Countable Borel Equivalence Relations, Markers, and Shift Equivalence

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X, Y will denote standard Borel spaces.

An equivalence relation E is countable if all classes $[x]_E$ are countable. E is Borel if $E \subseteq X \times X$ is Borel.

X/E is the quotient space of equivalence classes.

Example

If $E = \text{id}$, then $X/E \cong X$, a standard Borel space.

With AC, every set has a cardinality, and those of size $c = 2^\omega$ can be viewed as standard Borel spaces.

So, with AC, for every countable E, X/E is isomorphic to a standard Borel space.

However, we are interested in "definable" cardinalities, i.e., definable maps between spaces. Usually this means Borel.

Note that $X/\text{id} \cong X$ by a Borel map, namely, $f = \text{id}$.

Definition

If (X,E), (Y,F) are Borel equivalence relations, we say $E \leq F$ (E is reducible to F) if there is a Borel function $f\colon X \to Y$ such that

$$x\ E\ y \leftrightarrow f(x)\ F\ f(y).$$

X/E is Borel isomorphic to a standard Borel space iff $(X,E) \leq (\mathbb{R}, \mathrm{id})$.

When E is countable this equivalent to saying E has a Borel selector:

Definition

$S \subseteq X$ is a selector for E if for all x, $|S \cap [x]_E| = 1$.

Definition

E is smooth or tame if $E \leq \mathrm{id}$.

When E is smooth, then X/E is Borel isomorphic to a standard Borel space, and in this case the "Borel cardinalities" are completely understood.

Namely, if $A \subseteq X$ is Borel, then either A is countable or contains a perfect subset. Any two Borel sets in a Polish space of the same cardinality are Borel isomorphic.

So, for countable E on an uncountable Polish space X, there is up to Borel isomorphism only one smooth equivalence relation, id.

Theorem (Silver)

If E is a $\mathbf{\Pi}^1_1$ equivalence relation on a Polish space X, then E has either countable many or perfectly many equivalence classes.

Corollary

If E is a Borel equivalence relation with uncountably many classes, then $\mathrm{id} \leq E$.

Let $\{n\}$ be a Borel equivalence relation with n classes. Likewise for $\{\omega\}$.

For general Borel E we have the following initial segment of the equivalence relations:

$$\{1\} \leq \{2\} \cdots \leq \{\omega\} \leq \mathrm{id}$$

Let E_0 be the equivalence relation of eventual equality on 2^ω:

$$x \; E_0 \; y \leftrightarrow \exists n \, \forall n \geq n \, (x(m) = y(m)).$$

Fact

E_0 is bireducible with the Vitali equivalence relation on $\mathbb{R}$.

Theorem (Harrington-Kechris-Louveau)

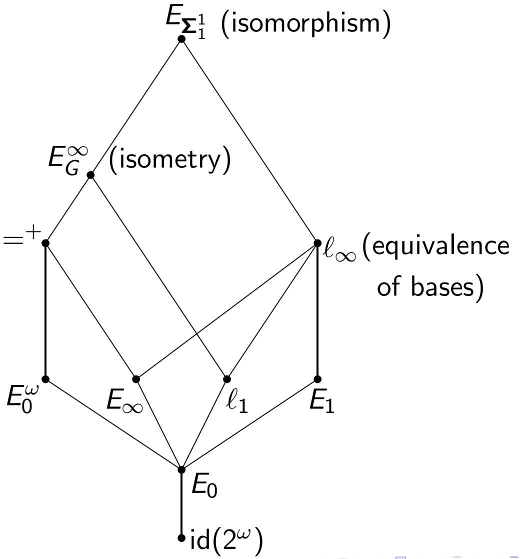
Let E be a Borel equivalence relation on a Polish space X Then either $E \leq id$ or $E_0 \leq E$ (in fact $E_0 \sqsubseteq E$).

So, for general Borel E we have:

$$\{1\} \leq \{2\} \leq \cdots \leq \{\omega\} \leq \text{id} \leq E_0.$$

General Borel equivalence relations can arise in many different ways.

