Colorings and Supercolorings on Countable Groups

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December 2011
Paris

We consider the shift action of a countable group G on the compact Polish space $X = 2^G$.

$$g \cdot x(h) = x(g^{-1}h)$$

This is a continuous action of G on the space 2^G. Let $[x]$ denote the orbit of x under this action.

We let $F(2^G)$ denote the free part of the space:

$$x \in F(2^G) \Leftrightarrow \forall g \neq 1_G \; (g \cdot x \neq x)$$

$F(2^G)$ is an invariant G_δ in X.

A subflow means a closed invariant subset of 2^G. Every orbit closure $[x]$ is a subflow.

We say a subflow $S \subseteq 2^G$ is a free subflow if $S \subseteq F(2^G)$.

We say S is a minimal subflow if S does not properly contain any subflows. So, $S = \overline{[x]}$ for all $x \in A$.

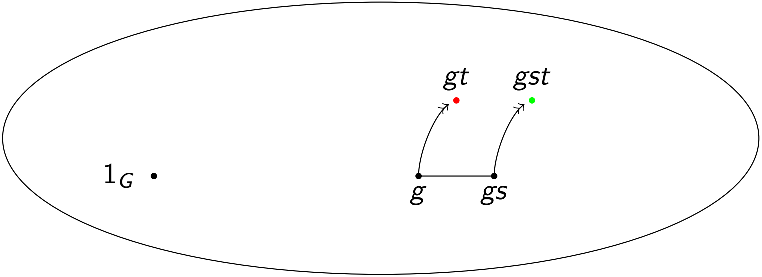
We say $x, y \in 2^G$ are orthogonal, $x \perp y$ if $\overline{[x]} \cap \overline{[y]} = \emptyset$.

2-colorings

Definition

A 2-coloring of a group G is an $x\colon G \to \{0,1\}$ satisfying the following: for every $s \neq 1_G$, there is a finite $T = T(s) \subseteq G$ such that:

$$\forall g \in G\ \exists t \in T\ (x(gt) \neq x(gst)).$$



Remark

The notion of 2-coloring is due to GJS and independently to Glasner-Uspensky.

The significance of 2-colorings is contained in the following.

Theorem (GJS; independently Glasner, Uspensky)

$x \in 2^G$ is a 2-coloring iff $\overline{[x]} \subseteq F(2^G)$.

Proof:

Suppose $x \in 2^G$ is a 2-coloring. We show that $\overline{[x]} \subseteq F(2^G)$.

Suppose $y = \lim g_n^{-1} \cdot x \in \overline{[x]}$, and let $s \neq 1_G$. Let $T = T(s)$ be as in a 2-coloring. Let n be large enough so that $y \restriction T \cup sT = g_n^{-1} \cdot x \restriction T \cup sT$.

Let $t \in T$ be such that $x(g_n s) \neq x(g_n st)$. Then $y(t) = x(g_n t) \neq x(g_n st) = y(st) = s^{-1} \cdot y(t)$.

So, $y \in F(2^G)$.

Suppose next that $\overline{[x]} \subseteq F(2^G)$.

Let $s \neq 1_G$. For $y \in \overline{[x]}$ let $t(y) \in G$ be least such that $y(t) \neq s^{-1} \cdot y(t) = y(st)$. The map $y \mapsto t(y)$ is continuous from $\overline{[x]}$ to G.

By compactness of $\overline{[x]}$ there is a finite $T \subseteq G$ containing the range of this map on $\overline{[x]}$.

In particular, this T works for all $y = g^{-1} \cdot x \in [x]$, and this shows that x is a 2-coloring: for all $g \in G$ there is a $t \in T$ such that

$$x(gt) = g^{-1} \cdot x(t) \neq s^{-1} \cdot g^{-1} \cdot x(t) = x(gst)$$

□

The construction of a 2-coloring involves putting a new kind of marker structure on an arbitrary countable group.

Question

What kinds of marker structures can we put on arbitrary groups. What about special families of groups?

