# Shift Equivalence Relations and Markers

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Throughout $G$ will denote a countable group, $X$ a standard Borel space. $2^G$ is a compact Polish space.

We consider the *shift action* of $G$ on $2^G$:

$$g \cdot x(h) = x(g^{-1}h).$$

This is a continuous action of $G$.

Let $F(2^G)$ denote the free part of $2^G$ under this action:

$$F(2^G) = \{x \in 2^G : \forall g \neq 1 \; g \cdot x \neq x\}.$$

$F(2^G)$ is a $G_\delta$ in $2^G$, so is a Polish space in the subspace topology.

Recall the Feldman-Moore Theorem:

### Theorem

*Let $E$ be a countable Borel equivalence relation on a standard Borel space $X$. Then $E$ is generated by the Borel action of a countable group $G$.*

Thus, we can study countable equivalence relation "group by group."
Recall the simplest non-trivial equivalence relation are the hyperfinite ones:

### Definition

$E$ is hyperfinite if $E = \bigcup_n E_n$ where each $E_n$ is finite.

Say a group is hyperfinite if all its Borel actions are hyperfinite.

**Theorem (Slaman-Steel)**

*$E$ is hyperfinite iff $E$ is induced by a Borel action of $\mathbb{Z}$.*

Can study equivalence relations from several points of view:

- Borel bi-reducibility
- Orbit equivalence
- Isomorphism of group actions

Basic technique in study of countable group actions is the notion of Marker sets and Marker regions.

We consider these notions both for equivalence relations and for groups.

# Markers For Equivalence Relations

Formally, a marker set is a Borel set $S \subseteq X$. By "marker regions" be mean formally a Borel finite subequivalence relation of $E$.

Usually we construct a sequence $S_n$ of marker sets, and $S_n$ is associated to a marker region decomposition (note: marker regions easily give marker sets).

We try to get marker regions with some kind of geometric or combinatorial properties.

# Markers For Groups

A marker set is a set $S \subseteq G$.

Marker regions means the classes of a finite equivalence relation on $G$.

note: Marker sets (regions) for $E(2^G)$ give Marker sets (regions) for $G$

If we can get marker regions that totally cohere, then this shows $E$ is hyperfinite.

This was the original proof of the hyperfiniteness of $\mathbb{Z}^n$ actions. More precisely, this produced Borel marker regions $R_n$ which cohered and such that $\bigcup_n R_n$ has finite index in $E$.

This suffices by a result of (J-Kechris-Louveau).

In J-Kechris-Louveau this was extended to show all finitely generated groups of polynomial growth have hyperfinite actions. These coincide (Gromov) with the almost nilpotent groups.

## Conjecture (Kechris)

The actions of an amenable group are hyperfinite.

In [Boykin-J] we use “regular” (i.e., rectangular) marker regions in a different way to give another proof of the hyperfiniteness of $\mathbb{Z}^n$ actions. We in fact showed:

## Theorem (Boykin-J)

*There is a continuous embedding from* $2^{\mathbb{Z}^n}$ *into* $E_0$*.*

The new idea was to use rectangular marker regions, and make them “anti-cohere.”

With Gao we extended this up actions of abelian groups, but we lose continuity:

## Theorem (Gao-J)

*Every action of an abelian group is hyperfinite.*

To illustrate the ideas, we sketch the proof in the simplest setting: show there is a continuous embedding from $F(2^{\mathbb{Z}})$ into $E_0$.

First we get (relatively) clopen marker sets (we do this step for $\mathbb{Z}^n$):

- $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$, each $S_i$ relatively clopen in $F(2^{\mathbb{Z}})$.
- There are distances $d_0 \gg d_1 \gg d_2 \gg \cdots$ such that:
    1. $\forall x, y \in S_n \; \rho(x, y) > d_n$.
    2. $\forall x \in X \; \exists y \in S_n \; \rho(x, y) \leq d_n$.

The definition of $S_n$ is an $\omega$-length construction, constructing a maximal set $S_n = \bigcup_i S_n^i$ satisfying (1).

