Stable Homogeneous Trees and Self-Justifying Systems

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Homogeneous Trees

The notion of homogeneous tree was formulated independently by Kunen and Martin, then isolated and given in its current form by Kechris.

Definition

A measure on a set X is a countably additive ultrafilter on X.

Recall that under AD every ultrafilter on a set X is a measure.

Definition

A tree T is homogeneous if there are measures $\{\mu_s \colon s \in \omega^{<\omega}\}$ on T_s satisfying:

- If $s \sqsubseteq s'$ then $\mu_{s'}$ projects to μ_s.
- (homogeneity property)
  For all $x \in \omega^\omega$ if T_x is illfounded then for any $A_{x \restriction n}$ with $\mu_{x \restriction n}(A_{x \restriction n}) = 1$, there is an f such that $\forall n \, f \restriction n \in A_n$.

Note: This is equivalent to saying that the direct limit of the ultrapowers by the measures $\mu_{x \restriction n}$ is wellfounded.

Fact (ZF, Martin): If $A = p[T]$ where T is homogeneous, then A is determined.

Weakly Homogeneous Trees

Definition (Weakly Homogeneous, strong form)

T is weakly homogeneous if there are measures $\mu_{s,t}$ on T_s defined for a set of (s,t) in a tree, and sets $A_{s,t} \subseteq T_s$ satisfying:

- $\mu_{s,t}(A_{s,t}) = 1$.
- If s' immediately extends s and $A_{s,t}$ is defined, then the $A_{s',t'}$ for t' extending t partition $T_{s'} \cap \{\vec{\alpha} \colon \pi_{s',s}(\vec{\alpha}) \in A_{s,t}\}$.
- If $(s',t') \sqsupseteq (s,t)$ then $\pi_{(s',t'),(s,t)}(\mu_{(s',t')}) = \mu_{(s,t)}$.
- (homogeneity property)
  For any $x, y \in \omega^\omega$, if $\mu_{x\restriction n, y\restriction n}$ is defined for all n and $\exists f\, \forall n\, (x \restriction n, f \restriction n) \in A_{x\restriction n, y\restriction n}$, then for any $B_{x\restriction n, y\restriction n}$ with $\mu_{x\restriction n, y\restriction n}(B_{x\restriction n, y\restriction n}) = 1$ we have that $\exists g\, \forall n\, (x \restriction n, g \restriction n) \in B_{x\restriction n, y\restriction n}$.

Definition (Weakly Homogeneous, weak form)

There are measures $\mu_{s,t}$ with $\mu_{s,t}(T_s) = 1$ satisfying:

- If $(s', t') \sqsupseteq (s, t)$ then $\pi_{(s',t'),(s,t)}(\mu_{s',t'}) = \mu_{s,t}$.
- For any $x \in \omega^\omega$, if T_x is illfounded then $\exists y \in \omega^\omega$ such that $\langle \mu_{x\restriction n, y\restriction n} \rangle$ has wellfounded direct limit: for any $\mu_{x\restriction n, y\restriction n}(B_{x\restriction n, y\restriction n}) = 1$ there is a g such that $\forall n\ (x \restriction n, g \restriction n) \in B_{x\restriction n, y\restriction n}$.

Alternate Version: There is a countable set $\mathcal{M}$ of measures such that whenever T_x is illfounded then there is a sequence $\mu_1, \mu_2, \ldots$ of measures in $\mathcal{M}$ with $\mu_n(T_{x\restriction n}) = 1$ such that $\langle \mu_{x\restriction n} \rangle$ has wellfounded direct limit.

Fact (Woodin)

In ZFC the strong and weak forms of the definition are equivalent.

Fact

The weak form of the definition is enough to make the Martin-Solovay construction work.

Fact

The strong form of the definition s equivalent to saying that there is a homogeneous tree $\bar{T}$ on $\omega \times \omega \times \kappa$ which is isomorphic to T (under a bijection of $\omega \times \kappa$ with κ).

So, a set $A \subseteq \omega^\omega$ is weakly homogeneously Suslin iff it is the projection of a strongly homogeneously Suslin set.

