Disintegration of σ-finite measures

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In 1950 Maharam asked whether every disintegration of a σ-finite measure is necessarily uniformly σ-finite.

The answer has been shown to be yes under special conditions on the disintegration.

We show that it is consistent that the answer is no. Specifically, we show the answer is no assuming $V = L$.

We do this by constructing, assuming $V = L$, a $\mathbf{\Pi}^1_1$ set in the plane with certain special properties.

Let $(X, \mathcal{B}(X))$, $(Y, \mathcal{B}(Y))$ be standard Borel spaces with their σ-algebras of Borel sets. Let $\varphi\colon Y \to X$ be measurable and μ, ν be measures on $\mathcal{B}(Y)$, $\mathcal{B}(X)$ respectively.

Definition

A disintegration of μ with respect to (ν, ϕ) is a family, $\{\mu_x : x \in X\}$, of measures on $(Y, \mathcal{B}(Y))$ satisfying:

1. $\forall B \in \mathcal{B}(Y),\ x \mapsto \mu_x(B)$ is $\mathcal{B}(X)$-measurable
2. $\forall x \in X,\ \mu_x(Y \setminus \phi^{-1}(x)) = 0$ and
3. $\forall B \in \mathcal{B}(Y),\ \mu(B) = \int \mu_x(B) d\nu(x)$.

Remark

The collection $\{\mu_x\colon x \in X\}$ is sometimes called a transition kernel.

Note that if we have a disintegration $\{\mu_x : x \in X\}$ of μ with respect to (ν, φ), then $\mu \circ \varphi^{-1} \ll \nu$.

The converse is also true according to the following theorem.

Theorem

Suppose $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ are standard Borel spaces, μ is a σ-finite measure on $\mathcal{B}(Y)$, ν is a σ-finite measure on $\mathcal{B}(X)$, and $\phi : Y \to X$ is a Borel measurable function. If $\mu \circ \phi^{-1} \ll \nu$ then there exists a σ-finite disintegration $\{\mu_x : x \in X\}$ of μ with respect to (ν, ϕ). Moreover this disintegration is unique in the sense that if $\{\hat{\mu}_x : x \in X\}$ is any σ-finite disintegration of μ with respect to (ν, ϕ), then there exists $N \subseteq X$ such that $\nu(N) = 0$ and $\forall x \notin N \; \mu_x = \hat{\mu}_x$.

We say the disintegration $\{\mu_x : x \in X\}$ is σ-finite if each μ_x is σ-finite.

We say the disintegration $\{\mu_x : x \in X\}$ is uniformly σ-finite if there exists a sequence, (B_n), from $\mathcal{B}(Y)$ such that

1. $\forall n \in \mathbb{N}\ \forall x \in X,\ \mu_x(B_n) < \infty$ and
2. $\forall x \in X,\ \mu_x(Y \setminus \bigcup_n B_n) = 0.$

Problem(Maharam)
Let $\{\mu_x : x \in X\}$ be a σ-finite disintegration of μ with respect to (ν, ϕ). Is this disintegration uniformly σ-finite?

Every σ-finite disintegration $\{\mu_x \colon x \in X\}$ of the σ-finite measure μ on Y with respect to (ν, φ) is "almost-everywhere" uniformly σ-finite in the following sense.

Fact

Suppose $\{\mu_x : x \in X\}$ is a σ-finite disintegration of the σ-finite measure μ with respect to (ν, ϕ). There exists a uniformly σ-finite disintegration $\{\hat{\mu}_x \colon x \in X\}$ of μ with respect to (ν, ϕ) such that $\mu_x = \hat{\mu}_x$ for ν-almost every $x \in X$.

Proof. Let $B_n \in \mathcal{B}(Y)$ with $Y = \bigcup_n B_n$ and $\mu(B_n) < \infty$ for all n.

Let $E = \bigcap_n \{x \in X \colon \mu_x(B_n) < \infty\}$. So, $\nu(X - E) = 0$.

Let $B'_n = B_n \cap \varphi^{-1}(E)$, so $\mu(B_n - B'_n) = 0$.

There is $E' \subseteq E$ with $\nu(Y - E') = 0$ such that for $x \in E'$, $\mu_x(Y - \bigcup_n B_n) = 0$.

Let

$$\hat{\mu}_x = \begin{cases} \mu_x & \text{if } x \in E' \\ 0 & \text{if } x \notin E' \end{cases}$$

So, Maharam's problem concerns the behavior on measure zero sets.

Proof assuming special set

We give the proof of the theorem assuming the existence of a special $\mathbf{\Pi}^1_1$ set in the plane. We assume henceforth that $X = Y = \omega^\omega$.

