New Partition Properties under AD

S. Jackson

Department of Mathematics
University of North Texas
jackson@unt.edu

ESI Workshop

June, 2009, Vienna

We work in the theory $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$.

Recall the Erdős-Rado Partition notation.

Definition

$\kappa \to (\kappa)^\lambda$ if for every partition $\mathcal{P} \colon (\kappa)^\lambda \to \{0, 1\}$ of the increasing functions from λ to κ into two pieces, there is a homogeneous $H \subseteq \kappa$ of size κ.

We say κ has the strong partition property if $\kappa \to (\kappa)^\kappa$, and say κ has the weak partition property if $\kappa \to (\kappa)^\lambda$ for all $\lambda < \kappa$.

We usually use a reformulation of the partition property which uses c.u.b. homogeneous sets.

To state this, we need the notion of type of a function.

Definition

We say $f \colon \alpha \to \mathrm{On}$ has uniform cofinality ω if there is an $f' \colon \alpha \times \omega \to \mathrm{On}$ which is increasing in the second argument and $f(\beta) = \sup_n f'(\beta, n)$ for all $\beta < \alpha$.

Definition

$f \colon \alpha \to \mathrm{On}$ is of the correct type if f is increasing, everywhere discontinuous, and of uniform cofinality ω.

We can likewise define uniform cofinality ω_1, ω_2, etc.

General Types

More generally, for any $g\colon \alpha \to \mathrm{On}$ we can define f having uniform cofinality g: there is an

$$f' : \{(\beta,\gamma) : \beta < \alpha, \gamma < g(\beta)\}$$

with $f(\beta) = \sup_\gamma f'(\beta,\gamma)$.

We use frequently the “almost everywhere” versions of these notions with respect to some measure μ on $\mathrm{dom}(f)$.

Partition Relations

Definition

$\kappa \overset{\text{c.u.b.}}{\longrightarrow} (\kappa)^\lambda$ iff for every partition $\mathcal{P}$ of the function $f \colon \lambda \to \kappa$ of the correct type, there is a c.u.b. $C \subseteq \kappa$ which is homogeneous for $\mathcal{P}$.

The ordinary and c.u.b. version of the partition relation are essentially equivalent:

Fact

$$\kappa \overset{\text{c.u.b.}}{\longrightarrow} (\kappa)^\lambda \Rightarrow \kappa \to (\kappa)^\lambda$$

$$\kappa \to (\kappa)^{\omega\cdot\lambda} \Rightarrow \kappa \overset{\text{c.u.b.}}{\longrightarrow} (\kappa)^\lambda$$

Definition

$A \subseteq \omega^\omega$ is κ-Suslin if there is a tree $T \subseteq \omega \times \kappa$ such that $A = p[T] = \{x \colon \exists f \in \kappa^\omega \in (x, f) \in [T]\}$. Let $S(\kappa) =$ the κ-Suslin sets.

κ is a Suslin cardinal if $S(\kappa) - \bigcup_{\lambda<\kappa} S(\lambda) \neq \emptyset$.

Theorem (Steel-Woodin)

The Suslin cardinals are closed below their supremum.

Assuming $V = L(\mathbb{R})$, there is a largest Suslin cardinal δ^2_1 (Martin-Steel), and the Suslin cardinals are c.u.b. in δ^2_1.

The first ω many Suslin cardinals are:

$$\lambda_1 = \omega,\ \boldsymbol{\delta}^1_1 = \omega_1,\ \lambda_3 = \omega_\omega,\ \boldsymbol{\delta}^1_3 = \omega_{\omega^\omega+1},\ \lambda_5 = \omega_{\omega^{\omega^\omega}},\ \boldsymbol{\delta}^1_5 = \omega_{\omega^{\omega^\omega}+1},\ \dots, \lambda_{2n+1},\ \boldsymbol{\delta}^1_{2n+1}, \dots$$

Here $\boldsymbol{\delta}^1_{2n+1} = (\lambda_{2n+1})^+$, and λ_{2n+1} is a cardinal of cofinality ω (Kechris).