From the third (weakest) form of the definition of weakly homogeneous we can get the second (weak) form easily by restricting to appropriate measure one sets:

Let $\{\mu_n\}$ be as in the third definition. We may assume the sequence of measures is closed under projections. For each $s \in \omega^{<\omega}$, let $\mu_{s,t}$ enumerate those measures in the sequence which satisfy $\mu(T_s) = 1$.

For each s, t let $A_{s,t}$ be such that $\mu_{s,t}(A_{s,t}) = 1$ and $\mu_{s,t'}(A_{s,t}) = 0$ for all $t' \neq t$. We may assume that $A_{s,t} \cap A_{s,t'} = \emptyset$ if $t \neq t'$.

Furthermore, we may assume that $\pi_{s,t}^{s',t'}(A_{s',t'}) \subseteq A_{s,t}$ when (s', t') extends (s, t).

Thinning the tree, we may assume that $T_s = \bigcup_t A_{s,t}$.

A Warm-Up Result

We can also show that the strong and weak forms of the definition are equivalent:

Theorem (ZF + AD)

Let T be a tree on $\omega \times \alpha$ for $\alpha < \Theta$ and assume there is a countable family of measures $\{\mu_1, \mu_2, \dots\}$ which witness the weak homogeneity of T by the third definition. Then there is a $T' \subseteq T$ with $p[T] = p[T]$ which is homogeneous by the strong definition (T' is the restriction of T to certain measure one sets with respect to the μ_n).

The proof of the fact is a warm-up for the main theorem, so we present it.

Fix measure one sets $A_{s,t} \subseteq T_s$ as above and measures $\mu_{s,t}$ on T_s with $A_{s,t} \cap A_{s,t'} = \emptyset$ if $t \neq t'$ and satisfying the weak form of the definition (we assume that $\bigcup_t A_{s,t} = T_s$ w.l.o.g.):

If T_x is ill-founded then there is a $y \in \omega^\omega$ such that the direct limit of the $\mu_{x \restriction n, y \restriction n}$ is well-founded.

We must thin the $A_{s,t}$ to measure one sets $B_{s,t}$ such that whenever T_x is ill-founded then for all y, if $\exists f\, \forall n\, f \restriction n \in B_{x \restriction n, y \restriction n}$ then the direct limit of the $\mu_{x \restriction n, y \restriction n}$ is well-founded.

We use from AD and $\alpha < \Theta$ that there is a fine measure ν on $\mathcal{P}_{\omega_1}(\bigcup_n(\mathcal{P}(\alpha^n)))$.

If $\sigma \in \mathcal{P}_{\omega_1}(\bigcup_n(\mathcal{P}(\alpha^n)))$, let

$$A^\sigma_{s,t} = A_{s,t} \cap \bigcap \{A \in \sigma \colon \mu_{s,t}(A) = 1\}.$$

We claim that for ν almost all σ that the sets $A^\sigma_{s,t}$ will work as the $B_{s,t}$.

Suppose not. Then for ν almost all σ we define a tree V^σ as follows.

Let $T^\sigma \subseteq T$ consist of the $(s, \vec{\alpha}) \in T$ such that $\forall i < \mathrm{lh}(s)\ (s \restriction i, \vec{\alpha} \restriction i) \in A^\sigma_{s\restriction i, t\restriction i}$ (for the unique appropriate t).

$(s, \vec{\alpha}, \vec{\beta}) \in V^\sigma \leftrightarrow (s, \vec{\alpha}) \in T^\sigma$ and for the unique t such that $\mathrm{lh}(t) = \mathrm{lh}(s)$ and $\vec{\alpha} \in \bigcap_{i<\mathrm{lh}(s)} A_{s\restriction i, t\restriction i}$ we have for all $i < \mathrm{lh}(s)$ that:

$$j^{s\restriction i, t\restriction i}_{s\restriction i-1, t\restriction i-1}(\beta_{i-1}) > \beta_i.$$

For almost all σ, let $(x^\sigma, \vec{\alpha}^\sigma, \vec{\beta}^\sigma)$ be the left-most branch of V^σ. Let $y^\sigma \in \omega^\omega$ be the unique real corresponding to $\vec{\alpha}^\sigma$.