- The orbit equivalence relation from a Borel action of a Polish group G on a Polish space X.
  For example, the logic action of S_∞ on the models of a countable theory.
- If $\mathcal{I}$ is any Borel ideal on ω, $x \equiv y$ iff $x \triangle y \in \mathcal{I}$.
- $\mathcal{B}$ a separable Banach space with basis $\{e_1, e_2, \dots\}$. $X = \mathbb{R}^\omega$ with xEy iff $x - y \in \mathcal{B}$. For example $c_0, \ell_1, \ell_p, \dots, \ell_\infty$.
- E_1 on $(2^\omega)^\omega$: $\{x_n\} E_1 \{y_n\}$ iff $\exists k\ \forall \ell \geq k\ (x_\ell = y_\ell)$.
- $(E_0)^\omega$, the countable product of E_0.
- $x =^+ y$ iff $\{x_n\} = \{y_n\}$.


$E_{\boldsymbol{\Sigma}^1_1}$ (isomorphism)
E_G^∞ (isometry)
$=^+$
ℓ_∞ (equivalence of bases)
E_0^ω
E_∞
ℓ_1
E_1
E_0
$\mathrm{id}(2^\omega)$

If G is a countable group, then 2^G is a compact Polish space.

The (left) action of G on 2^G is given by:

$$g \cdot x(h) = x(g^{-1}h)$$

Equivalently, $g \cdot A = gA = \{ga \colon a \in A\}$, where $A \subseteq G$.

Example

$\mathbb{Z}^n$ acts by shifts on $2^{\mathbb{Z}^n}$. Equivalence classes can be viewed as n-dimensional grids of 0s and 1s (without specifying an origin).

We consider henceforth countable Borel equivalence relations.

Theorem (Feldman-Moore)

If E is a countable Borel equivalence relation, the E is induced by the Borel action of a countable group G.

Thus, it makes sense to study countable equivalence relations "group by group."

If G is a finite group, then E_G is smooth.

Definition

E is hyperfinite if E is an increasing union $E = \bigcup_n E_n$ where each E_n is finite (i.e., all classes are finite).

Consider the simplest infinite group $\mathbb{Z}$.

Theorem (Slaman-Steel)

The following are equivalent.

1. *E is hyperfinite.*
2. *E is induced by a Borel action of $\mathbb{Z}$.*
3. *All the E (infinite) classes can be uniformly $\mathbb{Z}$ ordered.*
4. *$E \leq E_0$.*

In particular, $\mathbb{Z}$-actions give rise to hyperfinite equivalence relations.

Question

For which countable groups G are the Borel actions of G necessarily hyperfinite?

Theorem (Weiss)

If E is induced by a Borel action of $\mathbb{Z}^n$, then E is hyperfinite.

G is amenable if G has an invariant probability measure. Equivalent to the existence of a Fölner sequence.

Fact

If G is non-amenable then there is a free action of G which is not hyperfinite.

Conjecture (Kechris)

If G is amenable, then every Borel action of G is hyperfinite.

The conjecture has some credibility due to the following results.

Theorem (Connes-Feldman-Weiss)

If E is an equivalence relation induced by the action of an amenable group with an invariant probability measure μ, then E is hyperfinite μ-almost everywhere.

Theorem (Gao-J)

Every Borel action of a countable abelian group is hyperfinite.

The proof of the abelian result gives new information, even in the simplest case of $G = \mathbb{Z}$.

Theorem

There is a continuous embedding from $2^{\mathbb{Z}}$ into E_0.

In fact, we get:

Theorem

There is a continuous embedding f from $2^{\mathbb{Z}}$ into E_0 such that if $y \in 2^{\mathbb{Z}}$ is a positive shift of x, then $f(y)$ is a positive shift under the odometer action of $f(x)$.

This generalized to $(\omega^\omega)^{\mathbb{Z}}$ which then shows:

Corollary

If (X, E) is induced by the continuous action of $\mathbb{Z}$ on a 0-dimensional Polish space X, then there is a continuous embedding from (X, E) to $(2^\omega, E_0)$.

So, E_0 is universal for continuous actions of $\mathbb{Z}$ on 0-dimensional Polish spaces.

In fact:

Corollary

Let π be a free auto-homeomorphism of a 0-dimensional Polish space X. Then π is topologically isomorphic to the action of the odometer on a subspace of 2^ω.

Proof uses the construction of nice marker regions.

Definition

A Marker set for (X, E) is a Borel set $M \subseteq X$ with $M \cap [x]_E \neq \emptyset$ for all $x \in X$.
A set of marker regions for (X, E) is a Borel finite subequivalence relation $R \subseteq E$.
M is associated to R if $|M \cap [x]_R| = 1$ for all $x \in X$.

Note: Every set of marker regions has an associated marker set.

