Stable Homogeneous Trees

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April 30, 2011
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We consider the problem of getting the complete scale and Suslin cardinal analysis from just $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$.

We recall the basic definitions and results.

$A \subseteq \omega^\omega$ is κ-Suslin if there is a tree T on $\omega \times \kappa$ with

$$\begin{aligned} A = p[T] &= \{x \colon \exists f \in \kappa^\omega (x, f) \in [T]\} \\ &= \{x \colon \exists f \in \kappa^\omega \ \forall n \ (x \restriction n, f \restriction n) \in T\} \end{aligned}$$

We let $S(\kappa)$ be the collection of κ-Suslin sets, $S(<\kappa) = \bigcup_{\lambda<\kappa} S(\lambda)$.

We say κ is a Suslin cardinal if $S(\kappa) - S(<\kappa) \neq \emptyset$.

A semi-scale on $A \subseteq \omega^\omega$ is a sequence of norms $\varphi_n \colon A \to \mathrm{On}$ such that if $x_i \in A$, $x_i \to x$, and $\varphi_n(x_i)$ are eventually equal to λ_n, then $x \in A$.

A scale on A is a semi-scale which also satisfies $\varphi_n(x) \leq \lambda_n$.

Fact

For all cardinals κ, A is κ-Suslin iff A admits a scale into κ.

We may assume wlog that the norms of a (semi) scale are regular, that is, they map onto an ordinal.

If $A = p[T]$ is a Suslin representation for A, we get a semi-scale φ_n^T on A by: $\varphi_n^T(x) = \ell_x(n)$, where ℓ_x is the left-most branch of T_x.

Conversely, given a semi-scale φ_n on A, let $(s, \vec{\alpha}) \in T_\varphi$ iff $\exists x \sqsupseteq s\ \forall n < \mathrm{lh}(s)\ \varphi_n(x) = \alpha_n$.

If φ is a scale on A, then $(\varphi_0(x), \varphi_1(x), \dots)$ is the leftmost branch of T_φ.

If $A = p[T]$ we can get a scale on A by

$$\varphi_n(x) = \langle \ell_x(0), \ell_x(1), \dots, \ell_x(n-1) \rangle$$

We have the following facts.

Fact

*For any tree T, we have: $T_{\varphi_T} \subseteq T$ and $[T_{\varphi_T}] = [T]$.
If $T = T_\varphi$ is the tree of a scale φ_n then $T_{\varphi_T} = T$.*

Fact

For any scale φ_n on A we have: $\varphi = \varphi_{T_\varphi}$.

Fact

Both operations $T \mapsto T_{\varphi_T}$ and $\varphi \mapsto \varphi_{T_\varphi}$ are idempotent. [For any tree T we have $\varphi_T = \varphi_{T_{\varphi_T}}$.]

Recall the definition of a $\boldsymbol{\Gamma}$ scale.

Definition

A norm $\varphi\colon A \to \mathrm{On}$ is a $\boldsymbol{\Gamma}$-norm if the following norm relations are in $\boldsymbol{\Gamma}$:

$$x <^* y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \varphi(x) < \varphi(y))$$
$$x \leq^* y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \varphi(x) \leq \varphi(y))$$

Each $A_{\leq\alpha} = \{x \in A\colon \varphi(x) \leq \alpha\}$ is in $\boldsymbol{\Delta}$.

If $\boldsymbol{\Delta}$ is closed under $\wedge$, then each $A_\alpha = \{x \in A\colon \varphi(x) = \alpha\}$ is in $\boldsymbol{\Delta}$ as well.

Definition

φ_n is a $\boldsymbol{\Gamma}$-scale if all of the norm relation $<^*_n$, $\leq^*_n$ are in $\boldsymbol{\Gamma}$.

Definition

$\boldsymbol{\Gamma}$ has the scale property, scale($\boldsymbol{\Gamma}$), if every $A \in \boldsymbol{\Gamma}$ admits a $\boldsymbol{\Gamma}$ scale.

The scale property is the main structural representation for sets $A \subseteq \omega^\omega$ in descriptive set theory.

If scale($\boldsymbol{\Gamma}$) and $\wedge\boldsymbol{\Delta} = \boldsymbol{\Delta}$, then T_φ is a tree (φ a $\boldsymbol{\Gamma}$-scale on a $\boldsymbol{\Gamma}$ set A) on an ordinal $\leq \delta(\boldsymbol{\Gamma})$ where:

Definition

$\delta(\boldsymbol{\Gamma})$ = the supremum of the lengths of the $\boldsymbol{\Delta}$ prewellorderings of ω^ω.

Homogeneous Trees

The notion of homogeneous tree was formulated independently by Kunen and Martin, then isolated and given in its current form by Kechris.