By countable additivity we may fix x, y so $x = x^\sigma$, $y = y^\sigma$ for almost all σ.

The sequence $\mu_{x\restriction i, y\restriction i}$ has ill-founded direct limit (consider $\vec{\beta}^\sigma$ for almost all σ).

Fix measure one sets $A'_{x\restriction i, y\restriction i} \subseteq A_{x\restriction i, y\restriction i}$ witnessing the illfoundedness of the direct limit.

Fix σ such that $x = x^\sigma$, $y = y^\sigma$, and the $A'_{x\restriction i, y\restriction i} \in \sigma$.

Then $\vec{\alpha}^\sigma$ is a path through the $A^\sigma_{x\restriction i, y\restriction i} \subseteq A'_{x\restriction i, y\restriction i}$, a contradiction.

Stable Homogeneous Trees

Let $T, \{\mu_s\}$ of $T, \{\mu_{s,t}\}$ be a homogeneous or weakly homogeneous tree.

If T_x is wellfounded, we let $f_x \colon T_x \to \mathrm{On}$ be the ranking function (using the Brouwer-Kleene ordering).

If $\vec{A} = \{A_s\}$ (or $\{A_{s,t}\}$ are measure one sets, we let $T^{\vec{V}}$ be the tree restricted to these measure one sets, that is,

$$(s, \vec{\alpha}) \in T^{\vec{A}} \leftrightarrow (s, \vec{\alpha}) \in T \wedge \forall i < \mathrm{lh}(s)\ (s \restriction i, \vec{\alpha} \restriction i) \in A_{s \restriction i, t \restriction i}$$

(for the unique appropriate t determined by $s, \vec{\alpha}$).

We let $f_x^{\vec{A}}$ denote the ranking function on $T_x^{\vec{A}}$.

We let $(f_x^{\vec{A}})_s$ (or $(f_x^{\vec{A}})_{s,t}$ in the weakly homogeneous case) denote the sub-function on T_s (or $T_{s,t}$) induced by $f_x^{\vec{A}}$. If x (or x, y) are understood, we let $(f_x^{\vec{A}})_n$ denote $(f_x^{\vec{A}})_{s\restriction n}$ (or $(f_x^{\vec{A}})_{s\restriction n, t\restriction n}$).

We let $[f_x^{\vec{A}}]_{\mu_{x\restriction n}}$ denote the ordinal represented by $f_x^{\vec{A}}$ w.r.t. the measure $\mu_{x\restriction n}$ (or $\mu_{x\restriction n,t}$ in the weakly homogeneous case).

Definition

The homogeneous tree (T, μ_s) (or weakly homogeneous tree $(T, \mu_{s,t})$) is stable if there are measure one sets A_s w.r.t. the μ_s (or $A_{s,t}$) such that for any $x \in B = \omega^\omega - p[A]$ and any measure one sets $B_{x\restriction n}$ (w.r.t. $\mu_{x\restriction n}$) we have that

$$[(f_x^{\vec{A}})_n]_{\mu_{x\restriction n}} \leq [(f_x^{\vec{B}})_n]_{\mu_{x\restriction n}}$$

for all n.

[For weak homogeneity we have

$$[(f_x^{\vec{A}})_{x\restriction n,t}]_{\mu_{x\restriction n,t}} \leq [(f_x^{\vec{B}})_{x\restriction n,t}]_{\mu_{x\restriction n,t}}$$

for all n, t].

Main theorem

Theorem (ZF + AD + DC)

Every homogeneous tree $(T, \{\mu_s\})$ on $\alpha < \Theta$ is stable. Likewise, every weakly homogeneous tree $(T, \mu_{s,t})$ on $\alpha < \Theta$ is stable.

Proof uses ideas from the Martin-Woodin proof that all trees are weakly homogeneous.

sketch of proof: Let ν be a fine measure on $\mathcal{P}_{\omega_1}(\cup_n \mathcal{P}(\kappa^n))$.

For $\sigma \in \mathcal{P}_{\omega_1}(\cup_n \mathcal{P}(\kappa^n))$, let $A^\sigma = \{A^\sigma_s\}_s$ where $A^\sigma_s = \cap\{A \in \sigma \colon \mu_s(A) = 1\}$.

