A $\Pi^0_3$ Completeness Phenomenon in Group Actions

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Let $X$ be a Polish space and $G$ a countable group.

Let $(g, x) \mapsto g \cdot x$ be a Borel action of $G$ on $X$.

The groups action induces a countable Borel equivalence relation $\sim$ on $X$:

$$x \sim y \text{ iff } \exists g \in G \ (g \cdot x = y)$$

**Theorem (Feldman-Moore)**

Every countable Borel equivalence relation is induced by the action of a countable group.
For $G$ a countable group, the space $2^G$ is a compact Polish space.

$G$ acts on $2^G$ by the shift-action:

$$g \cdot x(h) = x(g^{-1}h)$$

This is a continuous action of $G$ on the space $X = 2^G$. 
A notion of complexity is provided by the following notion of reduction:

**Definition**

\((X, E) \leq (Y, F)\) if there is a Borel function \(f : X \to Y\) such that

\[(x_1 E x_2) \leftrightarrow (f(x_1) F f(x_2)).\]

The simplest equivalence relations are the smooth or tame relations:

**Definition**

\((X, E)\) is smooth if \((X, E) \leq (Y, =) = \text{id}.$

Being smooth for a countable Borel equivalence relation is equivalent to having a Borel selector.
By Silver dichotomy, \( \text{id} \leq (X, E) \) for any Borel equivalence relation \( E \) on an uncountable Polish space \( X \).

**Definition**

\( E_0 \) on \( \omega^\omega \) (or \( 2^\omega \)) is defined by:

\[
(x E_0 y) \iff \forall^* n \ (x(n) = y(n))
\]

By the **Harrington-Kechris-Louveau** dichotomy, for any Borel equivalence relation \( E \), either \( E \leq \text{id} \) or \( E_0 \leq E \).
The Countable Borel equivalence relations $E \leq E_0$ are exactly the **hyperfinite** relations.

**Definition**

$E$ is hyperfinite if $E$ can be written as an increasing unions $E = \bigcup_n E_n$ where each $E_n$ is a **finite** Borel equivalence relation.

**Theorem (Slaman-Steel)**

The following are equivalent:

- $E$ is hyperfinite.
- $E \leq E_0$.
- The (infinite) $E$ classes can be uniformly $\mathbb{Z}$-ordered in a Borel manner.
We say the countable group $G$ is hyperfinite if all its Borel actions are hyperfinite.

**Question**
Which groups are hyperfinite?

$\mathbb{Z}$ is hyperfinite by Slaman-Steel.

**Theorem (Weiss)**
Each $\mathbb{Z}^n$ is hyperfinite.

**Theorem (Gao-J)**
All countable abelian groups are hyperfinite.
If every action of $G$ is hyperfinite, then $G$ must be amenable.

**Conjecture (Kechris)**

Every amenable group is hyperfinite.

**Fact**

*For $G = F_2$, the shift action of $G$ on $2^G$ is a universal countable Borel equivalence relation.*

Also, the shift action of $G$ on $\mathbb{R}^G$ is universal for the Borel actions of $G$. 
For $x \in 2^G$, let $[x]$ denote the equivalence class of $x$ under the shift action of $G$ on $X = 2^G$.

Let $F \subseteq X$ denote the free part of $2^G$, that is.

$$x \in F \iff \forall g \neq 1 (g \cdot x \neq x).$$

$F$ is a (dense) $G_\delta$ in $X$, so $F$ is a Polish space in the subspace topology.
Free Orbit Closures

We say $x \in F$ has free orbit closure if $[x] \subseteq F$.

The following definition was formulated independently by Glasner and Uspensky.

**Definition**

A 2-coloring on a countable group $G$ is a $c: G \to \{0, 1\}$ satisfying:

$$\forall s \in G \exists T \subseteq G^{<\omega} \forall g \in G \exists t \in T \ (c(gt) \neq c(gst)).$$
The following are equivalent.

- $x \in 2^G$ has free orbit closure.
- $x$ is a 2-coloring of $G$.
- If $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ are relatively closed subsets of $F$ with $M_i \cap [x] \neq \emptyset$ for all $i$, then $\bigcap_i M_i \neq \emptyset$.

