Reflections on Dan's Work

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The following question was asked by Steinhaus in the 50's.

Question

Does there exist a set $S \subseteq \mathbb{R}^2$ meeting every isometric copy L of the integer lattice $\mathbb{Z}^2$ in exactly one point?

Note that the one-dimensional version of the problem is trivial.

Peter Komjàt showed that the corresponding question for copies of $\mathbb{Z}$ is true.

Variations

The problem has many variations.

- We can ask it for any lattice L in $\mathbb{R}^n$ (L doesn't have to n-dimensional).
- More generally, L could be any set, even a finite set.
- We can restrict the class of isometries to particular subgroups.
- We can restrict the properties of the Steinhaus set S, e.g., we can ask for it to be bounded, measurable, etc.
- We can vary the requirements on the intersection of S and $\pi(L)$.

The problem has connections to number theory, geometry, set theory.

For $n \geq 4$ the problem in $\mathbb{R}^n$ for the standard lattice $\mathbb{Z}^n$ has a negative solution (D. Goldstein): consider the copy L of $\mathbb{Z}^n$ centered at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, , 0, \dots, 0)$.

Theorem (J., Mauldin)

(ZFC) There is a Steinhaus set $S \subseteq \mathbb{R}^2$ for the standard lattice $\mathbb{Z}^2$. Any Steinhaus set cannot have the Baire property.

Consider the version of the problem ($n = 2$ or 3) where we restrict the group of isometries to translations, and we require that no two points of S are at a lattice distance apart.

We call such a set a partial Steinhaus set.

We can specialize further to just the rational translates of $\mathbb{Z}^2$. In fact, we can consider translates by $\left(\frac{\mathbb{Z}}{p}\right)^n$. This gives a finitary version of the problem.

For $n = 2$ this holds for all primes p. Only $p \equiv 1 \pmod 4$ are non-trivial.

For $n \geq 4$, this fails for $p = 2$ by previous argument.

We have the following number-theoretic problem:

Question

For which n and p does there exist a partial Steinhaus set?

For $n = 3$, there is one for $p = 2, 3$. For $p = 5$ the answer is not clear.

For $n = 2, 3$, a root is a tuple $(\lambda_1, \dots, \lambda_n)$ with $\lambda_1^2 + \cdots + \lambda_n^2 = 0$. We usually consider the roots up to scaling.

For $n = 2$ there are (up to scaling) two roots if $p \equiv 1 \pmod 4$, otherwise no roots. For $n = 3$ there are (up to scaling) $p + 1$ roots.

Consider the p^n points in $\left(\frac{\mathbb{Z}}{p}\right)^n \cap [0,1)^n$.

Having a partial Steinhaus set is equivalent to having functions $\ell_1, \dots, \ell_n \colon \{0, \dots, p-1\}^n \to \mathbb{Z}$ such that for all $x_1 = \frac{\vec{i}_1}{p}$, $x_2 = \frac{\vec{i}_2}{p}$, if

$$z_1 = \frac{\vec{i}_1}{p} + \vec{\ell}(\vec{i}_1)$$
$$z_2 = \frac{\vec{i}_2}{p} + \vec{\ell}(\vec{i}_2)$$

then $\rho(z_1, z_2)$ is not a lattice distance.

Saying here that $\rho(z_1, z_2)$ is not a lattice distance is equivalent to saying that $\rho^2(z_1, z_2) \notin \mathbb{Z}$ by the following:

Fact

If an integer is the sum of two (or three) squares of rationals, then it is the sum of two (or three) squares of integers.

There is a cute proof of this fact due to Aubrey which works for both $n = 2, 3$.

The condition $\rho^2(z_1, z_2) \notin \mathbb{Z}$ holds automatically unless $z_2 - z_1$ is a root.

Given a root $\vec{\lambda}$, we say a thread is the equivalence class of points in I_p which differ by a multiple of $\vec{\lambda}$ (this only depends on $\vec{\lambda}$ up to scaling).

Each equivalence class has size p.