By an abstract marker structure $(\Delta, \mathcal{R})$ on G we mean:

- $\mathcal{R} = \{R\}$ is a pairwise disjoint collection of subsets of G ("marker regions").
- For each $R \in \mathcal{R}$, $|\Delta \cap R| = 1$ (Δ is the "marker points").
- $\cup\{\delta_R^{-1} R \colon R \in \mathcal{R}\}$ is finite, where $\{\delta_R\} = \Delta \cap R$.

Special Marker Structures

We call a marker structure $(\Delta, \mathcal{R})$:

- regular if there is a single $F \subseteq G$ such that $R = \delta_R F$ for all $R \in \mathcal{R}$.
- centered if $1_G \in \Delta$.
- total if $\cup \mathcal{R} = G$.

We usually represent regular structures as (Δ, F).

We say (Δ, F) is a tiling if it is regular and total.

We say a sequence $(\Delta_n, \mathcal{R}_n)$ of marker structures is:

- coherent if whenever $k \leq n$, $R_k \in \mathcal{R}_k$, $r_n \in \mathcal{R}_n$, and $R_k \cap R_n \neq \emptyset$, then $R_k \subseteq R_n$.
- cofinal if for every finite $A \subseteq \cup_n \cup \mathcal{R}_n$ we have that for large enough m that $A \subseteq R_m$ for some $R_m \in \mathcal{R}_m$.
- centered if $1_G \in \Delta_n$ for all n.

For centered regular (Δ_n, F_n), being cofinal is equivalent to: for finite $A \subseteq \cup_n \cup \Delta_n F_n$ we have $A \subseteq F_n$ for n large enough (since wlog $1_G \in A$).

Definition (Weiss)

G is MT (monotilable) if it has a cofinal sequence of tilings.

Definition (GJS)

G ic ccc if it has a coherent, cofinal, centered sequence of tilings.

Weiss showed all solvable and resiidually finite groups are MT, and this class is closed under group extensions.

We showed that all nilpotent and polycyclic groups (solvable with finitely generated quotients) are ccc.

We have no examples on non-ccc groups.

Remark

Every tiling (Δ, F) has a presentation as a centered tiling, so G is ccc iff it has a coherent, cofinal sequence of tilings.

In hyperfiniteness proofs we deal with total, but not regular marker structures.

In the coloring theory of group actions we deal with regular, but not total structures.

Remark

The above definitions extend in a natural way to equivalence relations in place of groups. Note that the existence of a type of marker structure on an equivalence relation generated by a free action of G implies one on G.

Theorem

Every countable group G has a 2-coloring. In fact, there is a perfect set of pairwise orthogonal 2-colorings on G.

So, every shift action of G on 2^G has a perfect set of pairwise disjoint free subflows.

The set of 2-colorings is also dense in 2^G.

The construction of a 2-coloring involves constructing a sequence (Δ_n, F_n) of regular, coherent, centered marker structures.

We construct a sequence (Δ_n, F_n) of regular marker structures which are coherent, cofinal, and centered.

We will have $F_0 \subseteq F_1 \subseteq \cdots$ and $\Delta_0 \supseteq \Delta_1 \supseteq \cdots$

Main Properties:

- (coherence) If $k \leq n$ and $\psi F_k \cap \gamma F_n \neq \emptyset$, then $\psi F_k \subseteq \gamma F_n$ (here $\psi_k \in \Delta_k$, $\gamma \in \Delta_n$).
- (uniformity) If $k < n$ and $\gamma, \sigma \in \Delta_n$, then $\gamma^{-1}(\Delta_k \cap \gamma F_n) = \sigma^{-1}(\Delta_k \cap \sigma F_n)$.
- (maximal disjointness) For each n, $\{\delta F_n \colon \delta \in \Delta_n\}$ is a maximal disjoint collection of copies of F_n

Each F_n region will be a union of copies of F_i for $i < n$.