Corollary

If $G$ has a 2-coloring, then every decreasing sequence $\{M_i\}$ of relatively closed, complete sections of $F$ has a non-empty intersection.
Suppose $x \in 2^G$ is a 2-coloring. Suppose $y = \lim_n g_n^{-1} \cdot x \in [x]$.

Let $1 \neq s \in G$, we show $s^{-1} \cdot y \neq y$.

Let $T \in G^{<\omega}$ be as in the definition of 2-coloring. Let $n$ be large enough so that $y$ and $g_n^{-1} \cdot x$ agree on $T \cup sT$. Let $t \in T$ be such that $x(g_nt) \neq x(g_nst)$. Then:

$$s^{-1} \cdot y(t) = y(st) = g_n^{-1} \cdot x(st) = x(g_nst) \neq x(g_nt) = g_n^{-1} \cdot x(t) = y(t)$$

So, $[x] \subseteq F$. 

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Suppose $\overline{x} \subseteq F$. Let $s \in G$. Let $T_n$ be the first $n$ group elements.

If $T_n$ doesn’t work for $s$, let $g_n \in G$ be a counterexample. So, $x(g_n t) = x(g_n st)$ for all $t \in T_n$.

Let $y$ be a limit of $\{g_n^{-1} \cdot x\}$, (w.l.o.g.) $y = \lim_n g_n^{-1} \cdot x \in F$.

But, for any $t \in G$ we have (for large enough $n$, $t \in T_n$):

$$s^{-1} \cdot y(t) = y(st) = g_n^{-1} \cdot x(st) = x(g_n st) = x(g_n t) = g_n^{-1} \cdot x(t) = y(t).$$

This contradicts $y \in F$. 
Theorem

Every countable group $G$ has the 2-coloring property.

The proof uses a construction of certain “marker set” for an arbitrary group $G$ that we will also use in our result here.
If \((X, E)\) is a Borel equivalence relation, a set of marker regions for \(E\) is a finite subequivalence relation \(R \subseteq E\).

A marker set for \(E\) is a Borel set \(M \subseteq X\) such that \(M \cap [x] \neq \emptyset\) for all \(x \in X\).

We say \(M\) is a marker set for the marker regions \(R\) if 
\[|M \cap [x]_R| = 1\] for all \(x \in X\).
For $G$ a countable group, a set of marker regions is a finite subequivalence relation $R$ of $G$.

A marker set is a non-empty $M \subseteq G$.

We say $R$ is a tiling of $G$ if there is a finite $F \subseteq G$ such that each $R$ class is of the form $xF$.

**Note:** Marker regions and sets for $E$ give marker regions and sets for $G$ (for free action).
Definition
A sequence \((\Delta_n, F_n)\) of tilings of \(G\) is **coherent** if each class \(\delta F_n\) of \(R_n\) is contained in a class \(\delta' F_{n+1}\) of \(R_{n+1}\). We say the sequence is **cofinal** if every finite subset of \(G\) is contained in some \(R_n\) class. We say it is **centered** if \(1 \in \Delta_n, 1 \in F_n\) for all \(n\).

Definition
\(G\) is a **c.c.c. group** if it has a coherent, cofinal, centered tiling.

Remark
There is no loss of generality in assuming the tilings are centered. In this case \(F_n \subseteq F_{n+1}\).
Question
Which groups are c.c.c. groups?

Theorem
Every nilpotent and every solvable group with finitely generated quotients is c.c.c. Every free product of non-trivial groups is c.c.c.

Is every solvable group c.c.c.? Is there a non-c.c.c. group?
Minimal flows

**Definition**

$x \in 2^G$ is **minimal** if whenever $y \in [x]$ then $[y] = [x]$.

We give a combinatorial reformulation of minimality.