Sets are $S_n^i$ relatively open, so also is $S_n$. Maximality gives (2) which also shows $S_n$ is relatively closed.

From these clopen marker sets, one next constructs clopen marker regions which are rectangular. In fact, they can made almost the same size (side lengths of either $d_n$ or $d_n + 1$).

## Question

Can you get Borel marker regions for $F(2^{\mathbb{Z}^n})$ which are almost the same size and almost lined-up?

Construction of the marker regions from the marker sets uses the "big marker-little marker" method, and a finite sequence of successive adjustments.

In case of $\mathbb{Z}$, this step is rather trivial.

Next we modify the marker regions to anti-cohere.

At each step when we produce marker regions $R^n$, we also produce an "orthogonal" set of marker regions $\tilde{R}^n$: no face of an $\tilde{R}^n$ rectangle is close to a parallel face of an $R^n$ rectangle.

For $\mathbb{Z}$ this just says the endpoints of each $\tilde{R}^n$ interval are not close to those of an $R^n$ interval

Close here means some fixed fraction of $d_n$ (a geometrical constant depending only on $n$).

The $\tilde{R}^n$ are produced by the same adjustment process as the $R^n$.

We now use the $R^n$ and $\tilde{R}^n$ to produce the final clopen marker regions $Q^n$.

We start with $R^n_n = R^n$, and we define the marker regions $R^n_{n-1}, \dots, R^n_0$, and we will set $Q^n = R^n_0$.

### Remark

In the $\mathbb{Z}^n$ case the $R^n_n, \dots, R^n_1$ become increasingly "fractal."

In going from $R^n_{i+1}$ to $R^n_i$ we add or subtract an interval of $\tilde{R}^i$ from the ends of each interval in $R^n_{i+1}$. This ensures that the new endpoints of each $R^n_i$ interval are a fraction of $d_i$ away those of each $R^i$ interval.

We assume w.l.o.g. that $d_i \gg \sum_{j<i} d_j$.

- Each $Q^n$ interval is $\sum_{j<i} d_j \ll d_n$ close to an $R^n$ interval.
- For $n > m$, the endpoints of each $Q^n$ interval are $d_m$ far from the endpoints of each $R^m$, and hence each $Q^m$ interval.

Then for any $x \sim y$, there are only finitely many $n$ such that an endpoint of a $Q^n$ marker region separates $x$ from $y$ (this follows from (2) above).

Thus, $x \sim y$ iff for all large enough $n$ we have $x \sim_{Q^n} y$. This gives a continuous embedding into $E_0$.

Proof can be extended to handle non-free part of $2^{\mathbb{Z}}$ as well (and likewise for $2^{\mathbb{Z}^n}$.

### Question

Does there exists a continuous embedding from $2^{\mathbb{Z}^{<\omega}}$ into $E_0$? Yes for free part.

### Question

How far can these regular marker arguments be extended?

### Question

Are there more algebraic, less geometrical, versions of these arguments?

This may be important for extending these arguments further.

We now turn to markers and marker regions for groups.

## Question

What kind of marker regions can we get for general groups?

Note that if $F(2^G)$ admits a certain marker structure, then this copies over to $G$, but not (obviously) conversely.

Say $G$ admits *rectangular marker regions* if $G = \bigcup_n S_n$ such that for each $n$, $G$ is a disjoint union of translates of $S_n$.

## Question

Which groups admit rectangular marker structures?

**Theorem**

*Every abelian group admits rectangular markers.*

**Lemma**

*If $K \trianglelefteq G$, $K \leq Z(G)$, and $K$, $G/K$ admit rectangular markers, then so does $G$.*

**Corollary**

Every nilpotent group admits rectangular markers.

We now turn to the topological properties of orbits by the shift action of $G$ on $2^G$ for countable $G$.

### Definition

$c \colon G \to \{0,1\}$ is a 2-*coloring* if

$$\forall s \in G \ \exists T \in G^{<\omega} \ \forall g \in G \ \exists t \in T \ (c(gt) \neq c(gst)).$$

This definition was formulated independently by Pestov (c.f. paper of Glasner and Uspenski).