F_n will be constructed inside a region H_n. The H_n grow sufficiently fast:

$$H_{n-1}(H_0^{-1}H_0)(H_1^{-1}H_1)\cdots(H_{n-1}^{-1}H_{n-1}) \subseteq H_n.$$

We interpolate between H_{n-1} and H_n a sequence of sets

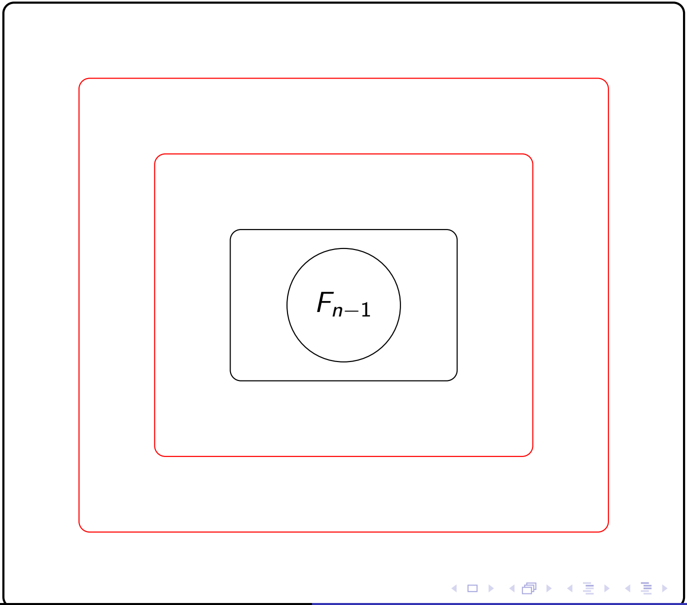
$$H_{n-1} \subseteq \beta(n,0) \subseteq \beta(n,1) \subseteq \cdots \subseteq \beta(n,n-1) = H_n$$

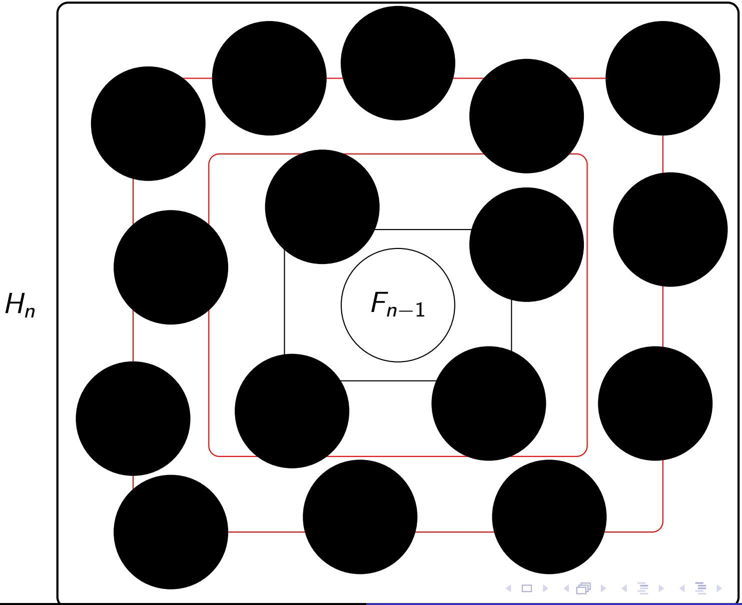
where $\beta(n,r) = \{g \colon g(H_{r+1}^{-1}H_{r+1})\cdots(H_{n-1}^{-1}H_{n-1}) \subseteq H_n\}$.