**Lemma**

$x$ is minimal iff for all $A \in G^{<\omega}$ there is a $T \in G^{<\omega}$ such that:

$$\forall g \in G \ \exists t \in T \ \forall a \in A \ (x(a) = x(gta)).$$
For \( G \) a countable group, a straightforward computation gives:

**Fact**

*The set of \( x \in 2^G \) which are 2-colorings of \( G \) is \( \Pi_3^0 \).*

**Fact**

*The set of \( x \in 2^G \) which are minimal is \( \Pi_3^0 \).*

So, the set of minimal 2-colorings is also \( \Pi_3^0 \).

**Question**

*When are these sets \( \Pi_3^0 \)-complete?*
It is fairly easy to see that the set of 2-colorings is $\Sigma^0_2$-hard, and the set of minimal colorings is $\Pi^0_3$-complete.

**Remark**
The set of 2-colorings is meager, in fact, the set of colorings that “block” a particular $s \neq 1$ is meager. It is also dense, so it cannot be $\Pi^0_2$.

**Remark**
The $\Pi^0_3$-completeness of the minimal colorings follows from our general marker region construction by a similar but easier argument to the main result.
Main Result

**Definition**

$G$ is a **flecc** group if there is a finite $A \subseteq G$ such that for all $1 \neq g \in G$, there is an $h \in G$ and $i \in \omega$ such that $hg^ih^{-1} \in A - \{1\}$.

The following is our main result.

**Theorem**

The set of 2-colorings is $\Pi^0_3$-complete iff $G$ is not a flecc group. If $G$ is a flecc group, the the set of 2-colorings is $\Sigma^0_2$. 

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Definition

The limit extended conjugacy class of \( g \in G \) is

\[
\text{lecc}(g) = \bigcap_{i \geq 1} \{ h^i g h^{-i} : h \in G, i \geq 1 \}
\]

for \( g \) of infinite order, and for \( g \) of finite order, every conjugacy class of any \( g^k \) where \( g^k \) has prime power order is a lecc of \( g \).

Fact

If \( \text{lecc}(g_1) \cap \text{lecc}(g_2) \neq \emptyset \), then \( \text{lecc}(g_1) = \text{lecc}(g_2) \).

Theorem

\( G \) is a flecc group iff \( G \) has only finitely many lecc classes.
Example
Every quasi-cyclic group $\mathbb{Z}_p^\infty$ is a flecc group.

Fact
Every abelian flecc group is a torsion group.

Fact
Any infinite sum of non-trivial groups is not a flecc group.

Theorem
An abelian group is a flecc group iff it is a finite sum of quasicyclic groups and reduced $p$-groups each with finitely many elements of order $p$. 
Corollary
The collection of abelian flecc groups is closed under products.

Question
Are the flecc groups closed under products?

Fact
The collection of flecc groups is not closed under quotients or subgroups.

For the first, take a (Zippin) group $G = \langle x, y_1, y_2, \ldots \rangle$ where $y_i^p = x$, $x^p = 1$.

For the second, use the fact that every torsion-free group is a subgroup of a group $G$ with 2 conjugacy classes (and so only finitely many lecc classes). So, every torsion-free group is a subgroup of a lecc group.
Proof that $G$ is flecc implies the set of 2-colorings is $\Sigma^0_2$.

Fact

- If $T$ blocks the element $s^i$, then $s$ is blocked by $T \cup sT \cup \cdots \cup s^{i-1}T$.
- If $T$ blocks $s$, then $hsh^{-1}$ is blocked by $hT$.

Let $G$ be flecc, and $A$ as in the definition of flecc. For each $s \in A$, let $T_s$ block $s$. By the above facts, there is a finite $T$ which blocks all $s \in G$.
So, $x$ is a 2-coloring of $G$ iff

$$\exists T \ \forall s \neq 1 \in G^{<\omega} \ \forall g \in G \ (\exists t \in T \ x(gt) \neq x(gst)).$$
We construct marker regions (partial tilings) \((\Delta_n, F_n)\) for a general countable group \(G\).