Definition

A measure on a set X is a countably additive ultrafilter on X.

Recall that under AD every ultrafilter on a set X is a measure.

Definition

A tree T is homogeneous if there are measures $\{\mu_s : s \in \omega^{<\omega}\}$ on T_s satisfying:

- If $s \sqsubseteq s'$ then $\mu_{s'}$ projects to μ_s.
- (homogeneity property)
  For all $x \in \omega^\omega$ if T_x is illfounded then for any $A_{x\restriction n}$ with $\mu_{x\restriction n}(A_{x\restriction n}) = 1$, there is an f such that $\forall n\ f \restriction n \in A_n$.

Note: This is equivalent to saying that the direct limit of the ultrapowers by the measures $\mu_{x\restriction n}$ is wellfounded.

Fact (ZF, Martin): If $A = p[T]$ where T is homogeneous, then A is determined.

Weakly Homogeneous Trees

Definition (Weakly Homogeneous, strong form)

T is weakly homogeneous if there are measures $\mu_{s,t}$ on T_s defined for a set of (s,t) in a tree, and sets $A_{s,t} \subseteq T_s$ satisfying:

- $\mu_{s,t}(A_{s,t}) = 1$.
- If s' immediately extends s and $A_{s,t}$ is defined, then the $A_{s',t'}$ for t' extending t partition $T_{s'} \cap \{\vec{\alpha} \colon \pi_{s',s}(\vec{\alpha}) \in A_{s,t}\}$.
- If $(s',t') \sqsupseteq (s,t)$ then $\pi_{(s',t'),(s,t)}(\mu_{(s',t')}) = \mu_{(s,t)}$.
- (homogeneity property)
  For any $x, y \in \omega^\omega$, if $\mu_{x \restriction n, y \restriction n}$ is defined for all n and $\exists f \; \forall n \; (x \restriction n, f \restriction n) \in A_{x \restriction n, y \restriction n}$, then for any $B_{x \restriction n, y \restriction n}$ with $\mu_{x \restriction n, y \restriction n}(B_{x \restriction n, y \restriction n}) = 1$ we have that $\exists g \; \forall n \; (x \restriction n, g \restriction n) \in B_{x \restriction n, y \restriction n}$.

Definition (Weakly Homogeneous, weak form)

There are measures $\mu_{s,t}$ with $\mu_{s,t}(T_s) = 1$ satisfying:

- If $(s', t') \sqsupseteq (s, t)$ then $\pi_{(s',t'),(s,t)}(\mu_{s',t'}) = \mu_{s,t}$.
- For any $x \in \omega^\omega$, if T_x is illfounded then $\exists y \in \omega^\omega$ such that $\langle \mu_{x\restriction n, y\restriction n} \rangle$ has wellfounded direct limit: for any $\mu_{x\restriction n,y\restriction n}(B_{x\restriction n,y\restriction n}) = 1$ there is a g such that $\forall n\ (x \restriction n, g \restriction n) \in B_{x\restriction n,y\restriction n}$.

Alternate Version: There is a countable set $\mathcal{M}$ of measures such that whenever T_x is illfounded then there is a sequence $\mu_1, \mu_2, \dots$ of measure in $\mathcal{M}$ with $\mu_n(T_{x\restriction n}) = 1$ such that $\langle \mu_{x\restriction n} \rangle$ has wellfounded direct limit.

Fact (Woodin)

In ZFC the strong and weak forms of the definition are equivalent.

Fact

The weak form of the definition is enough to make the Martin-Solovay construction work.

Fact

The strong form of the definition s equivalent to saying that there is a homogeneous tree $\bar{T}$ on $\omega \times \omega \times \kappa$ which is isomorphic to T (under a bijection of $\omega \times \kappa$ with κ).

So, a set $A \subseteq \omega^\omega$ is weakly homogeneously Suslin iff it is the projection of a strongly homogeneously Suslin set.

Martin-Solovay Tree

Let $(T, \vec{\mu})$ be a weakly homogeneous tree with $A = p[T]$. The Martin-Solovay tree T' is a tree with

$$\omega^\omega - A = p[T']$$

defined as follows.

$(s, \vec{\alpha}) \in T'(T, \vec{\mu}, \lambda)$ iff $\exists f \colon T_s \to \lambda$ order-preserving w.r.t. the BK ordering on T_s such that $\forall i < \text{lh}(s)$

$$\alpha_i = [f^i]_{\mu_{s_i, t_i}}$$

where (s_i, t_i) is a reasonable enumeration of $\omega^{<\omega} \times \omega^{<\omega}$ and $f^i = f \restriction \{\vec{\alpha} \colon |\vec{\alpha}| = i \wedge (s_i, \vec{\alpha}) \in T\}$ if $s_i \sqsubseteq s$, and $f = 0$ otherwise.

