Partition Triples Under AD

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We assume the *axiom of determinacy* AD throughout. We work in the theory ZF $+$ AD $+$ DC.

We prove a polarized partition result for certain triples of cardinals and give applications to forcing over models of AD.

Main result is a strengthening/generalization of a result proved by Kechris.

To state the main AD result, we first recall some partition notation.

$(\kappa)^\lambda$ denotes the increasing functions from λ to κ.

Definition (Erdös-Rado)

$\kappa \to (\kappa)^\lambda$ means for every partition $\mathcal{P} \colon (\kappa)^\lambda \to \{0, 1\}$ there is a homogeneous $H \subseteq \kappa$ of size κ.

H is homogeneous if $\mathcal{P} \restriction (H)^\lambda$ is constant.

Definition

κ has the weak partition property if $\kappa \rightarrow (\kappa)^{\lambda}$ for all $\lambda < \kappa$.

κ has the strong partition property if $\kappa \rightarrow (\kappa)^{\kappa}$.

In ZFC no cardinal has an infinite exponent parition relation.

Let $\vec{\kappa} = (\kappa_\alpha)_{\alpha<\rho}$ be a sequence of regualr cardinals. Let $\vec{\lambda} = (\lambda_\alpha)_{\alpha<\rho}$ be a sequence or ordinals, $\lambda_\alpha \leq \kappa_\alpha$.

A block function from $\vec{\lambda}$ to $\vec{\kappa}$ is a function with domain $\oplus_{\alpha<\rho}\lambda_\alpha$ which sends the copy of λ_α into κ_α.

Definition (Polarized Partition Property)

$\vec{\kappa} \to (\vec{\kappa})^{\vec{\lambda}}$ if for every partition $\mathcal{P}$ of the block functions from $\vec{\lambda}$ to $\vec{\kappa}$ there is an $H = \oplus_{\alpha<\rho}H_\alpha$, where $H_\alpha \subseteq \kappa_\alpha$, which is homogeneous for $\mathcal{P}$.

If T is a tree on a set Z, $[T] = \{f \in Z^\omega : \forall n\, f \restriction n \in T\}$ (set of infinite branches through T).

Definition

We say $A \subseteq \omega^\omega$ is κ-Suslin if there is a tree $T \subseteq \omega \times \kappa$ such that $A = p[T] = \{x : \exists f \in \kappa^\omega\ (x, f) \in [T]\}$.

Let $S(\kappa)$ denote the collection of κ-Suslin sets.

Definition

κ is a Suslin cardinal if $S(\kappa) - \bigcup_{\lambda<\kappa} S(\lambda) \neq \emptyset$.

The Suslin cardinals are closely related to the projective ordinals.

In ZF:

- ω is the first Suslin cardinal and $S(\omega) = \boldsymbol{\Sigma}^1_1$.
- ω_1 is the next Suslin cardinal and $S(\omega_1) \supseteq \boldsymbol{\Sigma}^1_2$.

Recall δ^1_n is the supremum of the lengths of the $\boldsymbol{\Delta}^1_n$ prewellorderings of the reals.

With AD:

- $\delta^1_{2n+2} = (\delta^1_{2n+1})^+$
- $\delta^1_{2n+1} = \lambda^+_{2n+1}$, where $\mathrm{cof}(\lambda_{2n+1}) = \omega$
- δ^1_n is regular.

The first ω Suslin cardinals are:

$$\lambda_1,\ \boldsymbol{\delta}^1_1,\ \lambda_3,\ \boldsymbol{\delta}^1_3,\ \lambda_5,\ \boldsymbol{\delta}^1_5, \dots$$

where: $\lambda_1 = \omega$, $\lambda_3 = \aleph_\omega$, $\lambda_5 = \aleph_{\omega^{\omega^\omega}}, \dots$.