We have the following properties:

- \(1 \in F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots\).
- \(\Delta_0 \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots\), \(1 \in \Delta_n\).
- The \(\Delta_n\) translates of \(F_n\) are maximally disjoint in \(G\).
- \(F_n\) is a disjoint union of copies of \(F_{n-1}, \ldots, F_0\).
- The copies \(\gamma \Delta_k, k < n\) which intersect a copy \(\delta \Delta_n\) of \(F_n\) are exactly those of the form \(\delta \eta F_k\) where \(\eta \in F_n \cap \Delta_k\).
Dynamics of the Shift Action
Marker Regions for General $G$

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Each $F_n$ will have two distinguished points, $a_n$ and $b_n$.

The $a_n$ will be used to establish a marker identification property.

The $b_n$ will be free points used to make the coloring have desired properties (e.g., be a 2-coloring).

Marker Identification Property: There is a finite $A_n \subseteq G$ such that $\forall a \in A (c(ga) = c(a))$ iff $g \in \Delta_n$.

Fact

For any $A \in G^{<\omega}$ and coloring $c_A$ of $A$ with $c_A(1) = 1$, there is an $F_0 \supseteq A$ and $a_0, b_0 \in F_0$ and a coloring $c_{F_0}$ of $F_0 - \{a_0, b_0\}$ extending $c_A$ with the following identification property:

For any $\Delta_0 \subseteq G$ such that $\{\delta_0 F_0 : \delta_0 \in \Delta_0\}$ is pairwise disjoint, and any coloring $c \in 2^G$ extending the union of the $c_{F_0}$ copied to the translates $\delta_0 F_0$ (and $c = 0$ off this union), $c$ has the m.i.p. (using $F_0$).
Construction of $F_0$

Given $A \in G^{<\omega}$ and $c_A : A \to \{0, 1\}$.

Pick any distinct $a_0, b_0, x, y \notin A$. Let $B_2 = A \cup \{a_0, b_0, x, y\}$

Let $z \notin (B_2B_2^{-1} \cup B_2B_2)$. Let $B_3 = B_2 \cup \{z\}$.

Let $F_0 = B_3B_3$. Let $c_{F_0}(x) = c_{F_0}(y) = c_{F_0}(z) = 1 = c_A(1)$. Let $c_{F_0} = 0$ on $F_0 - B_3$.

This works.
Construction of $F_n$
To get a 2-coloring use the fact that there are exponentially many $(2^{\lambda n} \sim 2^{|H_n|})$ choices for the color of the $b_i$ in $F_n$ but only polynomially many colors to avoid (about $|H_n|^5$).

Given $g$, $gs$ with $s \in H_n$, by maximal disjointness of the $H_n$ find $\gamma \in F_n F_n^{-1}$ such that $g\gamma \in \Delta_n$. Then

$$gs\gamma = g\gamma (\gamma^{-1} s \gamma)$$

where $\gamma^{-1} s \gamma \in H_n^5$. 
Suppose $G$ is not a flecc group. Let $z \in 2^G$ be a 2-coloring as constructed.

For each $k$, let $s_k$ be such that $hs_k^\ell h^{-1} \not\in H_k$ for all $h \in G$ and all $\ell$ such that $s_k^\ell \neq 1$. Also, $s_k \not\in F_k F_k^{-1}$.

We construct for each $k$ another auxiliary $x_k \in 2^G$:

- $x_k$ will have period $s_k^{-1}$ (i.e., $x_k(g) = x_k(s_k g)$).
- $x_k$ will have the 2-coloring property for first $k$-many shifts.
- $x_k$ will be 0 off of a maximal disjoint collection of copies of $F_k$. It will follow the basic construction and so will the marker identification property for these copies.
Let $P \subseteq 2^{\omega \times \omega}$ be defined by:

$$\alpha \in P \iff \forall k \exists m \forall m > n \alpha(k, n) = 0.$$ 

For $u \in 2^{n \times n}$ we define a partial coloring $c_u$ of $G$, and then take

$$f(\alpha) = \bigcup_n c_{\alpha | n}$$

for our reduction function.

For each $n$, let $k = k(n)$ be least such that $\alpha(k, n) = 1$ (take $n$ if no such $k$).
Introduction
Dynamics of the Shift Action
Main Result

Marker Regions for General $G$

use $z$

$K_{0,k(0)}$

$K_{1,k(1)}$

$K_{n,k(n)}$

use $x_k$

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