We show that for ν almost all σ that the A^σ_s stabilize T. We suppose not.

Uniformly in σ we define a tree U^σ on $\omega \times \lambda^{<\omega} \sim \omega \times \lambda$ where $\lambda = \sup j_{\mu_s}(\kappa)$.

For $x \in B$ the tree U^σ_x attempts to produce ordinals $[f^i_j]_{\mu_{x \restriction i}}$ which witness that the sets A^σ have not yet attained the minimal ranking functions.

They do this, roughly speaking, by describing embeddings of T_x (on measure one sets) into proper initial segments of $T_x^{A^\sigma}$.

We set $((s(0),\dots,s(n-1)),(\beta_0,\dots,\beta_{n-1}))\in U^\sigma$ iff:

1.) $\beta_0 \in \omega$.

2.) Each β_i for $i>0$ codes a finite sequence of integers t_i extending s (roughly, a commitment that later extensions of s must follow t_i) along with a finite sequence of ordinals $(\beta^i_0,\dots,\beta^i_{|t_i|-1})$.

3.) s must be compatible with all t_i for $i<|s|$.

4.) Let f^i_j represent β^i_j with respect to $\mu_{s\restriction i}$. Then there are $\mu_{s\restriction i}$ measure one sets restricted to which the maps $(\eta_0,\dots\eta_{i-1})\mapsto(f^i_0(\vec{\eta}),\dots,f^i_{|t_i|-1})$ give an order-preserving from the BK ordering on T_{s_n} to the BK ordering on T^{A^σ}.

5.) For $i < \beta_0$, each f^i_j is almost everywhere the identity function, and for $i = \beta_0$ we have that almost everywhere that $(f^i_0(\vec{\eta}), \ldots, f^i_{|t_i|-1}(\vec{\eta}))$ is a properly less than $\vec{\eta}$ in $T_{|t_i|}$.

6.) We weave in the Martin-Solovay tree into U^σ as well. Say at every even level of U^σ we put in ordinals from the Martin-Solovay tree (so a branch through $(U^\sigma)_x$ also gives a branch through $(T')_x$ and so proves $x \in B$).

For any x, U^σ_x is illfounded iff $x \in B$ and there are measure one sets $B_{x\restriction n} \subseteq A^\sigma_{x\restriction n}$ (with respect to $\mu_{x\restriction n}$) such that for some m, $[f^{\vec{B}}_x]_{\mu_{x\restriction m}} < [f^{A^\sigma}_x]_{\mu_{x\restriction m}}$.

So, for ν almost all σ there is a leftmost branch $(x^\sigma, \vec{\beta}^\sigma)$ through U^σ. By countable additivity we may fix $x = x^\sigma$ on a measure one set.

Fix any σ where $x^\sigma = x$ and also σ contains measure one sets $C_{x\restriction n}$ (w.r.t. the $\mu_{x\restriction n}$) which attain the minimal values for $[f^{\vec{C}}_x]_{\mu_{x\restriction n}}$ (which we can do by countable choice).

The branch $\vec{\beta}^\sigma$ gives a proper embedding from $T^{\vec{D}}_x$ into $T^{A^\sigma}_x \subseteq T^{\vec{C}}_x$ for some measure one sets $D_{x\restriction n}$. This contradicts the choice of the $C_{x\restriction n}$.

For $(T, \{\mu_{s,t}\})$ weakly homogeneous the proof is essentially the same.

We put now $(s, \vec{\beta}) \in U^\sigma$ provided the following hold.

Now β_0 codes a sequence $t_0 \in \omega^{<\omega}$. Each β_i, $i > 0$, codes a finite sequence of tuples $(t^i_j, u^i_j, v^i_j, \beta^i_j)$ where $\mathrm{lh}(t^i_j) = i \leq n$, $\mathrm{lh}(u^i_j) = \mathrm{lh}(v^i_j)$.

s must be consistent with all the u^i_j. $\beta^i_j = [f^i_j]_{\mu_{s\restriction i, t^i_j}}$ $(\mathrm{lh}(t^i_j) = i)$. The f^i_j must almost everywhere give an order-preserving map from T_s to T^{A^σ}. For $t < t_0$, we have $u^i_j = t^i_j$ and f^i_j is the identity a.e.

