A $\mathbf{\Pi}^0_3$ Completeness Phenomenon in Group Actions

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Let X be a Polish space and G a countable group.

Let $(g, x) \mapsto g \cdot x$ be a Borel action of G on X.

The groups action induces a countable Borel equivalence relation $\sim$ on X:

$$x \sim y \text{ iff } \exists g \in G \ (g \cdot x = y)$$

Theorem (Feldman-Moore)

Every countable Borel equivalence relation is induced by the action of a countable group.

For G a countable group, the space 2^G is a compact Polish space.

G acts on 2^G by the shift-action:

$$g \cdot x(h) = x(g^{-1}h)$$

This is a continuous action of G on the space $X = 2^G$.

A notion of complexity is provided by the following notion of reduction:

Definition

$(X, E) \leq (Y, F)$ if there is a Borel function $f : X \to Y$ such that

$$(x_1 \, E \, x_2) \leftrightarrow (f(x_1) \, F \, f(x_2)).$$

The simplest equivalence relations are the smooth or tame relations:

Definition

(X, E) is smooth if $(X, E) \leq (Y, =) = \mathrm{id}$.

Being smooth for a countable Borel equivalence relation is equivalent to having a Borel selector.

By Silver dichotomy, id $\leq (X, E)$ for any Borel equivalence relation E on an uncountable Polish space X.

Definition

E_0 on ω^ω (or 2^ω) is defined by:

$$(x\, E_0\, y) \leftrightarrow \forall^* n\ (x(n) = y(n))$$

By the Harrington-Kechris-Louveau dichotomy, for any Borel equivalence relation E, either $E \leq$ id or $E_0 \leq E$.

The Countable Borel equivalence relations $E \leq E_0$ are exactly the hyperfinite relations.

Definition

E is hyperfinite if E can be written as an increasing unions $E = \bigcup_n E_n$ where each E_n is a finite Borel equivalence relation.

Theorem (Slaman-Steel)

The following are equivalent:

- *E is hyperfinite.*
- *$E \leq E_0$.*
- *The (infinite) E classes can be uniformly $\mathbb{Z}$-ordered in a Borel manner.*

We say the countable group G is hyperfinite if all its Borel actions are hyperfinite.

Question

Which groups are hyperfinite?

$\mathbb{Z}$ is hyperfinite by Slaman-Steel.

Theorem (Weiss)

Each $\mathbb{Z}^n$ is hyperfinite.

Theorem (Gao-J)

All countable abelian groups are hyperfinite.

If every action of G is hyperfinite, then G must be amenable.

Conjecture (Kechris)

Every amenable group is hyperfinite.

Fact

For $G = F_2$, the shift action of G on 2^G is a universal countable Borel equivalence relation.

Also, the shift action of G on $\mathbb{R}^G$ is universal for the Borel actions of G.

For $x \in 2^G$, let $[x]$ denote the equivalence class of x under the shift action of G on $X = 2^G$.

Let $F \subseteq X$ denote the free part of 2^G, that is.

$$x \in F \leftrightarrow \forall g \neq 1\ (g \cdot x \neq x).$$

F is a (dense) G_δ in X, so F is a Polish space in the subspace topology.

Free Orbit Closures

We say $x \in F$ has free orbit closure if $\overline{[x]} \subseteq F$.

The following definition was formulated independently by Glasner and Uspensky.

Definition

A 2-coloring on a countable group G is a $c \colon G \to \{0,1\}$ satisfying:

$$\forall s \in G \; \exists T \subseteq G^{<\omega} \; \forall g \in G \; \exists t \in T \; (c(gt) \neq c(gst)).$$

Theorem

The following are equivalent.

- $x \in 2^G$ *has free orbit closure.*
- x *is a 2-coloring of* G.
- *If* $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ *are relatively closed subsets of* F *with* $M_i \cap [x] \neq \emptyset$ *for all* i, *then* $\bigcap_i M_i \neq \emptyset$.

Corollary

If G has a 2-coloring, then every decreasing sequence $\{M_i\}$ of relatively closed, complete sections of F has a non-empty intersection.

sketch of proof

Suppose $x \in 2^G$ is a 2-coloring. Suppose $y = \lim_n g_n^{-1} \cdot x \in \overline{[x]}$.

Let $1 \neq s \in G$, we show $s^{-1} \cdot y \neq y$.

Let $T \in G^{<\omega}$ be as in the definition of 2-coloring. Let n be large enough so that y and $g_n^{-1} \cdot x$ agree on $T \cup sT$. Let $t \in T$ be such that $x(g_n t) \neq x(g_n st)$. Then:

$$s^{-1} \cdot y(t) = y(st) = g_n^{-1} \cdot x(st) = x(g_n st) \neq x(g_n t) = g_n^{-1} \cdot x(t) = y(t)$$

So, $\overline{[x]} \subseteq F$.