Stable Homogeneous Trees

Question

When is the semi-scale from the Martin-Solovay tree T' a scale?

We make the following definition for weakly and strongly homogeneous trees. For simplicity we state them for homogeneous trees.

For (T, μ_s) a homogeneous tree, if $\vec{A} = \{A_s \colon s \in \omega^{<\omega}\}$ with $A_s \subseteq T_s$, we let

$$T^{\vec{A}} = \{(s, \vec{\alpha}) \in T \colon \forall i \leq |s| \ \vec{\alpha} \restriction i \in A_{s\restriction i}\}.$$

Let T_x, $T^{\vec{A}}$ denote the sections of T and $T^{\vec{A}}$ at x.

Let f_x, $f_x^{\vec{A}}$ denote the rank functions on T_x, $T_x^{\vec{A}}$ (for T_x wellfounded). Likewise for $f_{x\restriction n}$ and $f_{x\restriction n}^{\vec{A}}$.

Definition

The homogeneous tree (T, μ_s) is stable if there are measure one sets A_s w.r.t. the μ_s such that for any $x \in B = \omega^\omega - A$ $(A = p[T])$ and any measure one sets $B_{x\restriction n}$ (w.r.t. $\mu_{x\restriction n}$) we have that

$$[f^{\vec{A}}_{x,n}]_{\mu_{x\restriction n}} \leq [f^{\vec{B}}_{x,n}]_{\mu_{x\restriction n}}$$

for all n.

Theorem

If (T, μ_s) is a stable homogeneous tree, then the semi-scale φ_n from the Martin-Solovay tree T' constructed from $T^{\vec{A}}$ is a scale. Here $\varphi_n(x) = [f^{\vec{A}}_x]_{\mu_{x\restriction n}}$.

Proof: Let $x \in B = \omega^\omega - A$. So, $T_x^{\vec{A}}$ is wellfounded.

Let $x_i \in B$ with $x_i \to x$ and assume each $\varphi_n(x_i)$ is eventually equal to λ_n. Say $\varphi_n(x_i) = \lambda_m$ for all $i \geq n$. Assume $x_i \restriction i = x \restriction i$.

For each n let $f_n \colon T^{\vec{A}}_{x_n \restriction n} = T^{\vec{A}}_{x \restriction n} \to \mathrm{On}$ be such that

$$\forall i < n \; \lambda_i = [f_n^i]_{\mu_{x \restriction i}}$$

where f_n^i is the subfunction of f_n induced by restriction.

Fix measure one sets $B_{x \restriction n}$ such that for all $i_1, i_2 \geq n$ we have

$$f^n_{m_1} \restriction B_{x \restriction n} = f^n_{m_2} \restriction B_{x \restriction n}.$$

Then the f_n restricted to the $\vec{B}$ define an order-preserving $f \colon T_x^{\vec{B}} \to \mathrm{On}$. By stability we have $[f_x^{\vec{A}}]_{\mu_{x \restriction n}} = [f_x^{\vec{B}}]_{\mu_{x \restriction n}}$ for all n, so φ is actually a scale.

Main theorem

Theorem (ZF + AD + DC)

Every homogeneous tree $(T, \{\mu_s\})$ on $\kappa < \Theta$ is stable.

Proof uses ideas from the Martin-Woodin proof that all trees are weakly homogeneous.

sketch of proof: Let ν be a fine measure on $\mathcal{P}_{\omega_1}(\cup_n \mathcal{P}(\kappa^n))$.

For $\sigma \in \mathcal{P}_{\omega_1}(\cup_n \mathcal{P}(\kappa^n))$, let $A^\sigma = \{A^\sigma_s\}_s$ where $A^\sigma_s = \cap\{A \in \sigma \colon \mu_s(A) = 1\}$.

We show that for ν almost all σ that the A^σ_s stabilize T. We suppose not.

Uniformly in σ we define a tree U^σ on $\omega \times \lambda^{<\omega} \sim \omega \times \lambda$ where $\lambda = \sup j_{\mu_s}(\kappa)$.

For $x \in B$ the tree $(U_\sigma)_x$ attempts to produce ordinals $[f^i_j]_{\mu_{x \restriction i}}$ which witness that the sets A^σ have not yet attained the minimal ranking functions.

They do this, roughly speaking, by describing embeddings of T_x (on measure one sets) into proper initial segments of $T_x^{A^\sigma}$.

We set $((s(0), \dots, s(n-1)), (\beta_0, \dots, \beta_{n-1})) \in U^\sigma$ iff:

1.) $\beta_0 \in \omega$.

