Partition Triples Under AD

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We assume the *axiom of determinacy* AD throughout. We work in the theory $\text{ZF} + \text{AD} + \text{DC}$.

We prove a polarized partition result for certain triples of cardinals and give applications to forcing over models of AD.

Main result is a strengthening/generalization of a result proved by Kechris.
To state the main AD result, we first recall some partition notation.  

\((\kappa)^\lambda\) denotes the increasing functions from \(\lambda\) to \(\kappa\).

**Definition (Erdős-Rado)**

\(\kappa \rightarrow (\kappa)^\lambda\) means for every partition \(\mathcal{P} : (\kappa)^\lambda \rightarrow \{0, 1\}\) there is a homogeneous \(H \subseteq \kappa\) of size \(\kappa\).

\(H\) is homogeneous if \(\mathcal{P} \upharpoonright (H)^\lambda\) is constant.
Definition

κ has the weak partition property if \( \kappa \rightarrow (\kappa)^\lambda \) for all \( \lambda < \kappa \).

κ has the strong partition property if \( \kappa \rightarrow (\kappa)^\kappa \).

In ZFC no cardinal has an infinite exponent partition relation.
Let \( \vec{\kappa} = (\kappa_\alpha)_{\alpha < \rho} \) be a sequence of regular cardinals. Let \( \vec{\lambda} = (\lambda_\alpha)_{\alpha < \rho} \) be a sequence of ordinals, \( \lambda_\alpha \leq \kappa_\alpha \).

A block function from \( \vec{\lambda} \) to \( \vec{\kappa} \) is a function with domain \( \bigoplus_{\alpha < \rho} \lambda_\alpha \) which sends the copy of \( \lambda_\alpha \) into \( \kappa_\alpha \).

**Definition (Polarized Partition Property)**

\( \vec{\kappa} \rightarrow (\vec{\kappa})\vec{\lambda} \) if for every partition \( \mathcal{P} \) of the block functions from \( \vec{\lambda} \) to \( \vec{\kappa} \) there is an \( H = \bigoplus_{\alpha < \rho} H_\alpha \), where \( H_\alpha \subseteq \kappa_\alpha \), which is homogeneous for \( \mathcal{P} \).
If $T$ is a tree on a set $Z$, $[T] = \{ f \in Z^\omega : \forall n f \upharpoonright n \in T \}$ (set of infinite branches through $T$).

**Definition**

We say $A \subseteq \omega^\omega$ is $\kappa$-Suslin if there is a tree $T \subseteq \omega \times \kappa$ such that $A = p[T] = \{ x : \exists f \in \kappa^\omega (x, f) \in [T] \}$.

Let $S(\kappa)$ denote the collection of $\kappa$-Suslin sets.

**Definition**

$\kappa$ is a **Suslin cardinal** if $S(\kappa) - \bigcup_{\lambda < \kappa} S(\lambda) \neq \emptyset$. 
The Suslin cardinals are closely related to the projective ordinals.

In ZF:
- $\omega$ is the first Suslin cardinal and $S(\omega) = \Sigma^1_1$.
- $\omega_1$ is the next Suslin cardinal and $S(\omega_1) \supseteq \Sigma^1_2$.

Recall $\delta^1_n$ is the supremum of the lengths of the $\Delta^1_n$ prewellorderings of the reals.

With AD:
- $\delta^1_{2n+2} = (\delta^1_{2n+1})^+$
- $\delta^1_{2n+1} = \lambda^+_{2n+1}$, where cof($\lambda_{2n+1}$) = $\omega$
- $\delta^1_n$ is regular.
The first $\omega$ Suslin cardinals are:

$$\lambda_1, \delta_1^1, \lambda_3, \delta_3^1, \lambda_5, \delta_5^1, \ldots$$

where: $\lambda_1 = \omega$, $\lambda_3 = \aleph_\omega$, $\lambda_5 = \aleph_{\omega^\omega}$, $\ldots$.