Let P be a closed subset of $X \times Y$ such that $\forall x \in X,\ P_x$ is nonempty and perfect and if $x \neq x'$, $P_x \cap P_{x'} = \emptyset$. We say G is a special coanalytic set for P provided $G \subseteq P$ is a $\mathbf{\Pi}^1_1$ set with the following properties:

1. $\forall x \in X\ |G_x| = \omega_0$,
2. G is not the union of countably many $\mathbf{\Pi}^1_1$ graphs over X,
3. For every $n \in \omega$ and for every $B \in \mathcal{B}(Y)$, $\{x \in X : |B \cap G_x| = n\} \in \mathcal{B}(X)$.
4. There exists a Borel set $H \subseteq X$ with such that $G \cap (H \times Y)$ is the union of countably many pairwise disjoint Borel graphs over H.

Proof assuming special set

Remark

Properties (1)-(3) are the main properties, (4) is for convenience. A set satisfying (1) and (2) can be constructed in ZFC.

Let $P = \{(x, y) \colon \ (y)_0 = x\}$.

Assume there is a special set $G \subseteq \omega^\omega \times \omega^\omega$ for P.

Let $\varphi \colon Y \to X$ be defined by $\varphi(y) = (y)_0$.

For $B \in \mathcal{B}(Y)$ and $x \in X$, define $\mu_x(B) = |B \cap G_x|$, that is, counting measure on G_x. From (1) we have that each μ_x is σ-finite.

Since $G \subseteq P$ we have that $\mu_x(Y - \varphi^{-1}(\{x\})) = 0$.

Let $B \in \mathcal{B}(Y)$. Then

$$\{x \in X \colon \mu_x(B) \geq n\} = \{x \in X \colon |B \cap G_x| \geq n\} \in \mathcal{B}(X)$$

as G is special.

Thus, $x \mapsto \mu_x(B)$ is $\mathcal{B}(X)$ measurable, so $\{\mu_x \colon x \in X\}$ is a transition kernel.

From (4) of special we have a Borel $H \subseteq X$ and Borel functions $f_n \colon X \to Y$ such that for $x \in H$ we have $G_x = \bigcup_n \{f_n(x)\}$. Each f_n is one-to-one over H.

Let ν be a Borel probability measure on X with $\nu(H) = 1$.

Define μ on $\mathcal{B}(Y)$ by:

$$\mu(B) = \int \mu_x(B)\, d\nu(x)$$

So, $\{\mu_x \colon x \in X\}$ is a σ-finite disintegration of μ with respect to (ν, φ).

We show that μ is σ-finite. Let $B_n = f_n(H) \subseteq Y$. Each B_n is Borel as f_n is one-to-one on H.

For $x \in H$, $G_x = \bigcup_n (B_n \cap G_x)$. Since ν concentrates on H it follows that

$$\begin{aligned} \mu(Y - \bigcup_n B_n) &= \int \mu_x(Y - \bigcup_n B_n)\, d\nu(x) \\ &= \int \mu_x(G_x - \bigcup_n B_n)\, d\nu(x) = 0. \end{aligned}$$

So, $\{\mu_x \colon x \in X\}$ is a σ-finite disintegration of the σ-finite measure μ with respect to (ν, φ).

However, this disintegration is not uniformly σ-finite.

For suppose $\{B_n\}$ witnessed $\{\mu_x \colon x \in X\}$ being uniformly σ-finite. So, for all $x \in X$ we have $\mu_x(B_n) < \infty$ and $\mu_x(Y - \bigcup_n B_n) = 0$.

For each n, $G \cap (X \times B_n)$ is $\mathbf{\Pi}^1_1$ with finite sections and is thus a countable union of $\mathbf{\Pi}^1_1$ graphs implying that $G = \bigcup_n G \cap B_n$ is a countable union of $\mathbf{\Pi}^1_1$ graphs, a contradiction.

This constructs a counterexample assuming the existence of a special $\mathbf{\Pi}^1_1$ set in the plane.

Construction of a special set

Assuming $V = L$ we now construct a special $\mathbf{\Pi}^1_1$ set in $\omega^\omega \times \omega^\omega$.

Let φ_n enumerate the formulas in the language of set theory. Let φ_m define the canaonical wellordering of L. For $x \in 2^\omega$ let $\mathrm{Th}_x = \{\varphi_n \colon x(n) = 1\}$.

The set of codes C is defined by $x \in C$ iff $x \in 2^\omega$, Th_x is a consistent, complete extension of $\mathrm{ZF}_N + V = L$, and $x(m) = 1$.