Suppose $\overline{[x]} \subseteq F$. Let $s \in G$. Let T_n be the first n group elements.

If T_n doesn't work for s, let $g_n \in G$ be a counterexample. So, $x(g_n t) = x(g_n st)$ for all $t \in T_n$.

Let y be a limit of $\{g_n^{-1} \cdot x\}$, (w.l.o.g.) $y = \lim_n g_n^{-1} \cdot x \in F$.

But, for any $t \in G$ we have (for large enough n, $t \in T_n$):

$$s^{-1} \cdot y(t) = y(st) = g_n^{-1} \cdot x(st) = x(g_n st) = x(g_n t) = g_n^{-1} \cdot x(t) = y(t).$$

This contradicts $y \in F$.

Theorem

Every countable group G has the 2-coloring property.

The proof uses a construction of certain "marker set" for an arbitrary group G that we will also use in our result here.

Marker Sets and Regions

If (X, E) is a Borel equivalence relation, a set of marker regions for E is a finite subequivalence relation $R \subseteq E$.

A marker set for E is a Borel set $M \subseteq X$ such that $M \cap [x] \neq \emptyset$ for all $x \in X$.

We say M is a marker set for the marker regions R if $|M \cap [x]_R| = 1$ for all $x \in X$.

For G a countable group, a set of marker regions is a finite subequivalence relation R of G.

A marker set is a non-empty $M \subseteq G$.

We say R is a tiling of G if there is a finite $F \subseteq G$ such that each R class is of the form xF.

Note: Marker regions and sets for E give marker regions and sets for G (for free action).

Definition

A sequence (Δ_n, F_n) of tilings of G is coherent if each class δF_n of R_n is contained in a class $\delta' F_{n+1}$ of R_{n+1}. We say the sequence is cofinal if every finite subset of G is contained in some R_n class. We say it is centered if $1 \in \Delta_n$, $1 \in F_n$ for all n.

Definition

G is a c.c.c. group if it has a coherent, cofinal, centered tiling.

Remark

There is no loss of generality in assuming the tilings are centered. In this case $F_n \subseteq F_{n+1}$.

Question

Which groups are c.c.c. groups?

Theorem

Every nilpotent and every solvable group with finitely generated quotients is c.c.c. Every free product of non-trivial groups is c.c.c.

Is every solvable group c.c.c.? Is there a non-c.c.c. group?

Minimal flows

Definition

$x \in 2^G$ is minimal if whenever $y \in \overline{[x]}$ then $\overline{[y]} = \overline{[x]}$.

We give a combinatorial reformulation of minimality.

Lemma

x is minimal iff for all $A \in G^{<\omega}$ there is a $T \in G^{<\omega}$ such that:

$$\forall g \in G \; \exists t \in T \; \forall a \in A \; (x(a) = x(gta)).$$

For G a countable group, a straightforward computation gives:

Fact

The set of $x \in 2^G$ which are 2-colorings of G is $\mathbf{\Pi}^0_3$.

Fact

The set of $x \in 2^G$ which are minimal is $\mathbf{\Pi}^0_3$.

So, the set of minimal 2-colorings is also $\mathbf{\Pi}^0_3$.

Question

When are these sets $\mathbf{\Pi}^0_3$-complete?

It is fairly easy to see that the set of 2-colorings is $\boldsymbol{\Sigma}^0_2$-hard, and the set of minimal colorings is $\boldsymbol{\Pi}^0_3$-complete.

Remark

The set of 2-colorings is meager, in fact, the set of colorings that "block" a particular $s \neq 1$ is meager. It is also dense, so it cannot be $\boldsymbol{\Pi}^0_2$.

Remark

The $\boldsymbol{\Pi}^0_3$-completeness of the minimal colorings follows from our general marker region construction by a similar but easier argument to the main result.

Main Result

Definition

G is a flecc group if there is a finite $A \subseteq G$ such that for all $1 \neq g \in G$, there is an $h \in G$ and $i \in \omega$ such that $hg^i h^{-1} \in A - \{1\}$.

The following is our main result.

Theorem

The set of 2-colorings is $\mathbf{\Pi}^0_3$-complete iff G is not a flecc group. If G is a flecc group, the the set of 2-colorings is $\mathbf{\Sigma}^0_2$.

flecc groups

Definition

The limit extended conjugacy class of $g \in G$ is

$$\text{lecc}(g) = \bigcap_n \{hg^{ni}h^{-1} \colon h \in G, i \geq 1\}$$

for g of infinite order, and for g of finite order, every conjugacy class of any g^k where g^k has prime power order is a lecc of g.

Fact

If $lecc(g_1) \cap lecc(g_2) \neq \emptyset$, then $lecc(g_1) = lecc(g_2)$.