Also: $S(\lambda_1) = \Sigma_1^1$, $S(\delta_1^1) = \Sigma_2^1$, $S(\lambda_3) = \Sigma_3^1$, $S(\delta_3^1) = \Sigma_4^1$, $\ldots$
Theorem (Steel-Woodin)

(AD) *The Suslin cardinals are closed below their supremum.*

(AD\(^+\)) *The Suslin cardinals are closed below \(\Theta\).*

We say a Suslin cardinal \(\kappa\) is a **limit Suslin** cardinal if \(\kappa\) is a limit of Suslin cardinals, otherwise \(\kappa\) is a **successor Suslin** cardinal.

In \(L(\mathbb{R})\) there is a largest Suslin cardinal \(\delta^2_1\) (Martin-Steel), and 
\(S(\delta^2_1) = \Sigma^2_1\) is closed under real quantification.

\(\delta^2_1\) is regular and a limit of Suslin cardinals, so \(\delta^2_1\) is an **inaccessible Suslin** cardinal.

In fact, the inaccessible Suslin cardinals are stationary in \(\delta^2_1\).
We now state our main partition result.

**Theorem (AD)**

*For all inaccessible Suslin cardinals $\kappa$ we have:*

$$
(\kappa, \kappa^+, \kappa^{++}) \rightarrow (\kappa, \kappa^+, \kappa^{++})^{(\kappa, \kappa, \kappa)}
$$

**Corollary**

$\kappa$, $\kappa^+$, $\kappa^{++}$ are all measurable.
We consider the possible cofinalities/measurability of $\aleph_1$, $\aleph_2$, $\aleph_3$.

- There are 3 possibilities of $\aleph_1$ ($\text{cof} = \omega$, regular, measurable).
- There are 4 possibilities for $\aleph_2$, and 5 possibilities for $\aleph_3$.
- 13 of the 60 total possibilities are “trivially inconsistent.” For example, $\aleph_1$ regular, $\text{cof}(\aleph_2) = \aleph_1$, and $\text{cof}(\aleph_3) = \aleph_2$.

**Theorem**

*Assuming suitable large cardinals, all of the remaining 47 patterns are consistent with ZF.*
We review a few more facts about pointclasses and Suslin cardinals.

- By a Wadge degree we mean the equivalence class of a pair $(A, A^c)$ if $A \not\equiv A^c$ (non-selfdual) or of $(A)$ if $A \equiv A^c$ (selfdual).
- The selfdual and non-selfdual degrees alternate. At limit ordinals of cofinality $\omega$ there is a selfdual degree, and for uncountable cofinality a non-selfdual degree.
- The separation property falls on exactly one of the sides of a non-selfdual pointclass.
By a *Levy class* we mean a non-selfdual pointclass closed under either $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$ (or both). Let $\Sigma^1_\alpha$, $\Pi^1_\alpha$ enumerate these classes.

**Fact (Kechris-Solovay-Steel)**

*If $\Delta$ is selfdual and closed under quantifiers, $\land, \lor$, then*

$$o(\Delta) = \delta(\Delta)$$

$$o(\Delta) = \sup\{|A|_w : A \in \Delta\}.$$  

$$\delta(\Delta) = \sup\{|<| : < \in \Delta\}.$$
Every Levy pointclass lies in a projective-like hierarchy.

Let $C = \{o(\Delta) : \exists^{\omega\omega} \Delta \subseteq \Delta, \land \Delta \subseteq \Delta\}$. 

Given a levy class $\Gamma$. let $\alpha$ be the large element of $C$ such that $\Delta_\alpha = \{A : o(A) < \alpha\} \subseteq \Gamma$.

$\Delta_\alpha$ is the base of a projective-like hierarchy containing $\Gamma$.