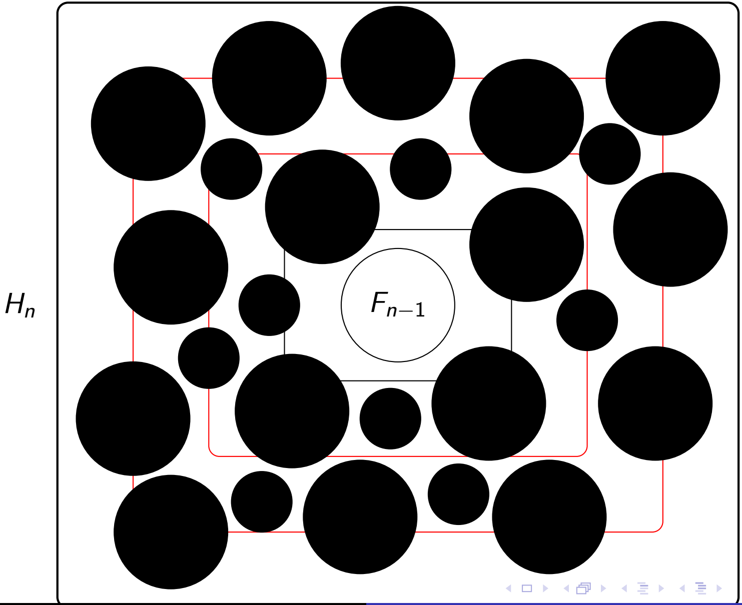
Going from F_{n-1} to F_n:

1. First put maximal disjoint collection of copies of F_{n-1} inside $H_n = \beta(n, n-1)$.
2. Next put maximal disjoint collection of copies of F_{n-2} inside of $\beta(n, n-2) - \bigcup F_{n-2}$ copies.
3. At step k, put maximal disjoint number of copies of F_k inside of $\beta(n, k) - \bigcup$ previous $F_{k+1}, \ldots, F_{n-1}$ copies.

H_n




H_n
F_{n-1}


H_n
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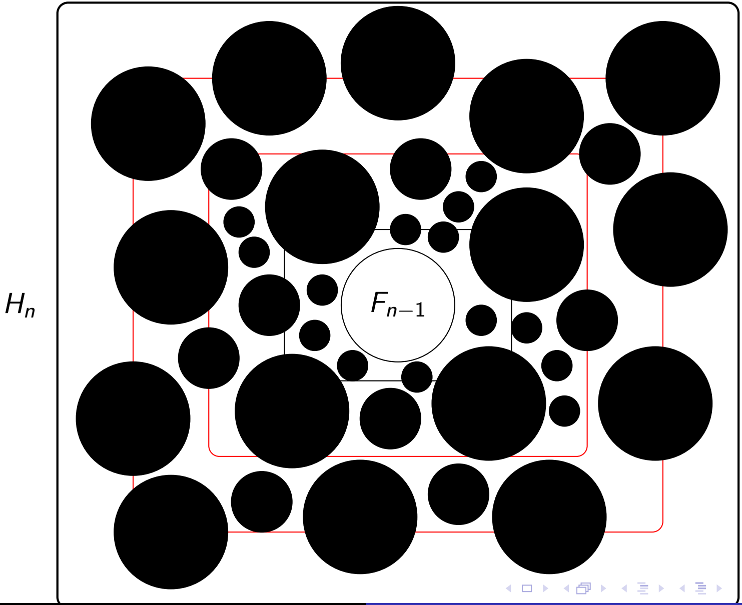

H_n
F_{n-1}

Main Point: If a copy of F_k meets a copy of F_l, $l > k$, then it must meet a copy of F_k inside F_l.

This guarantees the copies of F_k are maximally disjoint in G.

We label the copies of F_{n-1} inside of F_n by

$$\Lambda_n F_{n-1} = \lambda_1^n F_{n-1}, \dots, \lambda_{s(n)}^n F_{n-1},$$

$$\alpha_n F_{n-1}, \beta_n F_{n-1}, \gamma_n F_{n-1}.$$

F_n will have two distigiuished points a_n, b_n:

$$a_n = \alpha_n a_{n_1}$$
$$b_n = \beta_n b_{n-1}$$

We let $\Delta_n F_n$ be the copies of F_n placed in the construction.

