Complexity of a dynamics ideal

S. Jackson

Department of Mathematics
University of North Texas

March 2013
University of Mississippi

Let $f_i\colon \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \dots, k$, be contraction mappings, that is $\|f_i(x) - f_i(y)\| \leq c\|x - y\|$ where $c < 1$.

There is a unique compact set $K = K(f_1, \dots, f_n)$ such that $K = f_1(K) \cup \cdots f_k(K)$.

We say $\vec{f}$ satisfies the porosity condition if there is a $p > 0$ such that for all balls B of radius r there is a sub-ball $B' \subseteq B$ of radius p such that $B' \cap K = \emptyset$.

If we fix a ball $B(r)$ and a $c < 1$, then the set of $\vec{f}$ which are contractions of $B(r)$ with constant c and is a Polish space $F(r, c)$, and the set of $\vec{f}$ satisfying the porosity condition is a closed subset of $F(r, c)$.

We let $X = X^n$ denote the Polish space of closed subsets of $\mathbb{R}^n$. We let *IFS* denote the $K \in X$ of the form $K = K(\vec{f})$ for some iterated function system $\vec{f}$ satisfying the porosity condition.

We let I_{ifs} denote the σ-ideal generated by the sets in *IFS*. So, $I_{\text{ifs}} \subseteq X$ is a σ-ideal of closed subsets of $\mathbb{R}^n$.

We recall the following dichotomy theorem of Kechris and Woodin:

Theorem (Kechris,Woodin)

Let $\mathcal{I}$ be a $\mathbf{\Pi}^1_1$ σ-ideal of closed sets in a compact metric space. Then either $\mathcal{I}$ is $\mathbf{\Pi}^0_2$ or $\mathcal{I}$ is $\mathbf{\Pi}^1_1$-complete.

Every $F \in I_{\text{ifs}}$ is meager. However the meager ideal is $\mathbf{\Pi}^0_2$.

Theorem

I_{ifs} is $\mathbf{\Pi}^1_1$-complete.

A straightforward computation shows that I_{ifs} is $\mathbf{\Sigma}^1_2$.

If the $\vec{f}$ are similarities (i.e., $\rho(f_i(x), f_i(y)) = c\rho(x, y)$) of a compact set K, the $f_i(K)$ are pairwise disjoint, and $K - \bigcup_i f_i(K) \neq \emptyset$, then the porosity condition holds.

More generally, Let $\mathcal{P} = \{p_\alpha\}_{\alpha<\lambda}$ be a family of porosity functions, $p_\alpha \colon \omega \to \omega$.

Let $I^{\mathcal{P}}_{\text{ifs}}$ be the σ-ideal generated by the $K(\vec{f})$ which have a porosity function dominated by a $p_\alpha \in \mathcal{P}$.

Theorem

If $|\mathcal{P}| < \mathfrak{b}$ = bounding number of the continuum, then $I^{\mathcal{P}}_{ifs}$ is $\mathbf{\Pi}^1_1$ hard. If $\mathcal{P}$ is countable then $I^{\mathcal{P}}_{ifs}$ is $\mathbf{\Pi}^1_1$-complete.

Corollary

The ideal I^{C}_{ifs} of all $K(\vec{f})$ which have a computable porosity function is $\mathbf{\Pi}^1_1$-complete.

So we have:

- If $|\mathcal{P}| < \mathfrak{b}$, then $I_{\text{ifs}}^{\mathcal{P}}$ is $\mathbf{\Pi}_1^1$ hard.
- If $\mathcal{P}$ is countable then $I_{\text{ifs}}^{\mathcal{P}}$ is $\mathbf{\Pi}_1^1$-complete.
- If $\mathcal{P} = \omega^\omega$ then $I_{\text{ifs}}^{\mathcal{P}}$ is $\mathbf{\Pi}_2^0$.

Question

Are there any Δ_1^1 $\mathcal{P}$ for which $I_{\text{ifs}}^{\mathcal{P}}$ is $\mathbf{\Sigma}_2^1$-complete?

Question

For which Δ_1^1 $\mathcal{P}$ is $I_{\text{ifs}}^{\mathcal{P}}$ $\mathbf{\Pi}_2^0$, $\mathbf{\Pi}_1^1$, $\mathbf{\Sigma}_2^1$-complete?

We can extend the previous result a bit.

Theorem

Let $\mathcal{P}$ be be given and assume there is a $g \colon \omega \to \omega$ such that

$$\forall p \in \mathcal{P}\ \exists^\infty n\ \left(\frac{p(n+1)}{p(n)} < g(n)\right).$$

Then $I^{\mathcal{P}}_{ifs}$ is $\mathbf{\Pi}^1_1$-hard.

Corollary

There is an uncountable Borel $\mathcal{P}$ such that $I^{\mathcal{P}}_{\text{ifs}}$ is $\mathbf{\Pi}^1_1$-hard.

We first show that I_{ifs} is $\mathbf{\Pi}^1_1$-hard.