Fix a root $\vec{\lambda}$ and a starting point $\frac{\bar{i}}{p}$. This defines the thread $\frac{1}{p}(\bar{i} + e\vec{\lambda} \pmod{p})$, where $0 \leq e < p$.

Let $\vec{\eta}(e) = \frac{\bar{i}+e\vec{\lambda}-(\bar{i}+e\vec{\lambda} \pmod{p})}{p}$.

The condition becomes the following.

Define

$$\pi_{\vec{i}}^{\vec{\lambda}}(e) = \vec{\lambda} \cdot \vec{\ell}(e) - \lambda \cdot \vec{\eta}(e) + e\frac{\delta}{2} \pmod{p}$$

where $\vec{\ell}(e) = \vec{\ell}(\frac{\vec{i}+e\vec{\lambda} \pmod{p}}{p})$, and $\delta = \frac{||\lambda||}{p}$.

Then all of the $\pi_{\vec{i}}^{\vec{\lambda}}$ are permutations.

Note: If we take λ to be a root $\pmod{p^2}$ and use the "unwrapped points" to take the ℓ values, then we have $\pi_{\vec{i}}^{\vec{\lambda}}(e) = \vec{\lambda} \cdot \vec{\ell}(e)$.

Comments on 3D Problem

- Mauldin with Goldstein and McLinden gave the heuristic estimate for the expected number of 3D partial Steinhaus sets for a given p:

$$\frac{p!^{p^3+p^2}}{p^{p^4-2p^3}}$$

Prime	Expected Number
3	$1.35 \cdot 10^{15}$
5	$5.80 \cdot 10^{49}$
7	102.24
11	$1.08 \cdot 10^{-1438}$

- With Henkis and Lobe we observed that a certain overdetermined subset of the 3D system appears to be consistent.
- Every "pseudo" partial Steinhaus set is actually a partial Steinhaus set.

The 2D proof also uses a geometric lemma: the "three circles theorem" which is a result about 4-bar mechanical linkages.

The proof uses an induction over the size of the continuum, using the "hull" method. This method naturally suggests the above geometric lemma.

It is still open whether there can be a measurable Steinhaus set in $\mathbb{R}^2$, or whether there can be a bounded one. However (Croft), there cannot be a bounded measurable one.

Kolountzakis and Wolff showed that there is no measurable Steinhaus set for the standard lattice $\mathbb{Z}^3$ (or " almost Steinhaus set for $\mathbb{Z}^d$, $d \geq 3$). Kolountzakis and Papadimitrakis gave another, simpler, proof of this.

Mauldin and Yingst extended this to other lattices in $\mathbb{R}^d$, $d \geq 3$.

Let $A \in \mathrm{GL}(d, \mathbb{R})$, and let $L_A = A \cdot \mathbb{Z}^d$ be the corresponding lattice. Let $\hat{L}_A = L_{A^{-T}}$ be the dual lattice.

For $M \in \mathrm{GL}(d,\mathbb{R})$, let $\mathcal{D}(M) = \{||Mx||^2 \colon x \in \mathbb{Z}^d\}$.

Definition

Let $A, B \in \mathrm{GL}(d,\mathbb{R})$. $A \succ B$ if $\mathcal{D}(A) \supseteq \mathcal{D}(B)$. $A \succ_w B$ if $A \succ B$ and $\frac{\det(B)}{\det(A)} \notin \mathbb{Z}$. $A \succ_s B$ if $A \succ B$ and $\frac{\det(B)}{\det(A)} \notin \mathbb{Q}$.

Theorem (Mauldin, Yingst)

Let $A \in GL(d,\mathbb{R})$ and suppose $A^{-T} \succ_w B$ for some $B \in GL(d,\mathbb{R})$. Then there is no measurable (almost) Steinhaus set for the lattice L_A.

Mauldin, Yingst show that these results apply to the tetrahedral lattice in $\mathbb{R}^3$.

They give an example of a matrix in $\mathrm{GL}(d, \mathbb{R})$ which does not weakly dominate any matrix.

Extending this work of Mauldin and Yingst, Chan and Mauldin showed the following.