Here we use $(s \restriction i, t^i_j, (\eta_0, \dots, \eta_{i-1})) \mapsto ((u^i_j, v^i_j, (\cdots f^i_j(\vec{\eta}))).$

Martin-Solovay Tree

We apply our main theorem to the Martin-Solovay tree. Recall the definition.

Let $(T, \{\mu_{s,t}\})$ be a weakly homogeneous tree with $A = p[T]$. The Martin-Solovay tree $T' = T'(T, \{\mu_{s,t}\}, \lambda)$ is a tree with $p[T'] = \omega^\omega - A = p[T']$ defined as follows.

$(s, \vec{\alpha}) \in T'$ iff $\exists f \colon T_s \to \lambda$ order-preserving w.r.t. the BK ordering on T_s such that $\forall i < \mathrm{lh}(s)$

$$\alpha_i = [f^i]_{\mu_{s_i,t_i}}$$

where (s_i, t_i) is a reasonable enumeration of $\omega^{<\omega} \times \omega^{<\omega}$ and $f^i = f \restriction \{\vec{\alpha} \colon |\vec{\alpha}| = i \wedge (s_i, t_i, \vec{\alpha}) \in T\}$ if $s_i \sqsubseteq s$, and $f = 0$ otherwise.

For $x \in A'$, let

$$\varphi_i(x) = [(f_x)_{s_i,t_i}]_{\mu_{s_i,t_i}}$$

(recall (s_i, t_i) enumerates the pairs of equal length).

Recall f_x is the ranking function on T_x and $(f_x)_{s_i,t_t}$ is the appropriate subfunction of f_x.

The Martin-Solovay theorem says that $\{\varphi_i\}$ is a semi-scale on A'.

We can extract a scale from the Martin-Solovay tree T' by using left=most branches, but this is nor good enough for later purposes.

Two Questions About The Martin-Solovay Tree

Question

When is the semi-scale $\{\varphi_i\}$ from the Martin-Solovay tree a scale?

Question

When is the Martin-Solovay tree the tree of a scale?

For the first question, our theorem gives an answer.

Theorem

If (T, μ_s) is a stable homogeneous (or weakly homogeneous) tree, then the semi-scale $\{\varphi_i\}$ from the Martin-Solovay tree T' constructed from $T^{\vec{A}}$ is a scale. Here

Proof Suppose $x_m \in A'$ and $x_m \to x$. Suppose $\varphi_n(x_m)$ is eventually equal to λ_n for each n. Say $(\varphi_0(x_m), \dots, \varphi_m(x_m)) = (\lambda_0, \dots, \lambda_m)$.

For each m, the ranking functions f_{x_m} satisfy $[(f_{x_m})_{s_i,t_i}]_{\mu_{s_i,t_i}} = \lambda_i$ for all $i \leq m$.

As in the Martin-Solovay proof this gives measure one sets $\vec{A} = A_{x \restriction n,t}$ on which the f_{x_m} together are order-preserving on T_x.

From the stability property we know that

$$\varphi_n(x) = [(f_x)_{s_n,t_n}]_{\mu_{s_n,t_n}} = [(f_x^{\vec{A}})_{s_n,t_n}]_{\mu_{s_n,t_n}} \leq [(f_{x_n}^{\vec{A}})_{s_n,t_n}]_{\mu_{s_n,t_n}} = \lambda_n.$$

□

As for the second question: If T' is the tree of a scale $\{\psi_m\}$ then $\{\psi_n\}$ is given by the left-most branch of T'.

Our arguments show that the left-most branch of T' is given by $\{\varphi_n\}$ (assuming T has been stabilized), so $\vec{\psi} = \vec{\varphi}$.

So, this is equivalent to asking if for every $(s, \vec{\alpha}) \in T'$ there is an $x \in A'$ extending s with $\vec{\varphi}(x) \restriction \text{lh}(s) = \vec{\alpha}$.

Not much appears to be known about this.

Recall $\mathbf{\Gamma}$ has the scale property if every $A \in \mathbf{\Gamma}$ admits a scale $\{\varphi_n\}$ all of whose norms are in $\mathbf{\Gamma}$.