We define a coloring of $G = \bigcup_n \Delta_n \Lambda_n b_{n-1}$ inductively. We extend c_{n-1} to c_n by coloring all the points of $\Delta_{n-1} a_{n-1}$, $\Delta_{n-1} b_{n-1}$ except those in $\Delta_n a_n$, $\Delta_n b_n$, $\Delta_n \Lambda_n$.

We maintain local recognizability of the Δ_n points:

There is an $A \subseteq \gamma_{n-1} F_{n-1} \cap \mathrm{dom}(c_n)$ such that for any c extending c_n and any $g \in G$, if $c(ga) = c(a)$ for all $a \in A$, then $g \in \Delta_n$.

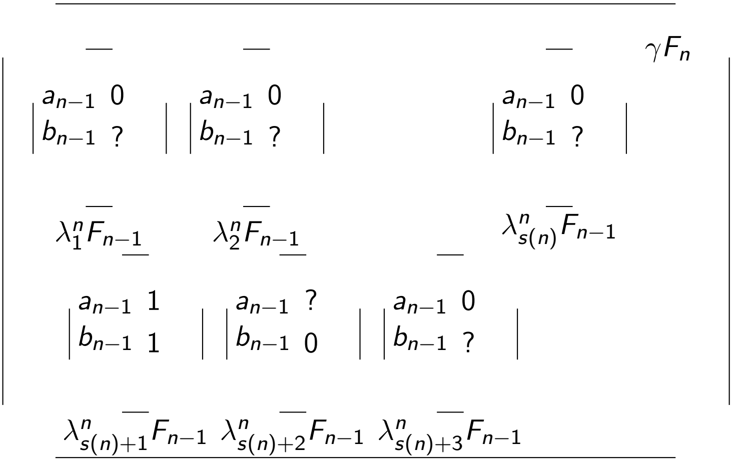


Figure: Extending c_{n-1} to c_n.

We extend $\bigcup_n c_n$ to a 2-coloring c. We color the points of $\boldsymbol{\Delta}_n\{\lambda_1^n, \dots, \lambda_{s(n)}^n\}b_{n-1}$ to take care of the coloring property for $s = g_n \in H_n$.

Let $g \in G$ and consider the pair g, gs. By maximal disjointness of F_n copies, $gf \in \boldsymbol{\Delta}_n$ for some $f \in F_n F_n^{-1}$. Done unless $gsf \in \boldsymbol{\Delta}_n$. In this case

$$gsf = gf(f^{-1}sf) \in (gf)F_nF_n^{-1}H_nF_nF_n^{-1}.$$

So there are about $|H_n|^5$ many points to consider, and there $2^{s(n)}$ many "colors" available, where $s(n)$ is linear in $|H_n|$.

Using these marker methods we can get other results connecting the algebraic properties of the group with the dynamics of the group action.

Theorem

There is a minimal 2-coloring, in fact the set of such x is dense.

It is easy to see that the set of 2 colorings is always $\mathbf{\Sigma}^0_2$ hard, and that it is a $\mathbf{\Pi}^0_3$ set.

Theorem

For any G, the set of minimal elements is $\mathbf{\Pi}^0_3$-complete.

For the set of 2-colorings, the complexity depends on the properties of the group.

flecc groups

Definition

G is flecc if there is a finite set $A \subseteq G - \{1_g\}$ such that for all $g \in G - \{1_G\}$ there is an $i \in \mathbb{Z}$ and an $h \in G$ such that $h^{-1}g^i h \in A$.

For $g \in G$, let ecc(g) $= \bigcup_n \{h^{-1}g^n h \colon h \in G\}$.

Let lecc(g) $= \bigcap_n \mathrm{ecc}(g^n) - \{1_G\}$ if g has infinite order and otherwise any $\mathrm{ecc}(g^k) - \{1_G\}$ where g^k is of prime order is a lecc of g.

Fact

Any two leccs which intersect must coincide.