This hierarchy falls into one of four types.
We specialize to the case of Suslin cardinals. Let $\alpha$ be a limit Suslin cardinal. Let $\Delta = \bigcup_{\lambda < \kappa} S(\lambda)$. Then $\Delta$ is at the base of a projective-like hierarchy.

**Type I** \( \text{cof}(\alpha) = \omega \). Let $\Sigma^\alpha_0$ be the collection of countable unions of sets in $\Delta$. Then $\Sigma^\alpha_0$, $\Pi^\alpha_1$, etc. have the scale property. $\alpha^+$ is a Suslin cardinal, and a $\Pi^\alpha_1$ scale on a $\Pi^\alpha_1$-complete set has norms of length $\kappa^+$. We have $S(\kappa) = \Sigma^\alpha_1$ and $S(\kappa^+) = \Sigma^\alpha_2$.

**Type II or III** \( \text{cof}(\alpha) > \omega \), $\Gamma$ not closed under real quantifiers. $\Gamma$ is the Steel class of Wadge rank $\alpha$. So, $\Delta = \Gamma \cap \check{\Gamma}$, and $\Gamma$ is closed under $\land$, $\forall^R$. Then $S(\kappa) = \exists^R \Gamma$, and $\text{scale}(\Gamma)$, $\text{scale}(S(\kappa))$ hold.

**Type IV** In this case, the pointclasses $\Gamma$, $\check{\Gamma}$ of Wadge degree $\kappa$ are closed under real quantification. Let $\Gamma$ be such that $\check{\Gamma}$ has the separation property. Then $\text{scale}(\Gamma)$, and $S(\kappa) = \Gamma$. 

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Of particular importance for the proof are the Type I hierarchies.

Fix now an inaccessible Suslin $\kappa$, with corresponding pointclass $\Gamma$ (of Wadge rank $\kappa$).

$k$ is a limit Suslin cardinal. Let $C$ be the c.u.b. set of limit Suslin cardinals below $\kappa$.

We are interested in the points of $C$ of cofinality $\omega$ (Type I hierarchy).
We have the following picture.

\[ \Sigma^\alpha_0 = \bigcup_{\lambda < \alpha} S(\lambda) \]

\[ S(\alpha) = \Sigma^\alpha_1, \quad S(\alpha^+) = \Sigma^\alpha_2 \]

scale(\(\Sigma^\alpha_0\), ) scale(\(\Pi^\alpha_1\))
Plan of the proof

Let $\mu$ be the $\omega$-cofinal normal measure on $\kappa$. ($\kappa$ has the strong partition property by Kechris-Kleinberg-Moschovakis-Woodin).

Let $\mu_\alpha$ be the $\omega$-cofinal normal measure on $\alpha^+$. We have $j_{\mu_\alpha}(\alpha^+) = \alpha^{++}$.

1. We show $[\alpha \mapsto \alpha^+]_\mu = \kappa^+$.
2. We show $\delta \equiv [\alpha \mapsto \alpha^{++}]_\mu \leq \kappa^{++}$.
3. We show for all $\theta < \omega_1$ that $(\kappa, \kappa^+, \delta) \rightarrow (\kappa, \kappa^+, \kappa^{++})^\theta$.
   It follows that $\delta$ is regular, so $\delta = \kappa^{++}$.
4. We show $(\kappa, \kappa^+, \delta) \rightarrow (\kappa, \kappa^+, \kappa^{++})^\theta$ for all $\theta < \kappa$.
5. Finally, we show $(\kappa, \kappa^+, \delta) \rightarrow (\kappa, \kappa^+, \kappa^{++})^\kappa$. 

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We define two trees $T^+$ and $T^{++}$ on $\omega \times \kappa$ to analyze the ultrapowers $[\alpha \mapsto \alpha^+]_\mu$ and $[\alpha \mapsto \alpha^{++}]_\mu$.