## Significance of the definition.

### Theorem

*$x \in 2^G$ is a 2-coloring iff $\overline{[x]} \subseteq F(2^G)$.*

Also, the 2-coloring property for $G$ gives a *marker compactness property* for $F(2^G)$:

### Theorem (MCP)

*Let $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$ be relatively closed complete sections of $F$. Then $\bigcap_n S_n \neq \emptyset$.*

### Remark

The MCP for $\mathbb{Z}$ shows we cannot get a continuous embedding from even $F(2^Z)$ into $E_0$ by constructing (as in Slaman-Steel for Borel) a decreasing set relatively clopen marker sets $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$.

## Theorem

*Every countable group $G$ has a 2-coloring.*

The proof uses two ideas:

- Construct suitable marker regions for the group $G$.
- Exploit polynomial vs. exponential growth.

We first describe the construction of the marker regions. Recall $G$ is a countable infinite group.

We inductively define marker sets $\Delta_n \subseteq G$ and finite sets $F_n \subseteq G$ (with $1 \in F_n$).

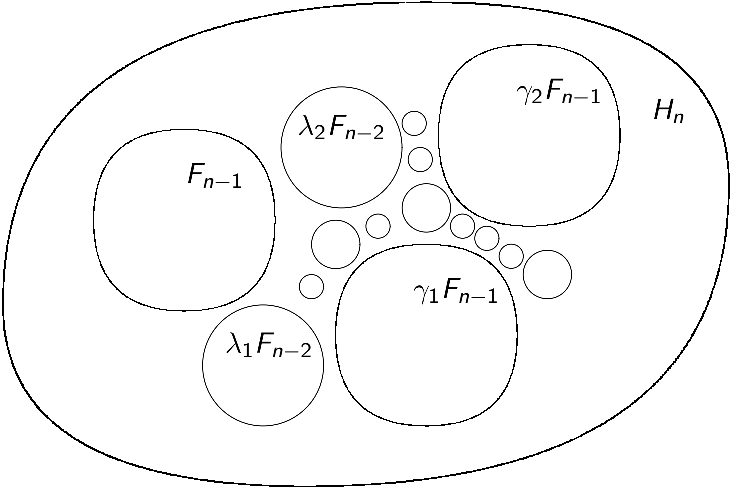
The $n$th level marker regions will be the translates $gF_n$ for $g \in \Delta_n$.

Will have:

- $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$
- $\boldsymbol{\Delta}_0 \supseteq \boldsymbol{\Delta}_1 \supseteq \boldsymbol{\Delta}_2 \supseteq \cdots$

Each $F_n$ region will be a union of copies of $F_i$ for $i < n$.

$F_n$ will be constructed inside a region $H_n$.


$\gamma_2 F_{n-1}$
$H_n$
$\lambda_2 F_{n-2}$
$F_{n-1}$
$\gamma_1 F_{n-1}$
$\lambda_1 F_{n-2}$

Will maintain two properties:

- (homogeneity) Within any copy $\gamma F_n$ of $F_n$, the points in $\mathbf{\Delta}_k$ ($k \leq n$) are precisely the translates $\gamma(\mathbf{\Delta}_k \cap F_n)$ of the points in $F_n$.
- (fullness) If a copy $\delta F_k$ intersects $\gamma F_n$ ($k \leq n$) then $\delta F_k \subseteq \gamma F_n$.

# Construction of the $F_n$

Start with $1 \in H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ sufficiently fact growing.

Say,

$$H_{n-1}(H_0^{-1}H_0)(H_1^{-1}H_1)\cdots(H_{n-1}^{-1}H_{n-1}) \subseteq H_n.$$

Assume $F_{n-1} \subseteq H_{n-1}$ is defined. We define $F_n \subseteq H_n$.