Recall $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$, and all the $\boldsymbol{\delta}^1_n$ are regular (measurable).

Definition

Given an increasing, discontinuous sequence of cardinals $\{\kappa_i\}_{i<\theta}$ and a sequence of ordinals $\{\lambda_i\}_{i<\theta}$, we say a block function from $\vec{\lambda}$ to $\vec{\kappa}$ is a $\vec{f} = \{f_i\}_{i<\theta}$ where $f_i \colon \lambda_i \to \kappa_i - \sup_{j<i} \kappa_j$. A block c.u.b. set is a $\vec{C} = \{C_i\}_{i<\theta}$ where C_i is c.u.b. in $\kappa_i - \sup_{j<i} \kappa_j$.

We next define the polarized partition property.

Definition

$\vec{\kappa} \to (\vec{\kappa})^{\vec{\lambda}}$ if for every partition $\mathcal{P}$ of the block functions $\vec{f} \colon \vec{\lambda} \to \vec{\kappa}$ into two pieces, there is a block c.u.b. set $\vec{C}$ homogeneous for $\mathcal{P}$.

For sequences of length 3 we will write $(\kappa_0, \kappa_1, \kappa_2) \to (\kappa_0, \kappa_1, \kappa_2)^{\lambda_0, \lambda_1, \lambda_2}$.

Main Results

Theorem (Apter, J, Löwe)

Let κ be an inaccessible Suslin cardinal. Then $(\kappa, \kappa^+, \kappa^{++}) \to (\kappa, \kappa^+, \kappa^{++})^{\kappa,\kappa,\kappa}$.

This extends a result of Kechris for the countable exponent case.

Theorem

For all regular κ with $\delta^1_{2n+1} < \kappa < \delta^1_{2n+3}$ we have $\kappa \to (\kappa)^{\delta^1_{2n+1}}$ but $\kappa \not\to (\kappa)^{\delta^1_{2n+2}}$.

Corollary

$\aleph_{\omega\cdot 2+1}, \aleph_{\omega^\omega+1}$ are regular cardinals without the weak partition property.

Application

In [Apter, J, Löwe] we used the first theorem to force over models of $\mathsf{ZF}+\mathsf{AD}$ to change the cofinalities of $\aleph_1, \aleph_2, \aleph_3$.

- There are 3 possibilities for $\aleph_1$ Namely, cof $=\omega$, r (regular, non-measurable), or m (measurable).
- There are 4 possibilities for $\aleph_2$, and 5 possibilities for $\aleph_3$.
- 13 of the 60 total possibilities are "trivially inconsistent." For example, $\aleph_1$ regular, $\mathrm{cof}(\aleph_2)=\aleph_1$, and $\mathrm{cof}(\aleph_3)=\aleph_2$.

Theorem (Apter, J, Löwe)

Assuming suitable large cardinals, all of the remaining 47 *cases are consistent with* ZF.

Types of Suslin Cardinals

By a Lévy pointclass we mean a pointclass Γ closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$ (or both),

The Wadge hierarchy of Lévy pointclasses falls into projective hierarchies of 4 types.

We specialize to the Suslin pointclasses. The limit Suslin cardinals κ correspond to the bases of projective hierarchies.

Note that if κ is a limit Suslin cardinal then $\Lambda = S(<\kappa)$ is a selfdual pointclass closed under quantifiers.

Also, $o(\Lambda) = \delta(\Lambda) = \kappa$.

If $\operatorname{cof}(\kappa) > \omega$, there is a nonselfdual pointclass $\boldsymbol{\Gamma}_S$ (Steel pointclass) of Wadge degree κ closed under $\forall^{\omega^\omega}$, $\wedge$, with $\operatorname{sep}(\check{\boldsymbol{\Gamma}}_S)$. Also, $\operatorname{scale}(\boldsymbol{\Gamma}_S)$.