2.) Each β_i for $i > 0$ codes a finite sequence of integers t_i extending s (roughly, a commitment that later extensions of s must follow t_i) along with a finite sequence of ordinals $(\beta^i_0, \dots, \beta^i_{|t_i|-1})$.

3.) s must be compatible with all t_i for $i < |s|$.

4.) Let f^i_j represent β^i_j with respect to $\mu_{s \restriction i}$. Then there are $\mu_{s \restriction i}$ measure one sets restricted to which the maps $(\eta_0, \dots \eta_{i-1}) \mapsto (f^i_0(\vec{\eta}), \dots, f^i_{|t_i|-1})$ give an order-preserving from the BK ordering on T_{s_n} to the BK ordering on T^{A^σ}.

5.) For $i < \beta_0$, each f^i_j is almost everywhere the identity function, and for $i = \beta_0$ we have that almost everywhere that $(f^i_0(\vec{\eta}), \dots, f^i_{|t_i|-1}(\vec{\eta}))$ is a properly less than $\vec{\eta}$ in $T_{|t_i|}$.

6.) We weave in the Martin-Solovay tree into U_σ as well. Say at every even level of U_σ we put in ordinals from the Martin-Solovay tree (so a branch through $(U_\sigma)_x$ also gives a branch through $(T')_x$ and so proves $x \in B$).

For any x, $(U^\sigma)_x$ is illfounded iff $x \in B$ and there are measure one sets $B_{x\restriction n} \subseteq A^\sigma_{x\restriction n}$ (with respect to $\mu_{x\restriction n}$) such that for some m, $[f_x^{\vec{B}}]_{\mu_{x\restriction m}} < [f_x^{A^\sigma}]_{\mu_{x\restriction m}}$.

So, for ν almost all σ there is a leftmost branch $(x^\sigma, \vec{\beta}^\sigma)$ through U^σ. By countable additivity we may fix $x = x^\sigma$ on a measure one set.

Fix any σ where $x^\sigma = x$ and also σ contains measure one sets $C_{x\restriction n}$ (w.r.t. the $\mu_{x\restriction n}$) which attain the minimal values for $[f_x^{\vec{C}}]_{\mu_{x\restriction n}}$ (which we can do by countable choice).

The branch $\vec{\beta}^\sigma$ gives a proper embedding from $T_x^{\vec{D}}$ into $T_x^{A^\sigma} \subseteq T_x^{\vec{C}}$ for some measure one sets $D_{x\restriction n}$. This contradicts the choice of the $C_{x\restriction n}$.

Application

We use the above result to complete the Scale/Suslin cardinal analysis developed by the Cabal, assuming ZF + AD + DC. Reader can consult chapter 3 of Handbook article "Structural Consequences of Determinacy."

Assume κ is a limit Suslin cardinal with $\boldsymbol{\Gamma} = \S(\kappa)$ closed under quantifiers. (Type IV case).

Let Λ be the selfdual Martin pointclass which constructs the nest Suslin cardinal. Martin defined this class, Steel gave another characterization.

Let $\lambda = o(\Lambda)$ =next Suslin cardinal after κ.

Let $\Sigma_0^\lambda = \bigcup_\omega \Lambda$. Define Π_n^λ, Σ_n^λ as usual.

Previous results showed scale($\boldsymbol{\Sigma}_{2n}^\lambda$), scale($\boldsymbol{\Pi}_{2n+1}^\lambda$) for $n \geq 1$ with corresponding Suslin cardinals

$$\delta_1 = \delta(\boldsymbol{\Pi}_1^\lambda) = \lambda^+, \quad \lambda_3 = (\delta_3)^-, \quad \delta_3 = \delta(\boldsymbol{\Pi}_3^\lambda), \ldots$$

Previous arguments did nor give that Λ was a self-justifying system, nor the scale property at $\boldsymbol{\Sigma}_0^\lambda$, $\boldsymbol{\Pi}_1^\lambda$

Theorem

Λ is self-justifying and scale($\boldsymbol{\Sigma}_0^\lambda$), scale($\boldsymbol{\Pi}_1^\lambda$).

Proof: Let A be $\mathbf{\Gamma}$-complete and write $A = p[T]$ with T a tree on $\omega \times \kappa$. By Martin-Steel we may assume T is homogeneous. By theorem we may assume T has been stabilized.

For $x \in B = \omega^\omega - A$, let $\varphi_n(x) = [f_x]_{\mu_{x \restriction n}}$, where $f_x \colon T_x \to \mathrm{On}$ is the ranking function.

A direct computation shows φ_n is a Λ-norm. The $\{\varphi_n\}$ form a scale since T is stable.

This shows B admits a scale all of whose norms are in Λ. It follows that this also holds for all sets in Λ. The other results easily follow.