapsto \alpha^+]_{\mu} \leq \kappa^+$, and it follows easily that $[\alpha \mapsto \alpha^+]_{\mu} = \kappa^+$.

The second lemma shows that $\delta = [\alpha \mapsto \alpha^{++}]_{\mu} \leq \kappa^{++}$.
Construction of $T^+$

Fix a $\Gamma$-complete set $P$ and a $\Gamma$-scale $\{\varphi_n\}_{n \in \omega}$ on $P$. We use $\varphi = \varphi_0$ to code ordinals $< \kappa$.

Say $\alpha$ is strongly reliable if for all $\beta < \alpha$:

$$\sup\{\varphi_n(x) : x \in P \land \varphi_0(x) \leq \beta\} < \alpha$$

The set of strongly reliable ordinals is c.u.b. in $\kappa$. Assume $C \subseteq$ strongly reliables.

Let

$$(x, y) \in R \iff (x, y \in P \land \varphi(x) < \varphi(y)).$$

and

$$(x, y) \in R^\alpha \iff (x, y \in P \land \varphi(x) < \varphi(y) < \alpha).$$
\[ R^\alpha \in \Sigma_0^\alpha - \bigcup_{\lambda < \alpha} S(\lambda), \text{ and we uniformly have a } \Sigma_0^\alpha \text{ scale on } R^\alpha \] (essentially by restricting the scale \( \vec{\phi} \) to ordinals below \( \alpha \)).

Starting from this, we uniformly get \( \Sigma_1^\alpha \) universal sets \( B^\alpha \) and \( \Pi_1^\alpha \) sets \( Q^\alpha \) and \( \Pi_1^\alpha \) scales \( \vec{\psi}^\alpha \) on \( Q^\alpha \).

**Definition**

Let \( W = \{ x : \forall n \ (x)_n \in P \} \). \( x \in W \) will code the ordinal 
\[ |x| = \sup_n \varphi_0((x)_n). \]

The scale on \( P \) easily gives a scale on \( W \). Let \( T_W \) be the corresponding tree.
We first construct a tree $U$ on $\omega \times \omega \times \kappa$ with the following properties:

1. If $x \in W$ and $|x| = \alpha \in C$, then $U_{x,y}$ is wellfounded iff the $\Sigma_1^\alpha$ relation coded by $y$ is wellfounded.

2. For $x, y$ as above, $|U_{x,y} \upharpoonright \alpha| \geq |B^\alpha_y|$, the $\Sigma_1^\alpha$ relation coded by $y$.

Key Point: For $x, y$ as above, the entire tree $U_{x,y}$ is wellfounded (not just $U_{x,y} \upharpoonright \alpha$).

idea: $U$ is constructed as in the proof of the Kunen-Martin theorem, but we use the components of the real $x$ to verify the appropriate reals are in $B^{|x|}_y$. 

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Suppose \( f: \kappa \to \kappa \) and \( f(\alpha) < \alpha^+ \) for \( \alpha \in C \).

Consider the game \( G_f \):

\[
\begin{array}{c|c|c}
  & r & x, y \\
I & r & \\
II & x, y & \\
\end{array}
\]

II wins the run iff

\[
(r \in W) \rightarrow (x \in W \land B_y^{\vert x\vert} \text{ is wellfounded} \land \vert B_y^{\vert x\vert} \vert > f(\vert x\vert)).
\]
A boundedness argument shows that II has a winning strategy.

This suggests the following definition of the tree $T^+$ on $(\omega)^2 \times \kappa \times (\omega)^2 \times \kappa$:

$$(\sigma, r, \vec{\alpha}, x, y, \vec{\beta}) \in [T^+] \text{ iff:}$$

1. $(r, \vec{\alpha}) \in [T_W]$.
2. $\sigma(r) = (x, y)$
3. $(x, y, \vec{\beta}) \in [U]$.