Will have:

**Lemma**

*For any $f : \kappa \rightarrow \kappa$ with $f(\alpha) < \alpha^+$ there is a $x \in \omega^\omega$ with $T^+_x$ wellfounded and such that $\forall^*_\mu \alpha \ f(\alpha) < |T^+_x \upharpoonright \alpha|$.*

This immediately shows that $[\alpha \mapsto \alpha^+]_\mu \leq \kappa^+$, and the lower bound follows easily from the coding lemma.
For the tree $T^{++}$ we will have:

**Lemma**

*For any $f : \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$ there is a $x \in \omega^\omega$ with $T^{++}_x$ wellfounded such that $\forall^* \alpha \left( f(\alpha) < [\beta \mapsto |T^{++}_x \upharpoonright \beta|]_{\mu_\alpha} \right)$.*

The lemma and the normality of $\mu_\alpha$ show that $[\alpha \mapsto \alpha^{++}]_\mu \leq \kappa^{++}$.

These lemmas will also give a coding of the functions into $\kappa^+$ or $\kappa^{++}$ we use to get the partition relations.
Construction of $T^+$

Fix a $\Gamma$-complete set $P$ and a $\Gamma$-scale $\{\varphi_n\}_{n \in \omega}$ on $P$.
- we use $\varphi_0$ to code ordinals $< \kappa$.

Say $\alpha$ is strongly reliable if for all $\beta < \alpha$:

$$\sup\{\varphi_n(x) : x \in P \land \varphi_0(x) \leq \beta\} < \alpha$$

The set of strongly reliable ordinals is c.u.b. in $\kappa$.

Let $(x, y) \in R \iff (x, y \in P \land \varphi(x) < \varphi(y))$.

and

$(x, y) \in R^\alpha \iff (x, y \in P \land \varphi(x) < \varphi(y) < \alpha)$. 
Note that $R^\alpha \in \Sigma_0^\alpha - \bigcup_{\lambda > \alpha} S(\lambda)$.

Also, we uniformly have a $\Sigma_0^\alpha$ scale on $R^\alpha$ (essentially by restricting the scale $\vec{\varphi}$ to ordinals below $\alpha$).

Starting from this, we uniformly get $\Sigma_1^\alpha$ universal sets $B^\alpha$ and $\Pi_1^\alpha$ sets $Q^\alpha$ and $\Pi_1^\alpha$ scales $\vec{\psi}^\alpha$ on $Q^\alpha$.

Let $W = \{x : \forall n (x)_n \in P\}$. $x \in W$ will code the ordinal $|x| = \sup_n \varphi_0((x)_n)$. The scale on $P$ easily gives a scale on $W$. Let $V_W$ be the corresponding tree.
We first construct a tree $U$ on $\omega \times \omega \times \kappa$ with the following properties:

1. If $x \in W$ and $|x| \in C$, then $U_{x,y}$ is wellfounded iff the $\Sigma_1^\alpha$ relation coded by $y$ is wellfounded.

2. For $x, y$ as above, $|U_{x,y} \upharpoonright \alpha| \geq |B_y^{|x|}|$, the $\Sigma_1^\alpha (\alpha = |x|)$ coded by $y$.

Key Point: For $x, y$ as above, the entire tree $U_{x,y}$ is wellfounded (not just $U_{x,y} \upharpoonright \alpha$).

idea: $U$ is constructed as in the proof of the Kunen-Martin theorem, but we use the components of the real $x$ to verify the appropriate reals are in $B_y^{|x|}$. 
Suppose $f : \kappa \to \kappa$ and $f(\alpha) < \alpha^+$ for $\alpha \in C$.

Consider the game $G_f$:

\[
\begin{array}{c}
| & r \\
II & x, y \\
\end{array}
\]

II wins the run iff

\[(r \in W) \rightarrow (x \in W \land B_y^{\vert x \vert} \text{ is wellfounded} \land \vert B_y^{\vert x \vert} \vert > f(\vert x \vert)).\]
A boundedness argument shows that II has a winning strategy.