Theorem

If G is flecc then the set of 2-colorigs of G is $\mathbf{\Sigma}^0_2$. If G is non-flecc then the set of 2-colorings of G is $\mathbf{\Pi}^0_3$-complete.

Suppose G is flecc. Let $A \subseteq G - \{1_G\}$ be finite and witness G is flecc. Then $x \in 2^G$ is a 2-coloring iff for every $s \in A$ x blocks s, that is

$$\exists T \in G^{<\omega}\ \forall g \in G\ \exists t \in T\ [c(gt) \neq c(gst)].$$

This is because if x blocks $s^i \neq 1_G$ for some i then a blocks s, and x blocks s iff x blocks $h^{-1}sh$ for all $h \in G$.

Some Properties of Flecc Groups

- If $G \times H$ is flecc, then G, H are flecc.
- If G is flecc and H is a torsion flecc group, then $G \times H$ is flecc.
- If G is flecc and $H \trianglelefteq G$, then H is flecc.
- If T is is the torsion subgroup of a flecc group, then G/T is flecc.
- An abelian group is flecc iff it is a finite sum of quasicyclic groups $\mathbb{Z}(p^\infty)$ and finite cyclic groups.

Question

Is a product of two flecc groups a flecc group? Is the quotient of a flecc group a flecc group?

Question

If the set of minimal 2-colorings for a non-flecc group $\mathbf{\Pi}^0_3$-complete?

Topological Conjugacy

Definition

We say subflows $S_1, S_2 \subseteq 2^G$ are topologically conjugate if there is a homeomorphism $\varphi \colon S_1 \to S_1$ satisfying $\varphi(g \cdot x) = g \cdot \varphi(x)$.

Let $S(G) = \{A \subseteq 2^G \colon A \text{ is a subflow }\}$. $S(G)$ is a compact Polish space.

We consider also:

$$\begin{aligned} S_F(G) &= \text{ the set of free subflows} \\ S_M(G) &= \text{ the set of minimal subflows} \\ S_{MF}(G) &= \text{ the set of minimal free subflows} \end{aligned}$$

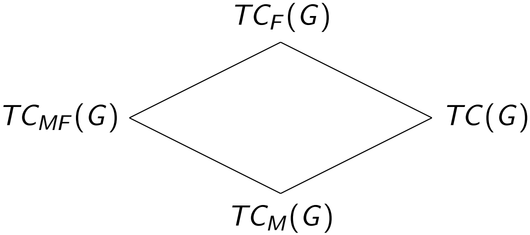
These sets are all standard Borel spaces.

Let $TC(G)$ be the topological conjugacy relation on $S(G)$, and let $TC_F(G)$, $TC_M(G)$, $TC_{MF}(G)$ be the restrictions to the above sets.

Fact

These are all countable Borel equivalence relations.

We have the following reductions:



Theorem

E_0 continuously embeds into $TC_{MF}(G)$ (and so into $TC_M(G)$, $TC_F(G)$, $TC(G)$).

Theorem

If G is locally finite then $TC(G)$ is Borel reducible to E_0 (and so also are $TC_{MF}(G)$, $TC_M(G)$, $TC_F(G)$).

Theorem

Let G be a nonlocally finite group. Then $TC_F(G)$ (and so $TC(G)$) is a universal countable Borel equivalence relation.

Remark

Clemens showed first that $TC(\mathbb{Z}^n)$ is a universal countable Borel equivalence relation, and also independently obtained the last two results.

Question

What is the complexity of $TC_M(G)$, $TC_{MF}(G)$?

Supercolorings

Definition

A super 2-coloring of G is a 2-coloring in which $|T(s)|$ is bounded independently of s.

2:coloring:

$$\forall s \neq 1_G \; \exists T(s) \in G^{<\omega} \; \forall g \in G \; \exists t \in T(s) \; (c(gt) \neq c(gst))$$

Super 2-coloring:

$$\exists n \; \forall s \neq 1_G \; \exists T(s) \in G^{n} \; \forall g \in G \; \exists t \in T(s) \; (c(gt) \neq c(gst))$$

Remark

We can't make $T(s)$ independent of s.