The proofs of the previous theorems use the construction of marker regions with nice geometric and definability properties.

These methods led to the following results.

Theorem

There is a continuous embedding from $2^{\mathbb{Z}^n}$ into E_0. Likewise for continuous action of $\mathbb{Z}^n$ on a 0-dimensional Polish space.

Theorem

There is a continuous embedding from the free part F of $2^{\mathbb{Z}^{<\omega}}$ into E_0.

Theorem

There is a Borel embedding from $2^{\mathbb{Z}^{<\omega}}$ into E_0.

Theorem

Every equivalence relation generated by the Borel action of an abelian group is hyperfinite.

To illustrate the ideas, we sketch the proof in the simplest setting: show there is a continuous embedding from $F(2^{\mathbb{Z}})$ into E_0.

First we get (relatively) clopen marker sets (we do this step for $\mathbb{Z}^n$):

- $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$, each S_i relatively clopen in $F(2^{\mathbb{Z}})$.
- There are distances $d_0 \gg d_1 \gg d_2 \gg \cdots$ such that:
  1. $\forall x, y \in S_n \ \rho(x, y) > d_n$.
  2. $\forall x \in X \ \exists y \in S_n \ \rho(x, y) \leq d_n$.

The definition of S_n is an ω-length construction, constructing a maximal set $S_n = \bigcup_i S_n^i$ satisfying (1).

Sets are S_n^i relatively open, so also is S_n. Maximality gives (2) which also shows S_n is relatively closed.

From these clopen marker sets, one next constructs clopen marker regions which are rectangular. In fact, they can made almost the same size (side lengths of either d_n or $d_n + 1$).

Question

Can you get Borel marker regions for $F(2^{\mathbb{Z}^n})$ which are almost the same size and almost lined-up?

Construction of the marker regions from the marker sets uses the "big marker-little marker" method, and a finite sequence of successive adjustments.

In case of $\mathbb{Z}$, this step is rather trivial.

Next we modify the marker regions to anti-cohere.

At each step when we produce marker regions R^n, we also produce an "orthogonal" set of marker regions $\tilde{R}^n$: no face of an $\tilde{R}^n$ rectangle is close to a parallel face of an R^n rectangle.

For $\mathbb{Z}$ this just says the endpoints of each $\tilde{R}^n$ interval are not close to those of an R^n interval

Close here means some fixed fraction of d_n (a geometrical constant depending only on n).

The $\tilde{R}^n$ are produced by the same adjustment process as the R^n.

We now use the R^n and $\tilde{R}^n$ to produce the final clopen marker regions Q^n.

We start with $R^n_n = R^n$, and we define the marker regions $R^n_{n-1}, \dots, R^n_0$, and we will set $Q^n = R^n_0$.

Remark

In the $\mathbb{Z}^n$ case the $R^n_n, \dots, R^n_1$ become increasingly "fractal."

In going from R^n_{i+1} to R^n_i we add or subtract an interval of $\tilde{R}^i$ from the ends of each interval in R^n_{i+1}. This ensures that the new endpoints of each R^n_i interval are a fraction of d_i away those of each R^i interval.

We assume w.l.o.g. that $d_i \gg \sum_{j<i} d_j$.

- Each Q^n interval is $\sum_{j<i} d_j \ll d_n$ close to an R^n interval.
- For $n > m$, the endpoints of each Q^n interval are d_m far from the endpoints of each R^m, and hence each Q^m interval.

Then for any $x \sim y$, there are only finitely many n such that an endpoint of a Q^n marker region separates x from y (this follows from (2) above).

Thus, $x \sim y$ iff for all large enough n we have $x \sim_{Q^n} y$. This gives a continuous embedding into E_0.

Proof can be extended to handle non-free part of $2^{\mathbb{Z}}$ as well (and likewise for $2^{\mathbb{Z}^n}$.

Question

Does there exists a continuous embedding from $2^{\mathbb{Z}^{<\omega}}$ into E_0? Yes for free part.

Question

How far can these regular marker arguments be extended?

Question

Are there more algebraic, less geometrical, versions of these arguments?

This may be important for extending these arguments further.

A Technical Question

- In the Slaman-Steel (Borel) embedding from $2^{\mathbb{Z}}$ to E_0, Borel marker sets $M_0 \supseteq M_1 \supseteq \cdots$ are constructed such that $\bigcap_n M_n = \emptyset$.
- For the continuous embedding from $2^{\mathbb{Z}}$ to E_0 we use clopen marker sets (on $F(2^{\mathbb{Z}})$) such that $|\bigcap_m M_n \cap [x]| = 0$ or 1 for all $x \in F(2^{\mathbb{Z}})$.