Fact (Steel)

If κ is regular, then $\boldsymbol{\Gamma}_S$ is closed under $\vee$, $\wedge$.

Wren κ is regular, we have boundedness of $\boldsymbol{\Delta}_S$ sets with respect to $\boldsymbol{\Gamma}_S$-norms on $\boldsymbol{\Gamma}_S$-complete sets.

- Type I $\mathrm{cof}(\kappa) = \omega$. Let $\boldsymbol{\Sigma}^{\kappa}_0 = \bigcup_{\omega}(S(<\kappa))$. Then $\mathrm{scale}(\boldsymbol{\Sigma}^{\kappa}_0)$, $\mathrm{scale}(\boldsymbol{\Pi}^{\kappa}_1)$, $\mathrm{scale}(\boldsymbol{\Sigma}^{\kappa}_2), \dots$. The Suslin cardinals are $\kappa = \lambda^{\kappa}_1$, $\kappa^+ = \boldsymbol{\delta}^{\kappa}_1$, λ^{κ}_3, $\boldsymbol{\delta}^{\kappa}_3, \dots$.
- Type II $\mathrm{cof}(\kappa) > \omega$, $\boldsymbol{\Gamma}_S$ not closed under $\vee$.
- Type III $\mathrm{cof}(\kappa) > \omega$, $\boldsymbol{\Gamma}_S$ closed under $\vee$ but not $\exists^{\omega^\omega}$ κ necessarily regular).
- Type IV $\mathrm{cof}(\kappa) > \omega$, $\boldsymbol{\Gamma}$ closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$.

For κ an inaccessible Suslin cardinal we are in Type III or Type IV.

The Type I hierarchies will play an important role in the proof.

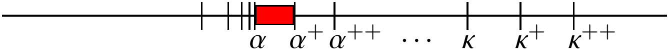
We fix the inaccessible Suslin cardinal κ. Let $C \subseteq \kappa$ be the c.u.b. set of limit Suslin cardinals.Let $C_\omega \subseteq C$ be the points of cofinality ω.

κ has the strong partition property by Kechris-Kleinberg-Moschovakis-Woodin.

Let μ be the ω-cofinal, normal measure on κ.

For $\alpha \in C_\omega$, let μ_α be the ω-cofinal, normal measure on α^+. We have $j_{\mu_\alpha}(\alpha^+) = \alpha^{++}$.

We have the following picture.



$$\boldsymbol{\Sigma}^{\alpha}_{0} = \bigcup_{\lambda<\alpha} S(\lambda)$$

$$S(\alpha) = \boldsymbol{\Sigma}^{\alpha}_{1}, \qquad S(\alpha^+) = \boldsymbol{\Sigma}^{\alpha}_{2}$$

$$\mathrm{scale}(\boldsymbol{\Sigma}^{\alpha}_{0},) \qquad \mathrm{scale}(\boldsymbol{\Pi}^{\alpha}_{1})$$

Plan of the proof

1. We show $[\alpha \mapsto \alpha^+]_\mu = \kappa^+$.
2. We show $\delta \doteq [\alpha \mapsto \alpha^{++}]_\mu \leq \kappa^{++}$.
3. We show for all $\theta < \omega_1$ that $(\kappa, \kappa^+, \delta) \rightarrow (\kappa, \kappa^+, \delta)^\theta$.
   It follows that δ is regular, so $\delta = \kappa^{++}$.
4. Finally, we show $(\kappa, \kappa^+, \delta) \rightarrow (\kappa, \kappa^+, \delta)^\kappa$

The Trees T^+ and T^{++}

We define two trees T^+ and T^{++} on $\omega \times \kappa$.

Lemma

For any $f\colon \kappa \to \kappa$ with $f(\alpha) < \alpha^+$ there is a $x \in \omega^\omega$ with T^+_x wellfounded and such that $\forall^*_\mu \alpha\ f(\alpha) < |T^+_x \restriction \alpha|$.