Theorem

G is a flecc group iff G has only finitely many lecc classes.

Example

Every quasi-cyclic group $\mathbb{Z}_{p^\infty}$ is a flecc group.

Fact

Every abelian flecc group is a torsion group.

Fact

Any infinite sum of non-trivial groups is not a flecc group.

Theorem

An abelian group is a flecc group iff it is a finite sum of quasicyclic groups and reduced p-groups each with finitely many elements of order p.

Corollary

The collection of abelian flecc groups is closed under products.

Question

Are the flecc groups closed under products?

Fact

The collection of flecc groups is not closed under quotients or subgroups.

For the first, take a (Zippin) group $G = \langle x, y_1, y_2, \dots \rangle$ where $y_i^p = x$, $x^p = 1$.

For the second, use the fact that every torsion-free group is a subgroup of a group G with 2 conjugacy classes (and so only finitely many lecc classes). So, every torsion-free group is a subgroup of a lecc group.

Proof that G is flecc implies the set of 2-colorings is $\mathbf{\Sigma}^0_2$.

Fact

- *If T blocks the element s^i, then s is blocked by $T \cup sT \cup \cdots \cup s^{i-1}T$.*
- *If T blocks s, then hsh^{-1} is blocked by hT.*

Let G be flecc, and A as in the definition of flecc. For each $s \in A$, let T_s block s. By the above facts, there is a finite T which blocks all $s \in G$.

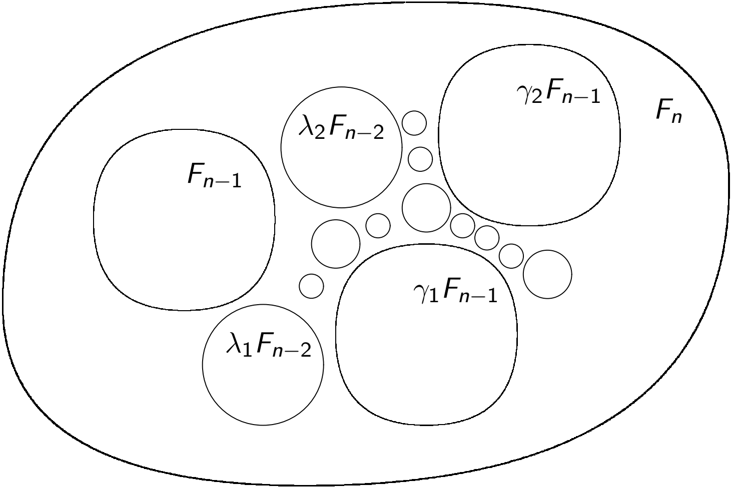
So, x is a 2-coloring of G iff

$$\exists T\ \forall s \neq 1 \ \in G^{<\omega}\ \forall g \in G\ (\exists t \in T\ x(gt) \neq x(gst)).$$

We construct marker regions (partial tilings) (Δ_n, F_n) for a general countable group G.

We have the following properties:

- $1 \in F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$.
- $\Delta_0 \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots$, $1 \in \Delta_n$.
- The Δ_n translates of F_n are maximally disjoint in G.
- F_n is a disjoint union of copies of $F_{n-1}, \dots, F_0$.
- The copies $\gamma\Delta_k$, $k < n$ which intersect a copy $\delta\Delta_n$ of F_n are exactly those of the form $\delta\eta F_k$ where $\eta \in F_n \cap \Delta_k$.


$\gamma_2 F_{n-1}$
F_n
$\lambda_2 F_{n-2}$
F_{n-1}
$\gamma_1 F_{n-1}$
$\lambda_1 F_{n-2}$

Each F_n will have two distinguished points, a_n and b_n.

The a_n will be used to establish a marker identification property.

The b_n will be free points used to make the coloring have desired properties (e.g., be a 2-coloring).

Marker Identification Property: There is a finite $A_n \subseteq G$ such that $\forall a \in A(c(ga) = c(a))$ iff $g \in \Delta_n$.

Fact

For any $A \in G^{<\omega}$ and coloring c_A of A with $c_A(1) = 1$, there is an $F_0 \supseteq A$ and $a_0, b_0 \in F_0$ and a coloring c_{F_0} of $F_0 - \{a_0, b_0\}$ extending c_A with the following identification property:

For any $\Delta_0 \subseteq G$ such that $\{\delta_0 F_0 \colon \delta_0 \in \Delta_0\}$ is pairwise disjoint, and any coloring $c \in 2^G$ extending the union of the c_{F_0} copied to the translates $\delta_0 F_0$ (and $c = 0$ off this union), c has the m.i.p. (using F_0).

Construction of F_0

Given $A \in G^{<\omega}$ and $c_A \colon A \to \{0,1\}$.