Also: $S(\lambda_1) = \boldsymbol{\Sigma}^1_1$, $S(\boldsymbol{\delta}^1_1) = \boldsymbol{\Sigma}^1_2$, $S(\lambda_3) = \boldsymbol{\Sigma}^1_3$, $S(\boldsymbol{\delta}^1_3) = \boldsymbol{\Sigma}^1_4, \dots$

Theorem (Steel-Woodin)

(AD) *The Suslin cardinals are closed below their supremum.*
(AD^+) *The Suslin cardinals are closed below* Θ*.*

We say a Suslin cardinal κ is a limit Suslin cardinal if κ is a limit of Suslin cardinals, otherwise κ is a successor Suslin cardinal.

In $L(\mathbb{R})$ there is a largest Suslin cardinal δ^2_1 (Martin-Steel), and $S(\delta^2_1) = \boldsymbol{\Sigma}^2_1$ is closed under real quantification.

δ^2_1 is regular and a limit of Suslin cardinals, so δ^2_1 is an inaccessible Suslin cardinal.

In fact, the inaccessible Suslin cardinals are stationary in δ^2_1.

We now state our main partition result.

Theorem (AD)

For all inaccessible Suslin cardinals κ we have:

$$(\kappa, \kappa^+, \kappa^{++}) \to (\kappa, \kappa^+, \kappa^{++})^{(\kappa,\kappa,\kappa)}$$

Corollary

κ, κ^+, κ^{++} are all measurable.

We consider the possible cofinalities/measurability of $\aleph_1, \aleph_2, \aleph_3$.

- There are 3 possibilities of $\aleph_1$ (cof $= \omega$, *regular*, measurable).
- There are 4 possibilities for $\aleph_2$, and 5 possibilities for $\aleph_3$.
- 13 of the 60 total possibilities are "trivially inconsistent." For example, $\aleph_1$ regular, $\mathrm{cof}(\aleph_2) = \aleph_1$, and $\mathrm{cof}(\aleph_3) = \aleph_2$.

Theorem

Assuming suitable large cardinals, all of the remaining 47 *patterns are consistent with* ZF*.*

We review a few more more facts about pointclasses and Suslin cardinals.

- By a Wadge degree we mean the equivalence class of a pair (A, A^c) if $A \not\equiv A^c$ (non-selfdual) or of (A) if $A \equiv A^c$ (selfdual).
- The selfdual and non-selfdual degrees alternate. At limit ordinals of cofinality ω there is a selfdual degree, and for uncountable cofinality a non-selfdual degree.
- The separation property falls on exactly one of the sides of a non-selfdual pointclass.

By a Levy class we mean a non-selfdual pointclass closed under either $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$ (or both). Let $\mathbf{\Sigma}^1_\alpha$, $\mathbf{\Pi}^1_\alpha$ enumerate these classes.

Fact (Kechris-Solovay-Steel)

If $\mathbf{\Delta}$ is selfdual and closed under quantifiers, $\wedge$, $\vee$, then $o(\mathbf{\Delta}) = \delta(\mathbf{\Delta})$

$o(\mathbf{\Delta}) = \sup\{|A|_w : A \in \mathbf{\Delta}\}$.

$\delta(\mathbf{\Delta}) = \sup\{|\prec| : \prec \in \mathbf{\Delta}\}$.

Every Levy pointclass lies in a projective-like hierarchy.

Let $C = \{o(\boldsymbol{\Delta})\colon \exists^{\omega^\omega} \boldsymbol{\Delta} \subseteq \boldsymbol{\Delta}, \wedge \boldsymbol{\Delta} \subseteq \boldsymbol{\Delta}\}$.

Given a levy class $\boldsymbol{\Gamma}$. let α be the larget element of C such that $\boldsymbol{\Delta}_\alpha = \{A\colon o(A) < \alpha\} \subseteq \boldsymbol{\Gamma}$.

$\boldsymbol{\Delta}_\alpha$ is the base of a projective-like hierarchy containing $\boldsymbol{\Gamma}$.

This hierarchy falls into one of four types.