Lemma

For any $f\colon \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$ there is a $x \in \omega^\omega$ with T^{++}_x wellfounded such that $\forall^*_\mu \alpha\ (f(\alpha) < [\beta \mapsto |T^{++}_x \restriction \beta|]_{\mu_\alpha})$.

The first lemma shows $[\alpha \mapsto \alpha^+]_\mu \leq \kappa^+$, and it follows easily that $[\alpha \mapsto \alpha^+]_\mu = \kappa^+$.

The second lemma shows that $\delta \doteq [\alpha \mapsto \alpha^{++}]_\mu \leq \kappa^{++}$.

Construction of T^+

Fix a $\boldsymbol{\Gamma}$-complete set P and a $\boldsymbol{\Gamma}$-scale $\{\varphi_n\}_{n\in\omega}$ on P. we use $\varphi = \varphi_0$ to code ordinals $< \kappa$.

Say α is strongly reliable if for all $\beta < \alpha$:

$$\sup\{\varphi_n(x) \colon x \in P \wedge \varphi_0(x) \leq \beta\} < \alpha$$

The set of strongly reliable ordinals is c.u.b. in κ. Assume $C \subseteq$ strongly reliables.

Let
$(x, y) \in R \leftrightarrow (x, y \in P \wedge \varphi(x) < \varphi(y)\}$.

and
$(x, y) \in R^\alpha \leftrightarrow (x, y \in P \wedge \varphi(x) < \varphi(y) < \alpha\}$.

$R^\alpha \in \boldsymbol{\Sigma}^\alpha_0 - \bigcup_{\lambda<\alpha} S(\lambda)$, and we uniformly have a $\boldsymbol{\Sigma}^\alpha_0$ scale on R^α (essentially by restricting the scale $\vec{\varphi}$ to ordinals below α).

Starting from this, we uniformly get $\boldsymbol{\Sigma}^\alpha_1$ universal sets B^α and $\boldsymbol{\Pi}^\alpha_1$ sets Q^α and $\boldsymbol{\Pi}^\alpha_1$ scales $\vec{\psi}^\alpha$ on Q^α.

Definition

Let $W = \{x \colon \forall n\, (x)_n \in P\}$. $x \in W$ will code the ordinal $|x| = \sup_n \varphi_0((x)_n)$.

The scale on P easily gives a scale on W. Let T_W be the corresponding tree.

We first construct a tree U on $\omega \times \omega \times \kappa$ with the following properties:

1. If $x \in W$ and $|x| = \alpha \in C$, then $U_{x,y}$ is wellfounded iff the $\boldsymbol{\Sigma}^\alpha_1$ relation coded by y is wellfounded.
2. For x, y as above, $|U_{x,y} \restriction \alpha| \geq |B^\alpha_y|$, the $\boldsymbol{\Sigma}^\alpha_1$ relation coded by y.

Key Point: For x, y as above, the entire tree $U_{x,y}$ is wellfounded (not just $U_{x,y} \restriction \alpha$).

idea: U is constructed as in the proof of the Kunen-Martin theorem, but we use the components of the real x to verify the appropriate reals are in $B^{|x|}_y$.

Suppose $f\colon \kappa \to \kappa$ and $f(\alpha) < \alpha^+$ for $\alpha \in C$.

Consider the game G_f:

I	r
II	x, y

II wins the run iff

$$(r \in W) \to (x \in W \wedge B_y^{|x|} \text{ is wellfounded } \wedge |B_y^{|x|}| > f(|x|).$$

A boundedness argument shows that II has a winning strategy.

This suggests the following definition of the tree T^+ on $(\omega)^2 \times \kappa \times (\omega)^2 \times \kappa$:

$(\sigma, r, \vec{\alpha}, x, y, \vec{\beta}) \in [T^+]$ iff:

1. $(r, \vec{\alpha}) \in [T_W]$.
2. $\sigma(r) = (x, y)$
3. $(x, y, \vec{\beta}) \in [U]$.