Say a lattice L_A is integral if the dot product of any two vectors (equivalently, columns of A are integers.

Theorem (Chan, Mauldin)

Let L_A be an integral lattice in R^d, $d \geq 3$. Then there is a B such that $A \prec_s B$.

So, they get:

Theorem (Chan, Mauldin)

If L is equivalent to an integral lattice in $\mathbb{R}^d$, $d \geq 3$, then there is no measurable (almost) Steinhaus set for L.

Finite Version

We can consider the version of the Steinhaus problem where we use an arbitrary set $A \subseteq \mathbb{R}^d$ in place of a lattice. In particular, A can be a finite set.

Consider the finite version in $\mathbb{R}^2$.

Conjecture

For every finite $A \subseteq \mathbb{R}^2$ with $|A| > 1$, there does not exist a set $S \subseteq \mathbb{R}^2$ meeting every isometric copy of A in exactly one point.

For $|A| = 2, 3$ the result is easily proved.

Gao, Miller showed that certain families of four-point sets satisfy the conjecture.

Then Gao's student Mingzhi Xuan proved the following.

Theorem (Xuan)

Every four-point set in the plane satisfies the conjecture, In 1991 Mauldin and J proved the following.

Theorem (ZF)

Let X, Y be uncountable Polish spaces. There is a $\mathbf{\Pi}^1_1$ set $P \subseteq X \times Y$ with all sections P_x countable, such that P is not the union of countably many $\mathbf{\Sigma}^1_2$ (partial) graphs of functions. In fact same conclusion for $\mathcal{B}(\mathbf{\Sigma}^1_2)$.

Assuming $\mathbf{\Pi}^1_1$-determinacy the least pointclass $\mathbf{\Gamma}$ was identified such that every $\mathbf{\Pi}^1_1$ with countable sections can be written as a countable union of graphs in $\mathbf{\Gamma}$. Namely,

$$\mathbf{\Gamma} = \bigcup_n \Game \omega \cdot n - \mathbf{\Pi}^1_1.$$

Recently, the method of proof used for this non-uniformization result has been used to get a result concerning a question of Maharam about σ-finite measures.

Let μ, ν be Borel measures on uncountable Polish spaces X, Y, and let $\phi\colon X \to Y$ be Borel.

Definition

A disintegration of μ with respect to ν, ϕ is a family $\{\mu_y\}_{y\in Y}$ on Borel measures on X satisfying:

(1) $\forall B \in \mathcal{B}(X)\ (y \mapsto \mu_y(B))$ is Borel.

(2) $\forall y \in Y\ (\mu_y(X - \phi^{-1}(y)) = 0$.

(3) $\forall B \in \mathcal{B}(X)\ (\mu(X) = \int \mu_y(B)\, d\nu)$

We say the disintegration $\{\mu_y\}_{y\in Y}$ is σ-finite if for every $y \in Y$, μ_y is a σ-finite measure.

We say the disintegration is uniformly σ-finite if there are Borel sets B_n in X satisfying:
(1) $\forall n\, \forall y \in Y\ \mu_y(B_n) < \infty$.
(2) $\forall y \in Y\ (\mu_y(X - \bigcup_n B_n) = 0$.

Question (Maharam)

Is every σ finite disintegration uniformly σ-finite?

Theorem (Backs, J, Mauldin)

Assuming $V = L$, there is a σ-finite disintegration which is not uniformly σ-finite.

The proof uses the following improvement to the earlier non-uniformization result.

Theorem

Assume $V = L$. Then there is a $\mathbf{\Pi}^1_1$ set $G \subseteq \omega^\omega \times \omega^\omega$ satisfying

1. $\forall x \in X \ |G_x| \leq \omega$
2. *G is not the union of countably many $\mathbf{\Pi}^1_1$ (or $\mathbf{\Sigma}^1_2$) graphs.*
3. $\forall n \in \omega \ \forall B \in \mathcal{B}(Y) \ \{x \in X \colon |G_x| = n\} \in \mathcal{B}(X)$