We specialize to the case of Suslin cardinals. Let α be a limit Suslin cardinal. Let $\boldsymbol{\Delta} = \bigcup_{\lambda<\kappa} S(\lambda)$. Then $\boldsymbol{\Delta}$ is at the base of a projective-like hierarchy.

Type I $\text{cof}(\alpha) = \omega$. Let $\boldsymbol{\Sigma}^{\alpha}_{0}$ be the collection of countable unions of sets in $\boldsymbol{\Delta}$. Then $\boldsymbol{\Sigma}^{\alpha}_{0}$, $\boldsymbol{\Pi}^{\alpha}_{1}$, etc. have the scale property. α^{+} is a Suslin cardinal, and a $\boldsymbol{\Pi}^{\alpha}_{1}$ scale on a $\boldsymbol{\Pi}^{\alpha}_{1}$-complete set has norms of length κ^{+}. We have $\mathbf{S}(\kappa) = \boldsymbol{\Sigma}^{\alpha}_{1}$ and $\mathbf{S}(\kappa^{+}) = \boldsymbol{\Sigma}^{\alpha}_{2}$.

Type II or III $\text{cof}(\alpha) > \omega$, $\boldsymbol{\Gamma}$ not closed under real quantifiers. $\boldsymbol{\Gamma}$ is the Steel classof Wadge rank α. So, $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$, and $\boldsymbol{\Gamma}$ is closed under $\wedge$, $\forall^{\mathbb{R}}$. Then $\mathbf{S}(\kappa) = \exists^{\mathbb{R}}\boldsymbol{\Gamma}$, and scale($\boldsymbol{\Gamma}$), scale($\mathbf{S}(\kappa)$) hold.

Type IV In this case, the pointclasses $\boldsymbol{\Gamma}$, $\check{\boldsymbol{\Gamma}}$ of Wadge degree κ are closed under real quantification. Let $\boldsymbol{\Gamma}$ be such that $\check{\boldsymbol{\Gamma}}$ has the separation property. Then scale($\boldsymbol{\Gamma}$), and $\mathbf{S}(\kappa) = \boldsymbol{\Gamma}$.

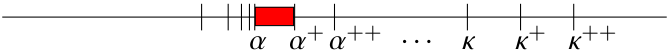
Of particular importance for the proof are the Type I hierarchies.

Fix now an inaccessible Suslin κ, with corresponding pointclass $\boldsymbol{\Gamma}$ (of Wadge rank κ).

κ is a limit Suslin cardinal. Let C be the c.u.b. set of limit Suslin cardinals below κ.

We are interested in the points of C of cofinality ω (Type I hierarchy).

We have the following picture.



$$\mathbf{\Sigma}^\alpha_0 = \bigcup_{\lambda<\alpha} S(\lambda)$$

$$S(\alpha) = \mathbf{\Sigma}^\alpha_1, \qquad S(\alpha^+) = \mathbf{\Sigma}^\alpha_2$$

$$\text{scale}(\mathbf{\Sigma}^\alpha_0,) \qquad \text{scale}(\mathbf{\Pi}^\alpha_1)$$

Plan of the proof

Let μ be the ω-cofinal normal measure on κ. (κ has the strong partition property by Kechris-Kleinberg-Moschovakis-Woodin).

Let μ_α be the ω-cofinal normal measure on α^+. We have $j_{\mu_\alpha}(\alpha^+) = \alpha^{++}$.

1. We show $[\alpha \mapsto \alpha^+]_\mu = \kappa^+$.
2. We show $\delta \doteq [\alpha \mapsto \alpha^{++}]_\mu \leq \kappa^{++}$.
3. We show for all $\theta < \omega_1$ that $(\kappa, \kappa^+, \delta) \to (\kappa, \kappa^+, \kappa^{++})^\theta$.
   It follows that δ is regular, so $\delta = \kappa^{++}$.
4. We show $(\kappa, \kappa^+, \delta) \to (\kappa, \kappa^+, \kappa^{++})^\theta$ for all $\theta < \kappa$.
5. Finally, we show $(\kappa, \kappa^+, \delta) \to (\kappa, \kappa^+, \kappa^{++})^\kappa$

We define two trees T^+ and T^{++} on $\omega \times \kappa$ to analyze the ultrapowers $[\alpha \mapsto \alpha^+]_\mu$ and $[\alpha \mapsto \alpha^{++}]_\mu$.