For this direction, it is enough to consider the case $n = 1$ (so $f_i \colon \mathbb{R} \to \mathbb{R}$).

We fix a reasonable homeomorphism h between ω^ω and $\text{Irr} \subseteq [0, 1]$.

Let $W \subseteq \omega^\omega$ be the set of wellfounded trees, so W is $\mathbf{\Pi}^1_1$-complete. (W is a subset of the Polish space Tr of trees on ω).

We define a continuous map $\pi \colon T \to K([0, 1])$ reducing W to I_{ifs}.

If $T \in W$ then $\pi(T)$ will be countable, and hence in I_{ifs}. If $T \notin W$ then we will have $\pi(T) = \overline{\pi_0(T)}$, where $\pi_0(T) \subseteq \text{Irr} \cap [0,1]$is a closed subset of Irr.

Note that if A is a closed subset of ω^ω, then $A \in I_{\text{ifs}}$ iff $\overline{A} \in I_{\text{ifs}}$.

So, we define $\pi_0 \colon Tr \to F(\omega^\omega) \approx F(\text{Irr})$.

Fix $T \in Tr$ and we define $U = \pi_0(T)$, a tree on ω.

The idea is to diagonalize out of all possible iterated function systems.

Fix a function $b \colon \mathbb{Q} \times \omega^{<\omega} \to \omega$ such that for rational porosity number p and $s \in \omega^{<\omega}$, we have that for any iterated function system $\vec{f}$ with porosity $\geq p$ there is a sequence $t \in \omega^{<\omega}$ of length $\leq b(p,s)$ such that $h(s^\frown t) \cap K(\vec{f}) = \emptyset$.

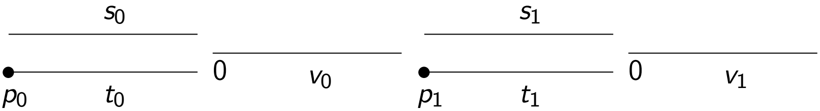
This uses the fact that h is reasonable. Actually, $b(p,s)$ depends only on p and $|s|$.

Given $u = (u(0), \dots, u(n))$, we will decode u into a sequence of finite sequences $u = ((s_0, t_0), v_0, (s_1, t_1), v_1, \dots)$ as follows. First write $u = u_0{}^\frown v_0{}^\frown u_1{}^\frown v_1, \dots$ where $v_0(0)$ is the first occurrence > 0 (if it exists) of a 0 in u, $u_0 = \langle s_0, t_0 \rangle$, $|v_0| = b(p, u_0)$, and $p = \frac{1}{t_0(0)}$.

Let $v_{i+1}(0)$ be the least occurrence (if it exists) of a 0 in u after $|u_0| + |v_0| + \cdots + |u_i| + |v_i| + 1$. Let $|v_{i+1}| = b(p, u_0{}^\frown v_0{}^\frown \cdots {}^\frown u_{i+1})$ where $p = \frac{1}{t_{i+1}(0)}$.

Let $s(u) = s_0{}^\frown s_1{}^\frown \cdots$ be the concatenation of the s_i sequences that u determines. Similarly define $t(u) = t_0{}^\frown t_1{}^\frown \cdots$.

Note that for any $u \in \omega^\omega$, the $s(u \restriction n)$ converge to an $s \in \omega^\omega$. This is because the v_i have bounded size (given by the b function), and each u_i has length at least 1. Similarly we get $t \in \omega^\omega$.



- s_i build a branch through T.
- t_i build an IFS.
- v_i get out of $K(\vec{f})$.

Given $u \in \omega^{<\omega}$ (and the tree T) we now describe the conditions which determine if $u \in U$.

For u to be in U we first require that $s(u) = s_0{}^\frown s_1{}^\frown \cdots \in T$.

This alone gives that $T \in W$ implies $\pi(T) \in I_{\text{ifs}}$ (as $\pi(T) \subseteq \mathbb{Q} \cap [0,1]$).

Let t_i' be the subsequence of $t(u) = t_){}^\frown t_1{}^\frown \cdots$ given by $t_i' = \bar{t}_{\langle i,0\rangle}{}^\frown \bar{t}_{\langle i,1\rangle}{}^\frown \cdots$, where the bar means to drop the first digit (which encodes the porosity number).

For each n fix a continuous surjection h_n from ω^ω to the Polish space of iterated function systems on $[0,1]$ with porosity $\geq \frac{1}{n}$.

For each i such that $t'_i \neq \emptyset$, $h_{1/t_i(0)}(t'_i)$ is consistent with being an iterated function system $\vec{f}$ such that $K(\vec{f}) \cap N_{u_0 \frown v_0 \frown \cdots \frown v_i} = \emptyset$.

That is, there is a $z \in \omega^\omega$ extending t'_i such that $h_{1/t_i(0)}(z)$ is an i.f.s. $\vec{f}$ as above.

This completes the second requirement on the sequence u for being in the tree U.