We say c is a super 2-coloring of degree n if the $T(s)$ have size n.

A topological definition:

Definition

For $A \subseteq 2^G$ we define the n-closure of A by:

$$y \in \overline{A}^n \Leftrightarrow \forall F \in G^n \; \exists x \in A \; (y \restriction F = x \restriction F).$$

- $\overline{A}^1 \supseteq \overline{A}^2 \supseteq \cdots \overline{A}^n \supseteq \cdots \supseteq \overline{A}^\infty = \overline{A}$.
- Each $\overline{A}^n$ is a closed set.
- Each $\overline{[x]}^n$ is a closed invariant set in 2^G (i.e., is a subflow).
- $\overline{[x]}^1 = 2^G$ unless x is the constant 0 or constant 1 element.

We say $x \in 2^G$ is an *n-free orbit* if $\overline{[x]}^n \subseteq F(2^G)$.

Fact

If G has super 2-coloring of degree n (i.e., $|T(s)| = n$), then there is an orbit $[x]$ with $\overline{[x]}^{2n} \subseteq F(2^G)$.

So, the existence of a super 2-coloring on G implies the existence of a $2n$-free orbit.

Proof.

Let $x \in 2^G$ be a super 2-coloring of degree n. Let $y \in \overline{[x]}^{2n}$. To show $y \in F(2^G)$, fix $s \neq 1_G$. Let $T = T(s)$ of size n be as in super 2-coloring.

Let $F = T \cup sT$. Let $g \in G$ be such that $y \restriction F = g^{-1} \cdot x \restriction F$. Let $t \in T$ be such that $x(gt) \neq x(gst)$. Then $y(t) = x(gt) \neq x(gst) = y(st)$, so $s^{-1} \cdot y \neq y$.

$\square$

So,

There is a super 2-coloring of degree n of G
$\Rightarrow$ There exists a $2n$-free orbit
$\Rightarrow$ There is a free orbit
$\Leftrightarrow$ There is a 2-coloring of G.

Question

Which groups admit super 2-colorings? Which shift actions admit n-free orbits?

Remark

It was open for a while whether $\mathbb{Z}$ has a super 2-coloring.

Connection with combinatorics

The failure of G to have a super 2-coloring inplies a Van DerWaerden-type result for colorings of G:

If $x\colon G \to \{0,1\}$, then for any $n \in \mathbb{N}$ there is an $s \in G$ and a $g \in G$ such that

$$x(g) = x(sg) = x(s^2g) = \cdots = x(s^ng).$$

For $G = \mathbb{Z}$ this statement implies VanDerWaerden's theorem (for two colorings).

For a general G we can define its supercoloring or n-free orbit degree.

- $n_f(G)$ is the least "integer" n in $\{2, 3, \dots\} \cup \{\infty\}$ such that G has an n-free orbit.
- $n_s(G)$, if it exists, is the least integer n such that G has a super 2-coloring of degree n.

So, if $n_s(G)$ exists, then $n_f(G)$ is finite and $n_f(G) \leq 2n_s(G)$.

We showed that $\mathbb{Z}$ has a 3-free orbit.

H. Li showed that $\mathbb{Z}$ has a supercoloring of degree 13.

So, $n_s(\mathbb{Z}) \leq 13$, $n_f(\mathbb{Z}) \leq 3$.

Other Results

- If $G = \oplus_{i=1}^{\infty} G_i$ with each G_i nintriviasl, then G has a super 2-coloring of degree 5.
- A free product or groups G_1 with $|G_0| \geq 5$ has a super 2-coloring of degree 6.
- If $\mathbb{Z} \trianglelefteq G$, then G has a 7-free orbit.

Also, $\mathbb{Z}$ has a perfect set of 5-free orbits.

Question

Does every group has a super 2 coloring (of some degree)? Does it have an n-free orbit?