Will have:

Lemma

For any $f \colon \kappa \to \kappa$ with $f(\alpha) < \alpha^+$ there is a $x \in \omega^\omega$ with T_x^+ wellfounded and such that $\forall_\mu^* \alpha\; f(\alpha) < |T_x^+ \restriction \alpha|$.

This immediately shows that $[\alpha \mapsto \alpha^+]_\mu \leq \kappa^+$, and the lower bound follows easily from the coding lemma.

For the tree T^{++} we will have:

Lemma

For any $f\colon \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$ there is a $x \in \omega^\omega$ with T_x^{++} wellfounded such that $\forall^*_\mu \alpha \; (f(\alpha) < [\beta \mapsto |T_x^{++} \restriction \beta|]_{\mu_\alpha})$.

The lemma and the normality of μ_α show that $[\alpha \mapsto \alpha^{++}]_\mu \leq \kappa^{++}$.

These lemmas will also give a coding of the functions into κ^+ or κ^{++} we use to get the partition relations.

Construction of T^+

Fix a $\mathbf{\Gamma}$-complete set P and a $\mathbf{\Gamma}$-scale $\{\varphi_n\}_{n\in\omega}$ on P.

– we use φ_0 to code ordinals $< \kappa$.

Say α is *strongly reliable* if for all $\beta < \alpha$:

$$\sup\{\varphi_n(x) \colon x \in P \wedge \varphi_0(x) \leq \beta\} < \alpha$$

The set of strongly reliable ordinals is c.u.b. in κ.

Let
$(x, y) \in R \leftrightarrow (x, y \in P \wedge \varphi(x) < \varphi(y)\}$.

and
$(x, y) \in R^\alpha \leftrightarrow (x, y \in P \wedge \varphi(x) < \varphi(y) < \alpha\}$.

Note that $R^\alpha \in \mathbf{\Sigma}^\alpha_0 - \bigcup_{\lambda > \alpha} S(\lambda)$.

Also, we uniformy have a $\mathbf{\Sigma}^\alpha_0$ scale on R^α (essentially by restricting the scale $\vec{\varphi}$ to ordinals below α).

Starting from this, we uniformy get $\mathbf{\Sigma}^\alpha_1$ universal sets B^α and $\mathbf{\Pi}^\alpha_1$ sets Q^α and $\mathbf{\Pi}^\alpha_1$ scales $\vec{\psi}^\alpha$ on Q^α.

Let $W = \{x \colon \forall n\, (x)_n \in P\}$. $x \in W$ will code the ordinal $|x| = \sup_n \varphi_0((x)_n)$. The scale on P easily gives a scale on W.Let V_W be the corresponding tree.

We first construct a tree U on $\omega \times \omega \times \kappa$ with the following properties:

1. If $x \in W$ and $|x| \in C$, then $U_{x,y}$ is wellfounded iff the $\mathbf{\Sigma}^{\alpha}_1$ relation coded by y is wellfounded.
2. For x, y as above, $|U_{x,y} \restriction \alpha| \geq |B^{|x|}_y|$, the $\mathbf{\Sigma}^{\alpha}_1$ $(\alpha = |x|)$ coded by y.

Key Point: For x, y as above, the entire tree $U_{x,y}$ is wellfounded (not just $U_{x,y} \restriction \alpha$).

idea: U is constructed as in the proof of the Kunen-Martin theorem, but we use the components of the real x to verify the appropriate reals are in $B^{|x|}_y$.

Suppose $f\colon \kappa \to \kappa$ and $f(\alpha) < \alpha^+$ for $\alpha \in C$.