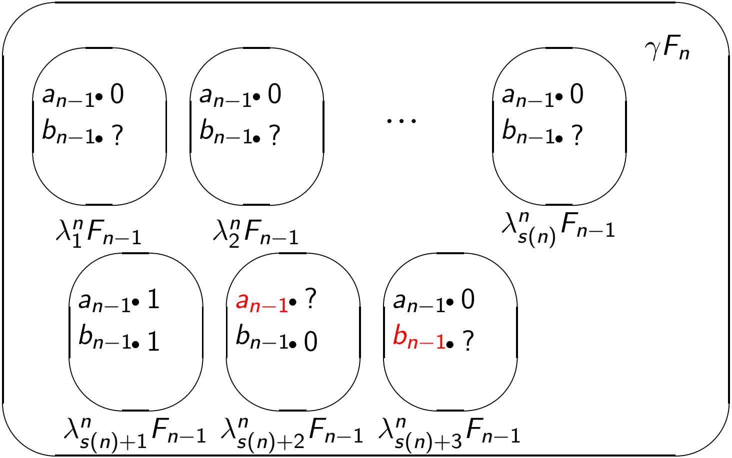
Pick any distinct $a_0, b_0, x, y \notin A$. Let $B_2 = A \cup \{a_0, b_0, x, y\}$

Let $z \notin (B_2 B_2^{-1} \cup B_2 B_2)$. Let $B_3 = B_2 \cup \{z\}$.

Let $F_0 = B_3 B_3$. Let $c_{F_0}(x) = c_{F_0}(y) = c_{F_0}(z) = 1 = c_A(1)$. Let $c_{F_0} = 0$ on $F_0 - B_3$.

This works.

Construction of F_n



To get a 2-coloring use the fact that there are exponentially many ($2^{\lambda_n} \sim 2^{|H_n|}$) choices for the color of the b_i in F_n but only polynomially many colors to avoid (about $|H_n|^5$).

Given g, gs with $s \in H_n$, by maximal disjointness of the H_n find $\gamma \in F_n F_n^{-1}$ such that $g\gamma \in \Delta_n$. Then

$$gs\gamma = g\gamma(\gamma^{-1}s\gamma)$$

where $\gamma^{-1}s\gamma \in H_n^5$.

The $\mathbf{\Pi}^0_3$ Construction

Suppose G is not a flecc group. Let $z \in 2^G$ be a 2-coloring as constructed.

For each k, let s_k be such that $hs_k^\ell h^{-1} \notin H_k$ for all $h \in G$ and all ℓ such that $s_k^\ell \neq 1$. Also, $s_k \notin F_k F_k^{-1}$.

We construct for each k another auxiliary $x_k \in 2^G$:

- x_k will have period s_k^{-1} (i.e., $x_k(g) = x_k(s_k g)$).
- x_k will have the 2-coloring property for first k-many shifts.
- x_k will be 0 off of a maximal disjoint collection of copies of F_k. It will follow the basic construction and so will the marker identification property for these copies.

The Reduction

Let $P \subseteq 2^{\omega\times\omega}$ be defined by:

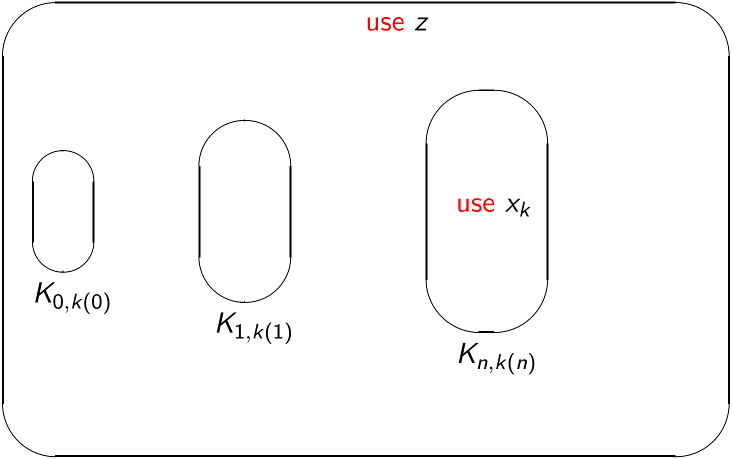
$$\alpha \in P \leftrightarrow \forall k\ \exists m\ \forall m > n\ \alpha(k,n) = 0.$$

For $u \in 2^{n\times n}$ we define a partial coloring c_u of G, and then take

$$f(\alpha) = \bigcup_n c_{\alpha\restriction n}$$

for our reduction function.

For each n, let $k = k(n)$ be least such that $\alpha(k,n) = 1$ (take n if no such k).


use z
use x_k
$K_{0,k(0)}$
$K_{1,k(1)}$
$K_{n,k(n)}$