Then T^+_σ is wellfounded and $|T^+_\sigma \restriction \alpha| > f(\alpha)$ for μ almost all α.

Construction of T^{++}

We first construct a tree V on $(\omega)^2 \times \kappa$ with the following properties:

1. $(x, y) \in [V]$ iff $x \in W$ and for all n, $(y)_n$ codes a $\boldsymbol{\Sigma}^{|x|}_1$ wellfounded relation $B^{|x|}_{(y)_n}$.
2. If $x \in W$, $|x| \in C$ then there is a c.u.b. $D \subseteq \alpha^+$ such that if $\gamma \in D$, $y \in \omega^\omega$ and for all n $(y)_n$ codes a $\boldsymbol{\Sigma}^{|x|}_1$ wellfounded relation of rank $< \gamma$, then $V_{x,y} \restriction \gamma$ is illfounded.

main point: We can translate the $\boldsymbol{\Pi}^{|x|}_1$ statement asserting the wellfoundedness of the $B^{|x|}_{(y)_n}$ into $\boldsymbol{\Pi}^{\beta}_1$ statements for any $\beta \geq |x|$ (use the $(x)_i$ as in the definition of U).

Suppose $x \in W$, $|x| = \alpha \in C$, and $g \colon \alpha^+ \to \alpha^+$. Play the game G_g:

I	z
II	w

II wins the run iff:

$$(\forall n\, B^{\alpha}_{(z)_n} \text{ is wellfounded }) \to (B^{\alpha}_{w} \text{ is wellfounded } \wedge |B^{\alpha}_{w}| > g(\sup_n |B^{\alpha}_{(y)_n}|))$$

By boundedness, II has a winning strategy τ for any G_g.

Suppose now $f\colon \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$.

Play the game G_f:

$$\begin{array}{ll} \mathrm{I} & r \\ \mathrm{II} & x, \tau \end{array}$$

r, x will be in W and τ will be strategy for a game G_g where $[g]_{\mu_\alpha} > f(\alpha)$, where $\alpha = |x|$.

More precisely, II wins the run iff:

$$\begin{aligned} r \in W \rightarrow (x \in W \wedge |x| = \alpha \geq |y|) \\ \wedge \forall z\, [\forall n\, B^{\alpha}_{(z)_n} \text{ is wellfounded } \rightarrow \\ B^{\alpha}_{\tau(z)} \text{ is wellfounded } \wedge |B^{\alpha}_{\tau(z)}| \geq g(\sup_n |B^{\alpha}_{(z)_n}|)] \end{aligned}$$

for some $g \colon \alpha^+ \rightarrow \alpha^+$ with $[g]_{\mu_\alpha} \geq f(\alpha)$.

II has a winning strategy σ for any f, and this suggests the definition of T^{++}:

$(\sigma, r, \vec{\alpha}, x, \tau, y, z, \vec{\beta}, \vec{\gamma}) \in T^{++}$ iff:

1. $(r, \vec{\alpha}) \in [T_W]$.
2. $\sigma(r) = (x, \tau)$.
3. $(x, y, \vec{\beta}) \in [V]$
4. $\tau(y) = z$.
5. $(x, a, \vec{\gamma}) \in [U]$

The properties of U and V show that T^{++} has the desired property.

The countable exponent θ case.

Fix a bijection $\pi\colon \omega \cdot \theta \to \omega$.

We code cofinally in κ^+, κ^{++} many ordinals using sections of our trees: T^+_x, T^{++}_x.

Suppose $\mathcal{P}$ is a partition of the block functions from $3 \times \theta$ to $(\kappa, \kappa^+, \kappa^{++})$.