Definition

We say a pointclass Λ closed under $\neg$ is self-justifying if every $A \in \Lambda$ admits a scale all of whose norms are in Λ.

We recall some of the scale/Suslin cardinal analysis assuming AD.

Let κ denote a limit Suslin cardinal (i.e., κ is a limit of Suslin cardinals).

Let $\boldsymbol{\Delta} = \{A : o(A) < \kappa\}$, so $\boldsymbol{\Delta}$ is closed under $\neg$, $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$.

We have the following facts from AD.

Type I.) $\mathrm{cof}(\kappa) = \omega$.
Let $\boldsymbol{\Sigma}_0 = \bigcup_\omega \boldsymbol{\Delta}$. Then scale($\boldsymbol{\Sigma}_0$), scale($\boldsymbol{\Pi}_1$), scale($\boldsymbol{\Sigma}_2$), ... The next Suslin cardinals are κ, κ^+, λ_3, $\delta_3 = (\lambda_3)^+$, ... Also, $S(\kappa) = \boldsymbol{\Sigma}_1$, $S(\kappa^+) = \boldsymbol{\Sigma}_2$, $S(\lambda_3) = \boldsymbol{\Sigma}_3$, etc.

Types II, III.) $\mathrm{cof}(\kappa) > \omega$, $\boldsymbol{\Gamma}$ not closed under real quantifiers. Let $\boldsymbol{\Pi}_{-1}$ be the Steel pointclass ($o(\boldsymbol{\Gamma}) = \kappa$), so $\boldsymbol{\Pi}_{-1}$ is closed under $\forall^{\omega^\omega}$ and $\mathrm{pwo}(\boldsymbol{\Pi}_{-1})$. Then $\mathrm{scale}(\boldsymbol{\Pi}_{-1})$, $\mathrm{scale}(\boldsymbol{\Sigma}_0)$, $\mathrm{scale}(\boldsymbol{\Pi}_1), \dots$. The next Suslin cardinals are $\kappa, \lambda_1, \delta 1, \lambda_3, \delta_3, \dots$. We have $S(\kappa) = \boldsymbol{\Sigma}_0$, $S(\lambda_1) = \boldsymbol{\Sigma}_1$, $S(\kappa_1) = \boldsymbol{\Sigma}_2, \dots$.

Type IV.) Let $\boldsymbol{\Gamma}$ be closed under quantifiers with $o(\boldsymbol{\Gamma}) = \kappa$ and $\mathrm{pwo}(\boldsymbol{\Gamma})$.

Then κ is regular and $\mathrm{scale}(\boldsymbol{\Gamma})$. We need to construct the next Suslin cardinal.

Recall we have several equivalent definitions of the envelope $\Lambda = \text{Env}(\mathbf{\Gamma}, \kappa)$, which constructs the next Suslin cardinal.

Definition (Martin)

Let $\mathcal{A} = \{A_\alpha\}_{\alpha<\alpha_0}$, each $A_\alpha \subseteq \omega^\omega$. Then $\overline{\mathcal{A}}$ is the set of $A \subseteq \omega^\omega$ such that for all countable $S \subseteq \omega^\omega$, there is an $\alpha < \alpha_0$ such that $S \cap A = S \cap A_\alpha$.
We let

$$\Lambda(\mathbf{\Gamma}, \kappa) = \{\overline{\mathcal{A}} : \mathcal{A} \subseteq \mathbf{\Gamma} \wedge \|\mathcal{A}\| \leq \kappa\}.$$

Facts about the Envelope

Assume $\mathbf{\Gamma}$ closed under quantifiers, $\kappa = o(\mathbf{\Delta})$. Let $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa)$.

- $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa) = \Lambda(\mathbf{\Delta}, \kappa)$.
- There is a single "universal" $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ such that every $A \in \Lambda$ is Wadge reducible to a set in $\overline{\mathcal{A}}$.
- There is a $\check{\mathbf{\Gamma}}$ set with no uniformization in Λ. In particular, a $\check{\mathbf{\Gamma}}$-complete set cannot have a scale all of whose norms are reducible to a fixed set $B \in \Lambda$.
- Every measure on κ has a code set reducible to a set in Λ.