Then $T^+_\sigma$ is wellfounded and $|T^+_\sigma | \uparrow \alpha > f(\alpha)$ for $\mu$ almost all $\alpha$. 
Construction of $T^{++}$

We first construct a tree $V$ on $(\omega)^2 \times \kappa$ with the following properties:

1. $(x, y) \in [V]$ iff $x \in W$ and for all $n$, $(y)_n$ codes a $\Sigma^1_{|x|}$ wellfounded relation $B_{(y)_n}^{|x|}$.

2. If $x \in W$, $|x| \in C$ then there is a c.u.b. $D \subseteq \alpha^+$ such that if $\gamma \in D$, $y \in \omega^\omega$ and for all $n$ $(y)_n$ codes a $\Sigma^1_{|x|}$ wellfounded relation of rank $< \gamma$, then $V_{x,y} \upharpoonright \gamma$ is illfounded.

**main point:** We can translate the $\Pi^1_{|x|}$ statement asserting the wellfoundedness of the $B_{(y)_n}^{|x|}$ into $\Pi^\beta_1$ statements for any $\beta \geq |x|$ (use the $(x)_i$ as in the definition of $U$).
Suppose $x \in W$, $|x| = \alpha \in C$, and $g: \alpha^+ \rightarrow \alpha^+$. Play the game $G_g$:

\[
\text{I } z \quad \text{II } w
\]

II wins the run iff:

\[
(\forall n B^\alpha_{(z)_n} \text{ is wellfounded } ) \rightarrow (B^\alpha_w \text{ is wellfounded } \land |B^\alpha_w| > g(\sup_{n} |B^\alpha_{(y)_n}|))
\]
By boundedness, II has a winning strategy $\tau$ for any $G_g$.

Suppose now $f : \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$.

Play the game $G_f$:

$$
\begin{array}{c|c}
| & r \\
\hline
| & x, \tau \\
\end{array}
$$

$r, x$ will be in $W$ and $\tau$ will be strategy for a game $G_g$ where $[g]_{\mu_\alpha} > f(\alpha)$, where $\alpha = |x|$. 
More precisely, II wins the run iff:

\[ r \in W \rightarrow (x \in W \land |x| = \alpha \geq |y|) \]

\[ \land \forall z \ [\forall n B^{\alpha}_{(z)_n} \text{ is wellfounded} \rightarrow \]

\[ B^{\alpha}_{\tau(z)} \text{ is wellfounded} \land |B^{\alpha}_{\tau(z)}| \geq g(\sup_n |B^{\alpha}_{(z)_n}|) \]

for some \( g : \alpha^+ \rightarrow \alpha^+ \) with \( [g]_{\mu_{\alpha}} \geq f(\alpha) \).
II has a winning strategy $\sigma$ for any $f$, and this suggests the definition of $T^{++}$:

$$(\sigma, r, \vec{\alpha}, x, \tau, y, z, \vec{\beta}, \vec{\gamma}) \in T^{++} \text{ iff:}$$

1. $(r, \vec{\alpha}) \in [T_W]$.
2. $\sigma(r) = (x, \tau)$.
3. $(x, y, \vec{\beta}) \in [V]$.
4. $\tau(y) = z$.
5. $(x, a, \vec{\gamma}) \in [U]$.

The properties of $U$ and $V$ show that $T^{++}$ has the desired property.
The countable exponent $\theta$ case.

Fix a bijection $\pi: \omega \cdot \theta \to \omega$.

We code cofinally in $\kappa^+, \kappa^{++}$ many ordinals using sections of our trees: $T^+_x$, $T^{++}_x$.

Suppose $\mathcal{P}$ is a partition of the block functions from $3 \times \theta$ to $(\kappa, \kappa^+, \kappa^{++})$.

Consider the game $G_{\mathcal{P}}$:

\begin{align*}
  I & : x, y, z \\
  II & : x', y', z'
\end{align*}
(1) If there is an \( j < \omega \cdot \theta \) such that \( (x)_{\pi(j)} \notin P_0 \) or \( (x')_{\pi(j)} \notin P_0 \), then player I wins iff for the least such \( j \), \( (x)_{\pi(j)} \in P_0 \).