Consider the game $G_{\mathcal{P}}$:

$$\begin{array}{ll} \text{I} & x, y, z \\ \text{II} & x', y', z' \end{array}$$

(1) If there is an $j < \omega \cdot \theta$ such that $(x)_{\pi(j)} \notin P_0$ or $(x')_{\pi(j)} \notin P_0$, then player I wins iff for the least such j, $(x)_{\pi(j)} \in P_0$.

(2) Suppose next that there is an $\alpha < \kappa$ such that one of the following holds.

(a) There is a $j < \omega \cdot \theta$ such that either $T^+_{(y)_{\pi(j)}} \restriction \alpha$ or $T^+_{(y')_{\pi(j)}} \restriction \alpha$ is illfounded.

(b) There is a $\beta < \alpha^+$ and a $j < \omega \cdot \theta$ such that either $T^{++}_{(z)_{\pi(j)}} \restriction \beta$ or $T^{++}_{(z')_{\pi(j)}} \restriction \beta$ is illfounded.

Let $\alpha < \kappa$ be least such that (a) or (b) above holds. If (a) holds, let j be least such that (a) holds for α and this j. In this case, Player I wins provided $T^+_{(y)_{\pi(j)}}$ is wellfounded. If (a) does not hold at α, but (b) does, let (β, j) be lexicographically least such that (b) holds. Player I wins in this case provided $T^{++}_{(z)_{\pi(j)}} \restriction \beta$ is wellfounded.

Assume II has a winning strategy τ.

We define c.u.b. sets $C_0 \subseteq \kappa$, $C_1 \subseteq \kappa^+$, and $C_2 \subseteq \kappa^{++}$.

For example, to define C_2 we define for $\alpha \in C$, $\beta, \gamma < \alpha^+$ and $j < \omega \cdot \theta$:

$$\begin{aligned} A_{\alpha,\beta,\gamma,j} = & \{(x,y,z) \,;\, \forall j\, ((x)_{\pi(j)} \in P_0 \wedge \varphi_0((x)_{\pi(j)}) < \alpha) \\ & \wedge \forall \alpha' < \alpha\; \forall \beta < (\alpha')^+\; \forall j\, (T^+_{(y)_{\pi(j)}} \restriction \alpha \text{ and } T^{++}_{(z)_{\pi(j)}} \restriction \beta \text{ are wellfoun} \\ & \wedge \forall j\, |T^+_{(y)_{\pi(j)}} \restriction \alpha| < \beta \wedge \forall (\beta', j') \leq_{\text{lex}} (\beta, j)\, (|T^{++}_{(z)_{\pi(j)}} \restriction \beta| \leq \eta)\}. \end{aligned}$$

We have: $A_{\alpha,\beta,\gamma,j} \in \mathbf{\Delta}^{\alpha}_1$.

Since τ is winning for Player II, for each $(x, y, z) \in A_{\alpha,\beta,\eta,j}$, if $\tau(x, y, x) = (x', y', z')$ then $\forall (\beta', j') \leq_{\text{lex}} (\beta, j)$ $T^{++}_{(z')_{\pi(j')}} \restriction \beta$ is wellfounded.

By boundedness,

$$\rho_2(\alpha, \beta, \eta, j) := \sup\{|T^{++}_{(z')_{\pi(j')}} \restriction \beta| ;\ (x', y', z') \in \tau[A_{\alpha,\beta,\eta,j}] \wedge j' \leq j\} < \alpha^+.$$

Let $C^{\alpha}_2 \subseteq \alpha^+$ be c.u.b. closed under ρ_2. The C^{α}_2 lift to $C_2 \subseteq \kappa^{++}$.

Exponent κ

We use generic codes (Kechris-Woodin) and the uniform coding lemma.

Let $U(Q, z, x, y)$ be universal for the syntactic class $\mathbf{\Sigma}_1(Q)$, where Q is a binary predicate symbol.

Can take

$$U(Q, z, x, y) \leftrightarrow \exists w \, (S(z, \langle x, y, w\rangle) \wedge \forall n \, W(((w)_n)_0, ((w)_n)_1)).$$

where S is universal $\mathbf{\Sigma}^1_1$.