For each $\varphi_n = \exists w\ \varphi(w, x_1, \dots, x_k)$ let $\tau_n(z)$ be the coresponding formula defining the skolem term, that is,

$$\tau_n(z) = (\exists w\ \varphi \wedge z \text{ is the } <_L \text{ such } w) \vee (\neg \exists w\ \varphi_n \wedge z = 0)$$

For $x \in C$, let M_x be the model whose elements are equivalence classes of terms $[\tau_n]$ where $\tau_n \equiv \tau_m$ iff $\mathrm{Th}_x \vdash \exists z\ [\tau_n(z) \wedge \tau_m(z)]$.

Let E_x be the natuaral binary relation on M_x: $[\tau_n] E_x [\tau_m]$ iff $\mathrm{Th}_x \vdash \exists z\ \exists w\ (\tau_n(z) \wedge \tau_m(w) \wedge z \in w)$.

So M_x is countable and $(M_x, E_x) \models \mathrm{ZF}_N + V = L$.

Also, $M_x = \mathrm{Hull}^{M_x}(\emptyset)$.

If (M_x, E_x) is wellfounded then there is an $\alpha < \omega_1$ such that $(M_x, E_x) \cong (L_\alpha, \epsilon)$.

Let π_x denote the transitive collapse on $\mathrm{wf}(M_x)$.

Fact

We have the following properties of the coding.

1. *If* $x, x' \in C$ *and* $M_x \cong L_\alpha \cong M_{x'}$, *then* $x = x'$.
2. *If* $L_{\alpha+1} \cap \omega^\omega \neq L_\alpha \cap \omega^\omega$, *then there is an* $x \in C$ *such that* $M_x \cong L_\alpha$.

(1) follows from the fact that M_x determines its theory. (2) follows from the fact that such an α projects to ω.

Define

$$V = \{x \in C \colon \omega \in \text{wf}(M_x)\}$$
$$U = \{x \in C \colon (M_x, E_x) \text{ is wellfounded } \}$$

So, $U \subseteq V \subseteq C$, $V, C \in \boldsymbol{\Delta}^1_1$, $U \in \boldsymbol{\Pi}^1_1$.

For $x \in \omega^\omega$, say β is good with respect to x if $L_\beta(x) \models \mathrm{ZF}_N + V = L$.

Recall $P = \{(x, y) \in \omega^\omega \times \omega^\omega : (y)_0 = x\}$.

Fix Borel functions $f_n : \omega^\omega \to \omega^\omega$ such that for all x and $n \neq m$ we have $f_n(x) \neq f_m(x)$, and $f_n(x) \in P_x$.

Define the set $G' \subseteq \omega^\omega \times \omega^\omega$ by $(x, y) \in G' \iff$

$$[x \notin V \wedge \exists n(y = f_n(x))] \vee [x \in V \wedge (x, y) \in P \wedge [\text{“}y \in M_x\text{''} \vee$$
$$\exists \text{ a well-founded extension } M \text{ of } M_x\ \exists \alpha < \omega_1$$
$$(M_x \subseteq M \cong L_\alpha \wedge y \in L_\alpha(x) \wedge$$
$$[\forall \gamma < \alpha(\neg(\gamma \text{ is good and a limit of good ordinals}) \vee$$
$$\exists \phi \in \Sigma^1_2\ \exists \tau > \gamma\, (L_\gamma(x) \models \neg\varphi \wedge L_\tau(x) \models \varphi))]])]].$$

Easily G' is $\mathbf{\Sigma}^1_2$ and all sections of G' are countable.

Let $G'(x,y) \leftrightarrow \exists z\ G(x,\langle y,z\rangle) \leftrightarrow \exists! z\ G(x,\langle y,z\rangle)$.

We may assume N is large enough so that if Ω' and Ω are the Σ^1_2, Π^1_1 statements defining G', G, then the above equivalences are theorems of ZF_N.

Clearly the sections of G are countable.

We verify property (3) of G.

Fix $B \subseteq Y = \omega^\omega$. Let $K_n = \{x \in X \colon |B \cap G_x| \geq n\}$. Let $b \in \omega^\omega$ be a Borel code for B. By $V = L$, let $\tau < \omega_1$ be such that $b \in L_\tau$.

It suffices to show that $K_n \cap V \in \boldsymbol{\Delta}^1_1$. Let

$$\psi(x) = \exists \text{ distinct } a_1, \ldots, a_n \in M_x \ (a_1, \ldots, a_n \in B).$$

ψ correctly computes K_n on $V - U$ by definition of G.

Let $V = D \cup E$ where $D = \{x \in V : b \in M_x\}$ and $E = V - D$.

If $x \in U \cap D$, then by $\boldsymbol{\Sigma}^1_1$ absoluteness we have that ψ still correctly computes K_n.

However, $U \cap E$ is countable. So, we have a Borel definition of $K_n \cap V$ which works except on a countable set. So, $K_n \cap V$ is Borel.