(2) Suppose next that there is an \( \alpha < \kappa \) such that one of the following holds.

(a) There is a \( j < \omega \cdot \theta \) such that either \( T^+_{(y)_{\pi(j)}} \upharpoonright \alpha \) or \( T^+_{(y')_{\pi(j)}} \upharpoonright \alpha \) is illfounded.

(b) There is a \( \beta < \alpha^+ \) and a \( j < \omega \cdot \theta \) such that either \( T^{++}_{(z)_{\pi(j)}} \upharpoonright \beta \) or \( T^{++}_{(z')_{\pi(j)}} \upharpoonright \beta \) is illfounded.

Let \( \alpha < \kappa \) be least such that (a) or (b) above holds. If (a) holds, let \( j \) be least such that (a) holds for \( \alpha \) and this \( j \). In this case, Player I wins provided \( T^+_{(y)_{\pi(j)}} \) is wellfounded. If (a) does not hold at \( \alpha \), but (b) does, let \( (\beta, j) \) be lexicographically least such that (b) holds. Player I wins in this case provided \( T^{++}_{(z)_{\pi(j)}} \upharpoonright \beta \) is wellfounded.
Assume II has a winning strategy $\tau$.

We define c.u.b. sets $C_0 \subseteq \kappa$, $C_1 \subseteq \kappa^+$, and $C_2 \subseteq \kappa^{++}$.

For example, to define $C_2$ we define for $\alpha \in C$, $\beta, \gamma < \alpha^+$ and $j < \omega \cdot \theta$:

$$A_{\alpha, \beta, \gamma, j} = \{(x, y, z) ; \forall j((x)_{\pi(j)} \in P_0 \land \varphi_0((x)_{\pi(j)}) < \alpha)$$

$$\land \forall \alpha' < \alpha \forall \beta < (\alpha')^+ \forall j(T^+_{(y)_{\pi(j)}} \upharpoonright \alpha \land T^{++}_{(z)_{\pi(j)}} \upharpoonright \beta \text{ are wellfounded})$$

$$\land \forall j \mid T^+_{(y)_{\pi(j)}} \upharpoonright \alpha \mid < \beta \land \forall (\beta', j') \leq \text{lex} (\beta, j) (\mid T^{++}_{(z)_{\pi(j)}} \upharpoonright \beta \mid \leq \eta)\}.$$
We have: \( A_{\alpha,\beta,\gamma,j} \in \Delta^\alpha_1 \).

Since \( \tau \) is winning for Player II, for each \((x, y, z) \in A_{\alpha,\beta,\eta,j} \), if \( \tau(x, y, x) = (x', y', z') \) then \( \forall (\beta', j') \leq_{\text{lex}} (\beta, j) \ T^{++}_{(z')\pi(j')} \upharpoonright \beta \) is wellfounded.

By boundedness,

\[
\rho_2(\alpha, \beta, \eta, j) := \sup \{|T^{++}_{(z')\pi(j')} \upharpoonright \beta| ; (x', y', z') \in \tau[A_{\alpha,\beta,\eta,j}] \wedge j' \leq j \} < \alpha^+.
\]

Let \( C^\alpha_2 \subseteq \alpha^+ \) be c.u.b. closed under \( \rho_2 \). The \( C^\alpha_2 \) lift to \( C_2 \subseteq \kappa^{++} \).
Exponent $\kappa$

We use generic codes (Kechris-Woodin) and the uniform coding lemma.

Let $U(Q, z, x, y)$ be universal for the syntactic class $\Sigma_1(Q)$, where $Q$ is a binary predicate symbol.

Can take

$$U(Q, z, x, y) \leftrightarrow \exists w \ (S(z, \langle x, y, w \rangle) \land \forall n \ W(((w)_n)_0, ((w)_n)_1)).$$

where $S$ is universal $\Sigma_1^1$.

Let $R'_\alpha$ code $\{(x, y) : \varphi(x) < \varphi(y) \leq \alpha\}$ and $\leq$ version.