Let R'_α code $\{(x, y) \colon \varphi(x) < \varphi(y) \leq \alpha\}$ and $\leq$ version.

We write $U_z(R'_\alpha)(x, y)$ for $U(R'_\alpha, z, x, y)$.

Uniform Coding Lemma says that if $A \subseteq \omega^\omega \times \omega^\omega$ with $\mathrm{dom}(A) = P$, then there is a $z \in \omega^\omega$ such that for all $\alpha < \kappa$, $U_z(R'_\alpha)$ is a choice subrelation for $A \restriction P_{\leq\alpha}$.

Let $\overline{U}_z(R'_\alpha)$ be a uniformization of $U_z(R'_\alpha)$ (using scale $\{\varphi_n\}$.

For $\alpha < \kappa$, let α' be the next reliable (w.r.t. $\{\varphi_n\}$).

Let $G \colon \kappa^\omega \to \omega^\omega$ be a generic coding function. So for any $s \in \kappa^\omega$, $x = G(\alpha^\frown s) \in P$, $\varphi(x) \leq \alpha$, and if s enumerates an honest set then $|x| = \alpha$.

Given reals x, y, z we say:

1. x codes a function at $\alpha < \kappa$ if $|a| = \alpha$ implies $\exists b \in P\ \overline{U}_x(R'_\alpha)(a, b)$. Also, if $|a| = |a'|$ then $|b| = |b'|$.
2. y is good at $\delta < \alpha \in C_\omega$ if $\forall^* a \in P_\delta\ \exists b\ \overline{U}_y(R'_\delta)(a, b)$ and $T^+_b \restriction \alpha$ is wellfounded (good at α if good at all $\delta < \alpha$).
3. z is good at $\delta < \alpha \in C_\omega, \beta < \alpha^+$ if $\forall^* a \in P_\delta\ \exists b\ \overline{U}_z(R'_\delta)(a, b)$ and $T^{++}_z \restriction \beta$ is wellfounded (good at α if good at all $\delta < \alpha$, $\beta < \alpha^+$).

$f_x(\alpha) =$ the unique value of $|b|$.

$g_y(\delta, \alpha) =$ least $\gamma < \alpha^+$ such that $\forall^* a \in P_\delta,\ |T^+_b \restriction \alpha| \leq \gamma$.

$h_z(\delta, \alpha, \beta) =$ least $\gamma < \alpha^+$ such that $\forall^* a \in P_\delta,\ |T^{++}_b \restriction \beta| \leq \gamma$.

I plays x, y, z, II plays x', y', z'.

Let α be least such that one of the following holds.

1. For some $\delta < \alpha$, y or y' is not good at (δ, α).
2. For some $\delta < \alpha, \beta < \alpha^+$, z or z' is not good at (δ, α, β).
3. x or x' does not code a function at α.

If (1) holds, then I wins iff for the least such δ, y is good at (δ, α). Suppose (1) does not hold, but (2) holds. Then I wins iff for the lexicographically least pair (β, δ) we have z is good at δ, α, β. If (1) and (2) don't hold, but (3) holds, then I wins iff x codes an ordinal at α.

Otherwise, we have functions f_x, $f_{x'}$, g_y, $g_{y'}$, h_z, $h_{z'}$.

- f_x, $f_{x'}$ together produce $F\colon \kappa \to \kappa$.
- g_y, $g_{y'}$ together produce $G\colon \kappa \to \kappa^+$.
- h_z, $h_{z'}$ together produce $H\colon \kappa \to \kappa^{++}$.

II wins the run in this case iff $\mathcal{P}(F, G, H) = 1$.

From a winning strategy τ for II, say, we define the homogeneous sets $C_0 \subseteq \kappa$, $C_1 \subseteq \kappa^+$ and $C_1 \subseteq \kappa^{++}$.