Consider the game G_f:

I	r
II	x, y

II wins the run iff

$$(r \in W) \to (x \in W \wedge B_y^{|x|} \text{ is wellfounded } \wedge |B_y^{|x|}| > f(|x|).$$

A boundedness argument shows that II has a winning strategy.

This suggests the following definition of the tree T^+ on $(\omega)^2 \times \kappa \times (\omega)^2 \times \kappa$:

$(\sigma, r, \vec{\alpha}, x, y, \vec{\beta}) \in [T^+]$ iff:

1. $(r, \vec{\alpha}) \in [T_W]$.
2. $\sigma(r) = (x, y)$
3. $(x, y, \vec{\beta}) \in [U]$.

Then T^+_σ is wellfounded and $|T^+_\sigma \restriction \alpha| > f(\alpha)$ for μ almost all α.

Construction of T^{++}

We first construct a tree V on $(\omega)^2 \times \kappa$ with the following properties:

1. $(x, y) \in [V]$ iff $x \in W$ and for all n, $(y)_n$ codes a $\mathbf{\Sigma}_1^{|x|}$ wellfounded relation $B^{|x|}_{(y)_n}$.
2. If $x \in W$, $|x| \in C$ then there is a c.u.b. $D \subseteq \alpha^+$ such that if $\gamma \in D$, $y \in \omega^\omega$ and for all n $(y)_n$ codes a $\mathbf{\Sigma}_1^{|x|}$ wellfounded relation of rank $< \gamma$, then $V_{x,y} \restriction \gamma$ is illfounded.

main point: We can translate the $\mathbf{\Pi}_1^{|x|}$ statement asserting the wellfoudedness of the $B^{|x|}_{(y)_n}$ into $\mathbf{\Pi}_1^{\beta}$ statements for any $\beta \geq |x|$ (use the $(x)_i$ as in the definition of U).

Suppose $x \in W$, $|x| = \alpha \in C$, and $g \colon \alpha^+ \to \alpha^+$. Play the game G_g:

I	z
II	w

II wins the run iff:

$$(\forall n\, B^\alpha_{(z)_n} \text{ is wellfounded }) \to (B^\alpha_w \text{ is wellfounded } \wedge |B^\alpha_w| > g(\sup_n |B^\alpha_{(y)_n}|))$$

By boundedness, II has a winning strategy τ for any G_g.

Suppose now $f\colon \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$.

Play the game G_f:

I	r
II	x, τ

r, x will be in W and τ will be strategy for a game G_g where $[g]_{\mu_\alpha} > f(\alpha)$, where $\alpha = |x|$.

More precisely, II wins the run iff:

$$\begin{aligned} r \in W \rightarrow (x \in W \wedge |x| = \alpha \geq |y|) \\ \wedge \forall z\, [\forall n\, B^\alpha_{(z)_n} \text{ is wellfounded } \rightarrow \\ B^\alpha_{\tau(z)} \text{ is wellfounded } \wedge |B^\alpha_{\tau(z)}| \geq g(\sup_n |B^\alpha_{(z)_n}|)] \end{aligned}$$

for some $g \colon \alpha^+ \rightarrow \alpha^+$ with $[g]_{\mu_\alpha} \geq f(\alpha)$.

II has a winning strategy σ for any f, and this suggests the definition of T^{++}:

$(\sigma, r, \vec{\alpha}, x, \tau, y, z, \vec{\beta}, \vec{\gamma}) \in T^{++}$ iff:

1. $(r, \vec{\alpha}) \in [T_W]$.
2. $\sigma(r) = (x, \tau)$.
3. $(x, y, \vec{\beta}) \in [V]$
4. $\tau(y) = z$.
5. $(x, a, \vec{\gamma}) \in [U]$

The properties of U and V show that T^{++} has the desired property.

The countable exponent θ case.

Fix a bijection $\pi\colon \omega\cdot\theta \to \omega$.