Question

Does there exists a sequence $M_0 \supseteq M_1 \supseteq \cdots$ of relatively clopen marker sets in $F(2^{\mathbb{Z}})$ with $\bigcap_n M_n = \emptyset$?

A Coloring Property

This question led to the formulation of the following property.

Definition

$c\colon G \to \{0,1\}$ is a 2-coloring if

$$\forall s \in G\ \exists T \in G^{<\omega}\ \forall g \in G\ \exists t \in T\ (c(gt) \neq c(gst)).$$

This definition was formulated independently by Pestov (c.f. paper of Glasner and Uspenski).

The following connects the coloring property with the dynamics of the shift action.

Theorem

$x \in 2^G$ is a 2-coloring iff $\overline{[x]} \subseteq F(2^G)$.

Note: Definition formulated independently by Pestov.

Also, the 2-coloring property for G gives a marker compactness property for $F(2^G)$:

Theorem (MCP)

Suppose G has the coloring property. Let $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$ be relatively closed complete sections of $F(2^G)$. Then $\bigcap_n S_n \neq \emptyset$.

Main Theorem

Theorem (Gao, J, Seward)

Every countable group has the 2-coloring property.

Note: Partial results were obtained independently also by Glasner and Uspenski.

Remark

By different arguments first showed the coloring property for abelian, solvable, and free groups, and for every group G with $\mathbb{Z} \trianglelefteq G$.

The proof uses two idea:

- Construct suitable marker regions for the group G.
- Exploit polynomial vs. exponential growth.

We first describe the construction of the marker regions. Recall G is a countable infinite group.

We inductively define marker sets $\Delta_n \subseteq G$ and finite sets $F_n \subseteq G$ (with $1 \in F_n$).

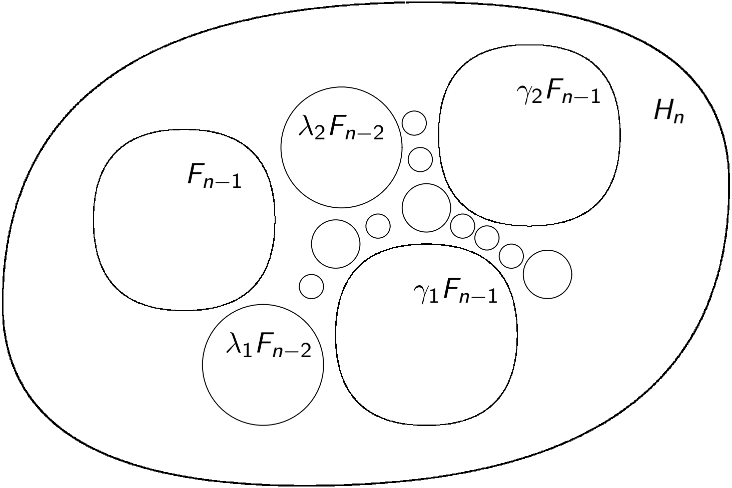
The nth level marker regions will be the translates gF_n for $g \in \Delta_n$.

Will have:

- $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$
- $\boldsymbol{\Delta}_0 \supseteq \boldsymbol{\Delta}_1 \supseteq \boldsymbol{\Delta}_2 \supseteq \cdots$

Each F_n region will be a union of copies of F_i for $i < n$.

F_n will be constructed inside a region H_n.


$\gamma_2 F_{n-1}$
H_n
$\lambda_2 F_{n-2}$
F_{n-1}
$\gamma_1 F_{n-1}$
$\lambda_1 F_{n-2}$

Will maintain two properties:

- (homogeneity) Within any copy γF_n of F_n, the points in $\boldsymbol{\Delta}_k$ $(k \leq n)$ are precisely the translates $\gamma(\boldsymbol{\Delta}_k \cap F_n)$ of the points in F_n.
- (fullness) If a copy δF_k intersects γF_n $(k \leq n)$ then $\delta F_k \subseteq \gamma F_n$.

We label the copies of F_{n-1} inside of F_n by

$$\lambda^n_1 F_{n-1}, \dots, \lambda^n_{s(n)} F_{n-1},$$

$$\lambda^n_{s(n)+1} F_{n-1}, \lambda^n_{s(n)+2} F_{n-1}, \lambda^n_{s(n)+3} F_{n-1}.$$

Each copy of an F_n will have two distinguished points, a_n and b_n.