We write $U_z(R'_\alpha)(x, y)$ for $U(R'_\alpha, z, x, y)$. 
Uniform Coding Lemma says that if $A \subseteq \omega^\omega \times \omega^\omega$ with $\text{dom}(A) = P$, then there is a $z \in \omega^\omega$ such that for all $\alpha < \kappa$, $U_z(R'_\alpha)$ is a choice subrelation for $A \upharpoonright P_{\leq \alpha}$.

Let $\overline{U}_z(R'_\alpha)$ be a uniformization of $U_z(R'_\alpha)$ (using scale $\{\varphi_n\}$).

For $\alpha < \kappa$, let $\alpha'$ be the next reliable (w.r.t. $\{\varphi_n\}$).

Let $G : \kappa^\omega \rightarrow \omega^\omega$ be a generic coding function. So for any $s \in \kappa^\omega$, $x = G(\alpha \triangleleft s) \in P$, $\varphi(x) \leq \alpha$, and if $s$ enumerates an honest set then $|x| = \alpha$. 
Given reals $x, y, z$ we say:

1. **$x$ codes a function** at $\alpha < \kappa$ if $|a| = \alpha$ implies 
   \[ \exists b \in P \overline{U}_x(R'_\alpha)(a, b). \] 
   Also, if $|a| = |a'|$ then $|b| = |b'|$.

2. **$y$ is good** at $\delta < \alpha \in C_\omega$ if $\forall^* a \in P_\delta \exists b \overline{U}_y(R'_\delta)(a, b)$ and 
   $T^+_b \upharpoonright \alpha$ is wellfounded (good at $\alpha$ if good at all $\delta < \alpha$).

3. **$z$ is good** at $\delta < \alpha \in C_\omega, \beta < \alpha^+$ if $\forall^* a \in P_\delta \exists b \overline{U}_z(R'_\delta)(a, b)$ 
   and $T^+_z \upharpoonright \beta$ is wellfounded (good at $\alpha$ if good at all $\delta < \alpha, \beta < \alpha^+$).

$f_x(\alpha) = \text{the unique value of } |b|$.

$g_y(\delta, \alpha) = \text{least } \gamma < \alpha^+ \text{ such that } \forall^* a \in P_\delta, |T^+_b \upharpoonright \alpha| \leq \gamma.$

$h_z(\delta, \alpha, \beta) = \text{least } \gamma < \alpha^+ \text{ such that } \forall^* a \in P_\delta, |T^{++}_b \upharpoonright \beta| \leq \gamma.$
I plays $x, y, z$, II plays $x', y', z'$.

Let $\alpha$ be least such that one of the following holds.

1. For some $\delta < \alpha$, $y$ or $y'$ is not good at $(\delta, \alpha)$.
2. For some $\delta < \alpha$, $\beta < \alpha^+$, $z$ or $z'$ is not good at $(\delta, \alpha, \beta)$.
3. $x$ or $x'$ does not code a function at $\alpha$.

If (1) holds, then I wins iff for the least such $\delta$, $y$ is good at $(\delta, \alpha)$.

Suppose (1) does not hold, but (2) holds. Then I wins iff for the lexicographically least pair $(\beta, \delta)$ we have $z$ is good at $\delta, \alpha, \beta$. If (1) and (2) don't hold, but (3) holds, then I wins iff $x$ codes an ordinal at $\alpha$. 

Otherwise, we have functions $f_x, f_x', g_y, g_y', h_z, h_z'$.

- $f_x, f_x'$ together produce $F : \kappa \to \kappa$.
- $g_y, g_y'$ together produce $G : \kappa \to \kappa^+$. 
- $h_z, h_z'$ together produce $H : \kappa \to \kappa^{++}$.

II wins the run in this case iff $P(F, G, H) = 1$.

From a winning strategy $\tau$ for II, say, we define the homogeneous sets $C_0 \subseteq \kappa$, $C_1 \subseteq \kappa^+$ and $C_1 \subseteq \kappa^{++}$. 

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