We clearly have that U is obtained continuously from T, and so the map π from Tr to $K([0,1])$ is also continuous.

As mentioned above, if $T \in W$ then U is also wellfounded and so $\pi(T) \subseteq \mathbb{Q}$ is in I_{ifs}.

Suppose now that T is illfounded. Fix $x \in [T]$. Let $K = \pi(T)$. Suppose $\{\vec{f}_n\}$ were a sequence of iterated function systems such that $K \subseteq \bigcup_n K(\vec{f}_n)$.

We define a particular $z \in [U]$ such that $h(z)$ (an element of K) is not in $\bigcup_n K(\vec{f}_n)$, a contradiction.

We require that $s(z) = x$, that is, the sequences $s_0, s_1, \dots$ defined by z build up x (here the $z \restriction k$ give the sequences u_k which determine the s_i, t_i, v_i).

We require that the t_i be such that $h_{\frac{1}{t_i(0)}}(t_i') = \vec{f}_i$. We also require that $t_i(0)$ be such that $\vec{f}_i$ has porosity constant $\geq \frac{1}{t_i(0)}$.

It remains to determine the lengths of the $u_i = \langle s_i, t_t \rangle$ and the sequences v_i (the lengths of the v_i are fixed by the b function and the earlier part of z).

We let the length of u_i be arbitrary with $|u_i| \geq 2$ (to code the porosity number for $\vec{f}_i$ and at least one more digit of x and the appropriate t_j'.

To get v_i, first we must take $v_i(0) = 0$ by definition of the v_i.

Let $w = u_0{}^\frown v_0{}^\frown \cdots {}^\frown u_i$ be the sequence we have determmined so far. $h(N_w)$ is a basic open set in Irr.

From the porosity condition for $\vec{f_i}$ (which has porosity constant $p_i \geq \frac{1}{t_i(0)}$), there is a sequence v_i of length $b(\frac{1}{t_i(0)}, w)$ such that $h(w{}^\frown v_i) \cap K(\vec{f_i}) = \emptyset$.

This completes the definition of z. We easily have that $z \in [U]$ since the $h_{\frac{1}{t_i(0)}}(t_i')$ are consistent with missing $N_{u_0{}^\frown v_0{}^\frown \cdots {}^\frown v_i}$.

So, $h(z) \in K$ but $h(z) \notin K(\vec{f_i})$ for all i, a contradiction.

We next show that $F \in I_{\text{ifs}}$ is $\boldsymbol{\Pi}^1_1$.

We may assume that $F \in B = B(\vec{0}, 1)$.

Suppose $F \in I_{\text{ifs}}$, say $F \subseteq \bigcup_n K_n$ where $K_n = K(\vec{f}_n)$.

By Baire category, there is an n such that $V_n \cap F \neq \emptyset$ and $V_n \cap F \subseteq K_m$ for some m.

For a fixed contraction constant $c < 1$, porosity constant $p > 0$, the space of i.f.s. $\vec{f}$ on B with constants c, p is compact.

Let $R(F, n, c, p, \vec{f})$ iff $(c(\vec{f}) \leq c) \wedge (p(\vec{f}) \geq p) \wedge (F \cap \overline{V_n} \subseteq K(\vec{f}))$.

R is a Δ^1_1 relation with compact sections.

As n, c, p are basically integers, if $F \cap V_n \subseteq K(\vec{f})$ for some f, then there is a $\vec{f} \in \Delta^1_1(F)$ witnessing this.

Let $R'(F, n, c, p, a)$ iff $[a \in \omega$ codes $\vec{f}_a \in \Delta^1_1(F) \wedge (R, F, n, c, p, \vec{f}_a)]$. So $R' \in \Pi^1_1$.

Let $\Psi\colon F \mapsto (n_F, c_F, p_F, a_F)$ be a Π^1_1 uniformization for R' (so the map is Δ^1_1 on $F \in I_{\text{ifs}}$).

For $F \in I_{\text{ifs}}$ we define the sequence of derivatives F_α by removing $n(F_\alpha)$ to get F_{α_1} (and take intersections at limit stages).

An effective transfinite recursion now gives a Δ^1_1 sequence $\vec{f}_n$ such that $F \subseteq \bigcup_n K(\vec{f}_n)$.

Set $(n, c, p, a) \prec (n', c', p', a')$ iff $\exists x \in \mathrm{LO}$ satisfying:

(1) $\langle n_F, c_F, p_F, a_F \rangle$ is the $\prec$-minimal element of $<_x$.

(2) for every $m \in \mathrm{dom}(<_x)$, $F_m = F - \bigcup\{V_n \colon \langle n, \dots \rangle <_x m\} \neq \emptyset$ and $m = \langle n(F_m), c(F_m), p(F_m), a(F_m) \rangle$.

(3) $(n, c, p, a) <_x (n', c', p', a')$.

By bounded quatification it follows that $I_{\mathrm{ifs}} \in \mathbf{\Pi}^1_1$.