Will have Marker Identification Property:
(MIP) There is a $A_n \subseteq F_{n-1}$ such that if $c(ga) = c(a)$ for all $a \in A_n$, then $g \in \Delta_n$.

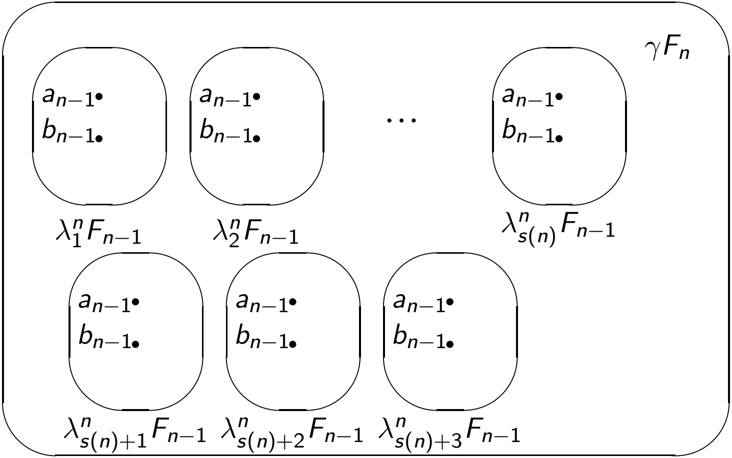


Figure: The labeling of the F_{n-1} copies inside an F_n copy

We define a coloring $c = \bigcup c_n$, which will then be extended to the 2-coloring c'.

c will color all points except those in

$$D = \bigcup_n \mathbf{\Delta}_n \{\lambda_1^n, \dots, \lambda_{s(n)}^n\} b_{n-1}.$$

In extending c_{n-1} to c_n we color the above points except for those in $\mathbf{\Delta}_n \lambda_1^n b_{n-1}, \dots, \mathbf{\Delta}_n \lambda_{s(n)}^n b_{n-1}$, and $\mathbf{\Delta}_n\{a_n, b_n\}$ where:

$$a_n \doteq \lambda_{s(n)+2}^n \, a_{n-1}$$
$$b_n \doteq \lambda_{s(n)+3}^n \, b_{n-1}.$$

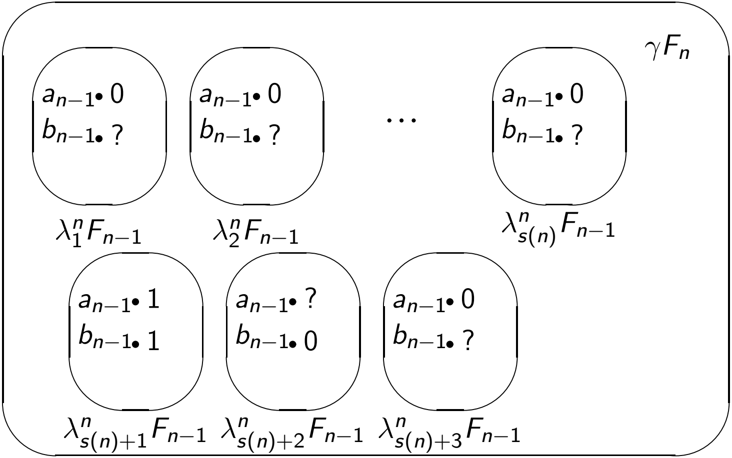


Figure: Extending c_{n-1} to c_n.

We extend c to c' by coloring the points of D so as to get a 2-coloring. Exploit polynomial versus exponential growth.

At stage n we extend c to points of $\mathbf{\Delta}_n\{\lambda_1^n, \dots, \lambda_{s(n)}^n\}b_{n-1}$ to take care of coloring property for $s = g_n \in H_n$.

Let $g \in G$ and consider the pair g, gs. By maximal disjointness of F_n copies, $gf \in \mathbf{\Delta}_n$ for some $f \in F_n F_n^{-1}$. Done unless $gsf \in \mathbf{\Delta}_n$. In this case

$$gsf = gf(f^{-1}sf) \in (gf)F_n F_n^{-1} H_n F_n F_n^{-1}.$$

So there are about $|H_n|^5$ many points to consider, and there $2^{s(n)}$ many "colors" available, where $s(n)$ is linear in $|H_n|$.