We code cofinally in κ^+, κ^{++} many ordinals using sections of our trees: T_x^+, T_x^{++}.

Suppose $\mathcal{P}$ is a partition of the block functions from $3\times\theta$ to $(\kappa, \kappa^+, \kappa^{++})$.

Consider the game $G_{\mathcal{P}}$:

$$\begin{array}{ll} \text{I} & x, y, z \\ \text{II} & x', y', z' \end{array}$$

(1) If there is an $j < \omega \cdot \theta$ such that $(x)_{\pi(j)} \notin P_0$ or $(x')_{\pi(j)} \notin P_0$, then player I wins iff for the least such j, $(x)_{\pi(j)} \in P_0$.

(2) Suppose next that there is an $\alpha < \kappa$ such that one of the following holds.

(a) There is a $j < \omega \cdot \theta$ such that either $T^{+}_{(y)_{\pi(j)}} \restriction \alpha$ or $T^{+}_{(y')_{\pi(j)}} \restriction \alpha$ is illfounded.

(b) There is a $\beta < \alpha^{+}$ and a $j < \omega \cdot \theta$ such that either $T^{++}_{(z)_{\pi(j)}} \restriction \beta$ or $T^{++}_{(z')_{\pi(j)}} \restriction \beta$ is illfounded.

Let $\alpha < \kappa$ be least such that (a) or (b) above holds. If (a) holds, let j be least such that (a) holds for α and this j. In this case, Player I wins provided $T^{+}_{(y)_{\pi(j)}}$ is wellfounded. If (a) does not hold at α, but (b) does, let (β, j) be lexicographically least such that (b) holds. Player I wins in this case provided $T^{++}_{(z)_{\pi(j)}} \restriction \beta$ is wellfounded.

Assume II has a winning strategy τ.

We define c.u.b. sets $C_{)} \subseteq \kappa$, $C_1 \subseteq \kappa^+$, and $C_2 \subseteq \kappa^{++}$.

For example, to define C_2 we define for $\alpha \in C$, $\beta, \gamma < \alpha^+$ and $j < \omega \cdot \theta$:

$$\begin{aligned} A_{\alpha,\beta,\eta,j} =& \{(x, y, z) \,;\, \forall j \, ((x)_{\pi(j)} \in P_0 \wedge \varphi_0((x)_{\pi(j)}) < \alpha) \\ & \wedge \forall \alpha' < \alpha \; \forall \beta \; < (\alpha')^+ \; \forall j \, (T^+_{(y)_{\pi(j)}} \restriction \alpha \text{ and } T^{++}_{(z)_{\pi(j)}} \restriction \beta \text{ are wellfoun} \\ & \wedge \forall j \, |T^+_{(y)_{\pi(j)}} \restriction \alpha| < \beta \wedge \forall (\beta', j') \leq_{\text{lex}} (\beta, j) \, (|T^{++}_{(z)_{\pi(j)}} \restriction \beta| \leq \eta)\}. \end{aligned}$$

We have: $A_{\alpha,\beta,\gamma,j} \in \mathbf{\Delta}^{\alpha}_1$.

Since τ is winning for Player II, for each $(x, y, z) \in A_{\alpha,\beta,\eta,j}$, if $\tau(x, y, x) = (x', y', z')$ then $\forall(\beta', j') \leq_{\text{lex}} (\beta, j)$ $T^{++}_{(z')_{\pi(j')}} \restriction \beta$ is wellfounded.

By boundedness,

$$\rho_2(\alpha, \beta, \eta, j) := \sup\{|T^{++}_{(z')_{\pi(j')}} \restriction \beta| \,;\, (x', y', z') \in \tau[A_{\alpha,\beta,\eta,j}] \wedge j' \leq j\} < \alpha^+.$$

Let $C^{\alpha}_2 \subseteq \alpha^+$ be c.u.b. closed under ρ_2. The C^{α}_2 lift to $C_2 \subseteq \kappa^{++}$.