This suggests the following definition of the tree $T^+$ on $(\omega)^2 \times \kappa \times (\omega)^2 \times \kappa$:

$$(\sigma, r, \vec{a}, x, y, \vec{b}) \in [T^+] \text{ iff:}$$

1. $(r, \vec{a}) \in [T_W]$.
2. $\sigma(r) = (x, y)$
3. $(x, y, \vec{b}) \in [U]$.

Then $T^+_\sigma$ is wellfounded and $|T^+_\sigma|^{\restriction \alpha} > f(\alpha)$ for $\mu$ almost all $\alpha$. 
Construction of $T^{++}$

We first construct a tree $V$ on $(\omega)^2 \times \kappa$ with the following properties:

1. $(x, y) \in [V]$ iff $x \in W$ and for all $n$, $(y)_n$ codes a $\Sigma_1$ wellfounded relation $B_{(y)_n}^{\vert x \vert}$.

2. If $x \in W$, $\vert x \vert \in C$ then there is a c.u.b. $D \subseteq \alpha^+$ such that if $\gamma \in D$, $y \in \omega^\omega$ and for all $n$, $(y)_n$ codes a $\Sigma_1$ wellfounded relation of rank $< \gamma$, then $V_{x, y \upharpoonright \gamma}$ is illfounded.

main point: We can translate the $\Pi_1^{\vert x \vert}$ statement asserting the wellfoundedness of the $B_{(y)_n}^{\vert x \vert}$ into $\Pi_1^\beta$ statements for any $\beta \geq \vert x \vert$ (use the $(x)_i$ as in the definition of $U$).
Suppose $x \in W$, $|x| = \alpha \in C$, and $g: \alpha^+ \to \alpha^+$. Play the game $G_g$:

\[ I \quad z \]
\[ II \quad w \]

II wins the run iff:

\[ (\forall n \ B^\alpha_{(z)_n} \text{ is wellfounded}) \to (B^\alpha_w \text{ is wellfounded} \land |B^\alpha_w| > g(\sup_n |B^\alpha_{(y)_n}|)) \]
By boundedness, II has a winning strategy $\tau$ for any $G_g$.

Suppose now $f : \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$.

Play the game $G_f$:

```
  I  r
  II x, \tau
```

$r, x$ will be in $W$ and $\tau$ will be strategy for a game $G_g$ where $[g]_{\mu\alpha} > f(\alpha)$, where $\alpha = |x|$. 

More precisely, II wins the run iff:

\[ r \in \mathcal{W} \rightarrow (x \in \mathcal{W} \land |x| = \alpha \geq |y|) \]
\[ \land \forall z \left( \forall n B^\alpha_{(z)_n} \text{ is wellfounded} \rightarrow \right. \]
\[ \left. \left( B^\alpha_{\tau(z)} \text{ is wellfounded} \land |B^\alpha_{\tau(z)}| \geq g(\sup_n |B^\alpha_{(z)_n}|) \right) \right] \]

for some \( g : \alpha^+ \rightarrow \alpha^+ \) with \( [g]_{\mu_\alpha} \geq f(\alpha) \).
II has a winning strategy $\sigma$ for any $f$, and this suggests the definition of $T^{++}$:

$$(\sigma, r, \vec{a}, x, \tau, y, z, \vec{\beta}, \vec{\gamma}) \in T^{++} \text{ iff:}$$

1. $(r, \vec{a}) \in [T_W]$.
2. $\sigma(r) = (x, \tau)$.
3. $(x, y, \vec{\beta}) \in [V]$.
4. $\tau(y) = z$.
5. $(x, a, \vec{\gamma}) \in [U]$.

The properties of $U$ and $V$ show that $T^{++}$ has the desired property.
The countable exponent $\theta$ case.

Fix a bijection $\pi: \omega \cdot \theta \to \omega$.