We interpolate between $H_{n-1}$ and $H_n$ a sequence of sets

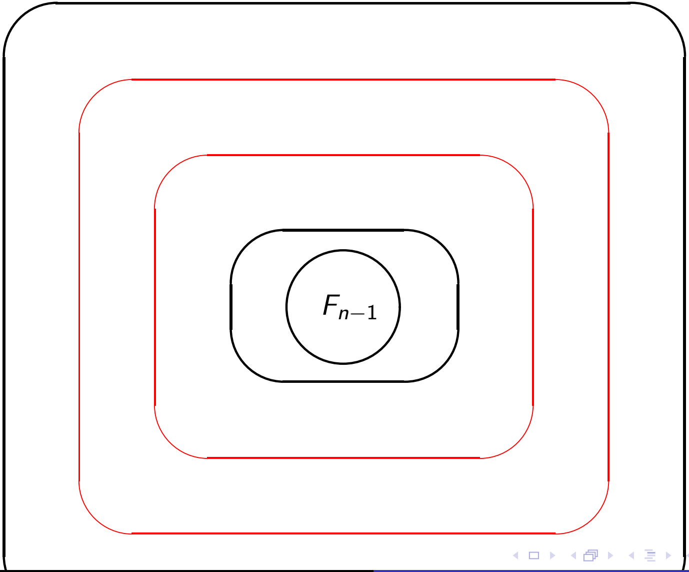
$$H_{n-1} \subseteq \beta(n,0) \subseteq \beta(n,1) \subseteq \cdots \subseteq \beta(n,n-1) = H_n$$

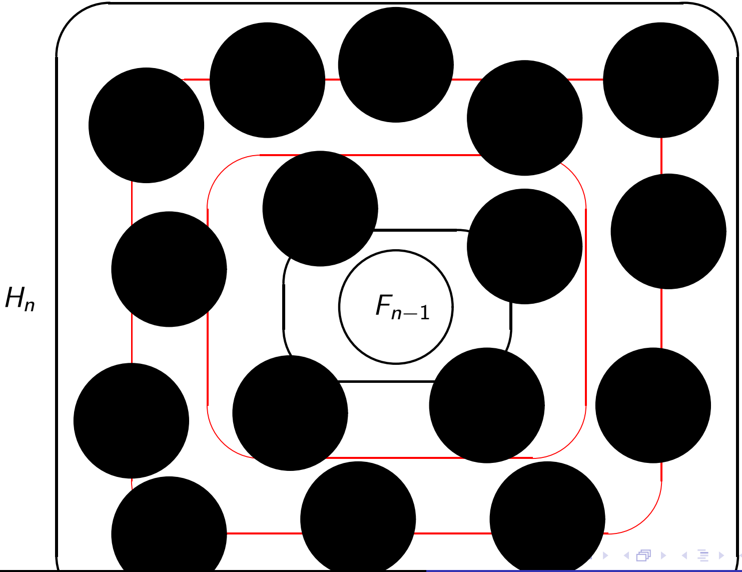
where $\beta(n,r) = \{g \colon g(H_{r+1}^{-1}H_{r+1})\cdots(H_{n-1}^{-1}H_{n-1}) \subseteq H_n\}$.

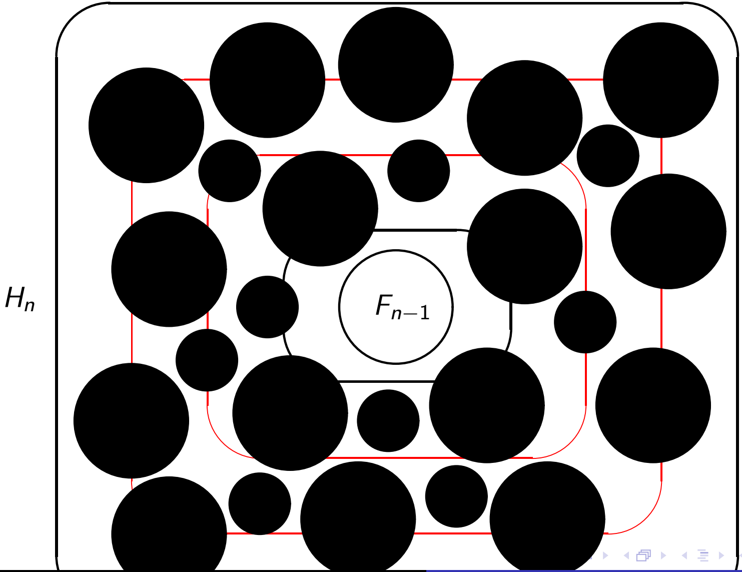
Going from $F_{n-1}$ to $F_n$:

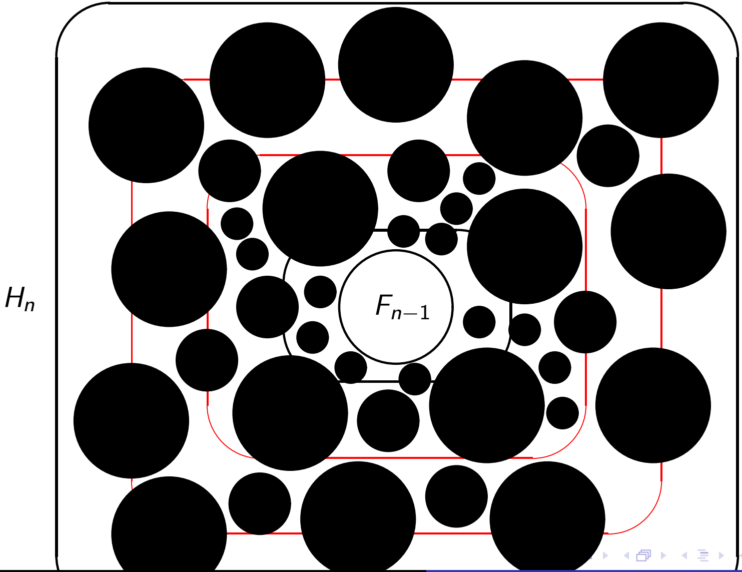
1. First put maximal disjoint collection of copies of $F_{n-1}$ inside $H_n = \beta(n, n-1)$.
2. Next put maximal disjoint collection of copies of $F_{n-2}$ inside of $\beta(n, n-2) - \bigcup F_{n-2}$ copies.
3. At step $k$, put maximal disjoint number of copies of $F_k$ inside of $\beta(n, k) - \bigcup$ previous $F_{k+1}, \ldots, F_{n-1}$ copies.

$H_n$




$H_n$
$F_{n-1}$


$H_n$
$F_{n-1}$


$H_n$
$F_{n-1}$

Main Point: If a copy of $F_k$ meets a copy of $F_l$, $l > k$, then it must meet a copy of $F_k$ inside $F_l$.

This guarantees the copies of $F_k$ are maximally disjoint in $G$.

We label the copies of $F_{n-1}$ inside of $F_n$ by

$$\lambda_1^n F_{n-1}, \ldots, \lambda_{s(n)}^n F_{n-1},$$

$$\lambda_{s(n)+1}^n F_{n-1}, \lambda_{s(n)+2}^n F_{n-1}, \lambda_{s(n)+3}^n F_{n-1}.$$

Each copy of an $F_n$ will have two distinguished points, $a_k$ and $b_k$.

Will have Marker Identification Property:
(MIP) There is a $A_n \subseteq F_{n-1}$ such that if $c(ga) = c(a)$ for all $a \in A_n$, then $g \in \Delta_n$.