Other characterizations of ENV

Steel gave an alternate definition of the envelope.

Fact

Let $\boldsymbol{\Gamma}$ be type IV and $\kappa = o(\boldsymbol{\Gamma})$. Then the following are equivalent:

- $A \in Env(\boldsymbol{\Gamma})$.
- *There is a $z_0 \in \omega^\omega$ s.t. for all countable σ containing z_0 we have $A \cap \sigma \in L(T, \sigma)$.*
- *There is a $z_0 \in \omega^\omega$ s.t. for all countable σ containing z_0 we have $A \cap \sigma$ is ordinal definable in $L(T, \sigma)$ from T, σ.*

Let A be a $\mathbf{\Gamma}$-complete set, and assume κ is not the largest Suslin cardinal.

Then from scale($\mathbf{\Gamma}$) and the Martin-Woodin theorem (every tree on an α below a Suslin cardinal is weakly homogeneous) we have that $A = p[T]$ where T is a weakly homogeneous tree on $\omega \times \kappa$ (in fact, by Martin-Steel we can take T to be homogeneous).

By our main theorem we may assume that T is stable. We assume w.l.o.g. that for $(s, \vec{\alpha}) \in T$ that $\alpha_0 \geq \max(\vec{\alpha})$.

Let $A' = \omega^\omega - A$, and T' the Martin-Solovay tree with $A' = p[T']$. Let $\{\varphi_n\}$ be the semi-scale from the homogeneous tree construction so, as T is stable, $\vec{\varphi}$ is actually a scale on A'.

From the previous facts, each norm φ_n is in Λ:

$$x <_n^* y \leftrightarrow T_x \text{ is wellfounded } \wedge \forall^*_{\mu_{s_n,t_n}} \vec{\alpha}\, \exists \beta < \kappa\, [|T_x(\vec{\alpha}) \leq \beta \wedge \neg(|T_y(\vec{\alpha}) \leq \beta)].$$

Lat $\lambda = o(\Lambda) =$ the supremum of the Λ prewellorderings.

So, every $\check{\Gamma}$ sets admits a scale with all norms of length less than λ. From this and the previous facts it follows that λ is the next Suslin cardinal after κ.

It follows that Λ is self-justifying:

Theorem

With $\boldsymbol{\Gamma}$, Λ as above, every $B \in \Lambda$ admits a scale all of whose norms lie in Λ.

Proof Fix a $\check{\boldsymbol{\Gamma}}$-complete set A' and a scale $\{\varphi_n\}$ on A' with all norm relations in Λ. Fix $B \in \Lambda$, and let $\gamma = o(B) < \lambda$. Fix a non-selfdual pointclass $\boldsymbol{\Gamma}_0 \subseteq \Lambda$ which is closed under $\exists^{\omega^\omega}$, $\forall^\omega$, pwo($\boldsymbol{\Gamma}_0$), and with $B \in \boldsymbol{\Delta}_0$.

So, $\boldsymbol{\Gamma}_0$ is closed under wellordered unions.

Fix $\delta >$ the supremum of the $\mathbf{\Gamma}_0$ prewellorderings. Fix n so that $|\varphi_n| > \delta$.

For each $\alpha < \delta$ let $A^n_\alpha = \{x \in A' \colon \varphi_n(x) \leq \alpha\}$.

We can't have that $o(A^n_\alpha) \leq \gamma$ for all $\alpha < \delta$ as this would give a δ increasing sequence of sets in $\mathbf{\Delta}_0$, and hence a $\mathbf{\Gamma}_0$ prewellodering of length δ, a contradiction.

Fix $\alpha < \delta$ so that $B \leq_W A^n_\alpha$.

Let $\{\psi_n\}$ be the restriction of $\{\varphi_n\}$ to A^n_α.

$\{\psi_n\}$ is easily a scale on A^n_α, and each ψ_n is a Λ-norm (Boolean combination of A^n_α with the norm relations for φ_n).

Since $B \leq_W A^n_\alpha$ it follows that B also has a scale all of whose norms lie in Λ.

□