We code cofinally in $\kappa^+, \kappa^{++}$ many ordinals using sections of our trees: $T^+_x, T^{++}_x$.

Suppose $\mathcal{P}$ is a partition of the block functions from $3 \times \theta$ to $(\kappa, \kappa^+, \kappa^{++})$.

Consider the game $G_{\mathcal{P}}$:

I $x, y, z$
II $x', y', z'$
(1) If there is an $j < \omega \cdot \theta$ such that $(x)_{\pi(j)} \notin P_0$ or $(x')_{\pi(j)} \notin P_0$, then player I wins iff for the least such $j$, $(x)_{\pi(j)} \in P_0$.

(2) Suppose next that there is an $\alpha < \kappa$ such that one of the following holds.

(a) There is a $j < \omega \cdot \theta$ such that either $T_{(y)_{\pi(j)}}^+ \upharpoonright \alpha$ or $T_{(y')_{\pi(j)}}^+ \upharpoonright \alpha$ is illfounded.

(b) There is a $\beta < \alpha^+$ and a $j < \omega \cdot \theta$ such that either $T_{(z)_{\pi(j)}}^{++} \upharpoonright \beta$ or $T_{(z')_{\pi(j)}}^{++} \upharpoonright \beta$ is illfounded.

Let $\alpha < \kappa$ be least such that (a) or (b) above holds. If (a) holds, let $j$ be least such that (a) holds for $\alpha$ and this $j$. In this case, Player I wins provided $T_{(y)_{\pi(j)}}^+$ is wellfounded. If (a) does not hold at $\alpha$, but (b) does, let $(\beta, j)$ be lexicographically least such that (b) holds. Player I wins in this case provided $T_{(z)_{\pi(j)}}^{++} \upharpoonright \beta$ is wellfounded.
Assume II has a winning strategy $\tau$.

We define c.u.b. sets $C_j \subseteq \kappa$, $C_1 \subseteq \kappa^+$, and $C_2 \subseteq \kappa^{++}$.

For example, to define $C_2$ we define for $\alpha \in C$, $\beta, \gamma < \alpha^+$ and $j < \omega \cdot \theta$:

$$A_{\alpha, \beta, \eta, j} = \{ (x, y, z) ; \forall j \ ((x)_{\pi(j)} \in P_0 \land \varphi_0((x)_{\pi(j)}) < \alpha)$$

$$\land \forall \alpha' < \alpha \forall \beta < (\alpha')^+ \forall j \ (T^+_y)_{\pi(j)} \upharpoonright \alpha \text{ and } T^{++}_{z)_{\pi(j)} \upharpoonright \beta \text{ are wellfounded}$$

$$\land \forall j \ |T^+_y)_{\pi(j)} \upharpoonright \alpha| < \beta \land \forall (\beta', j') \leq \text{lex } (\beta, j) \ (|T^{++}_{z)_{\pi(j)} \upharpoonright \beta| \leq \eta) \}. $$
We have: $A_{\alpha,\beta,\gamma,j} \in \Delta^\alpha_1$.

Since $\tau$ is winning for Player II, for each $(x, y, z) \in A_{\alpha,\beta,\eta,j}$, if $\tau(x, y, x) = (x', y', z')$ then $\forall (\beta', j') \leq_{\text{lex}} (\beta, j) \ T_{(z')}^{++} \upharpoonright \beta$ is wellfounded.

By boundedness,

$$\rho_2(\alpha, \beta, \eta, j) := \sup\{|T_{(z')}^{++} \upharpoonright \beta| ; (x', y', z') \in \tau[A_{\alpha,\beta,\eta,j}] \wedge j' \leq j\} < \alpha^+.$$ 

Let $C_{2}^\alpha \subseteq \alpha^+$ be c.u.b. closed under $\rho_2$. The $C_{2}^\alpha$ lift to $C_2 \subseteq \kappa^{++}$. 