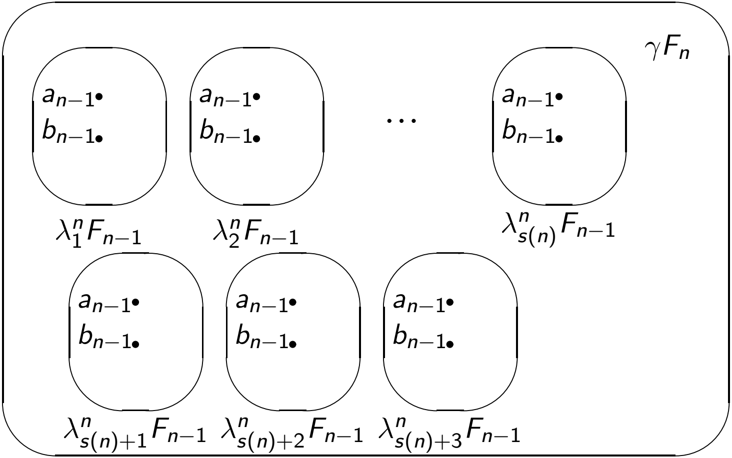



Figure: The labeling of the $F_{n-1}$ copies inside an $F_n$ copy

We define a coloring $c = \bigcup c_n$, which will then be extended to the 2-coloring $c'$.

$c$ will color all points except those in

$$D = \bigcup_n \boldsymbol{\Delta}_n \{\lambda_1^n, \dots, \lambda_{s(n)}^n\} b_{n-1}.$$

In extending $c_{n-1}$ to $c_n$ we color the above points except for those in $\boldsymbol{\Delta}_n \lambda_1^n, \dots, \boldsymbol{\Delta}_b \lambda_{s(n)}^n$, and $\boldsymbol{\Delta}_n \{a_n, b_n\}$ where:

$$a_n \doteq \lambda_{s(n)+2}^n \, a_{n-1}$$
$$b_n \doteq \lambda_{s(n)+3}^n \, b_{n-1}.$$

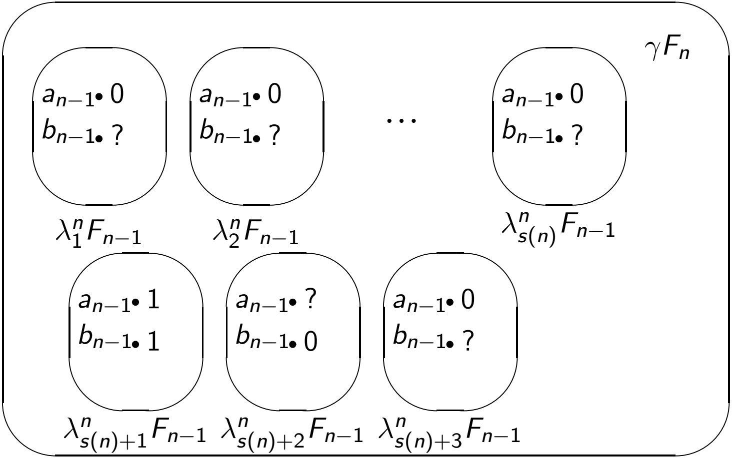



Figure: Extending $c_{n-1}$ to $c_n$.

We extend $c$ to $c'$ by coloring the points of $D$ so as to get a 2-coloring. Exploit polynomial versus exponential growth.

At stage $n$ we extend $c$ to points of $\mathbf{\Delta}_n\{\lambda_1^n, \dots, \lambda_{s(n)}^n\}b_{n-1}$ to take care of coloring property for $s = g_n \in H_n$.

Let $g \in G$ and consider the pair $g, gs$. By maximal disjointness of $F_n$ copies, $gf \in \mathbf{\Delta}_n$ for some $f \in F_n F_n^{-1}$. Done unless $gsf \in \mathbf{\Delta}_n$. In this case

$$gsf = gf(f^{-1}sf) \in (gf)F_nF_n^{-1}H_nF_nF_n^{-1}.$$

So there are about $|H_n|^5$ many points to consider, and there $2^{s(n)}$ many "colors" available, where $s(n)$ is linear in $|H_n|$.

# Some Tentative Results

**Theorem**

*The set of $x \in 2^G$ with $\overline{[x]} \subseteq F(2^G)$ is $\mathbf{\Pi}^0_3$-complete.*

**Theorem**

*The set of $x \in 2^G$ with $\overline{[x]} \subseteq F(2^G)$ and minimal is $\mathbf{\Pi}^0_3$-complete.*