Partition Properties Under AD

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CUNY
October, 2009

We work in the background theory $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$.

We recall the Erdős-Rado partition notation.

Definition

$\kappa \longrightarrow (\kappa)^\lambda$ iff for every partition $\mathcal{P} \colon (\kappa)^\lambda \to \{0, 1\}$ there is a homogeneous set $H \subseteq \kappa$ with $|K| = \kappa$.

We say κ has the strong partition property if $\kappa \longrightarrow (\kappa)^\kappa$.

We say κ has the weak partition property if $\forall \lambda < \kappa \; \kappa \longrightarrow (\kappa)^\lambda$.

In ZFC no κ has an infinite exponent partition property.

Polarized Relation

Suppose $\{\kappa_i\}_{i<\theta}$ is an increasing, discontinuous sequence of cardinals and $\{\lambda_i\}_{i<\theta}$ is a sequence of exponents ($\lambda_i \leq \kappa_i$).

A block function F from $\{\lambda_i\}$ to $\{\kappa_i\}$ is an increasing function from $\oplus_{i<\theta}\lambda_i$ to $\sup_{i<\theta}\kappa_i$ such that F restricted to the copy of λ_i maps to $\kappa_i - \sup_{j<i}\kappa_j$.

We say $\{\kappa_i\} \longrightarrow \{\kappa_i\}^{\lambda_i}$ if for all partitions $\mathcal{P}$ of the block functions from $\{\lambda_i\}$ to $\{\kappa_i\}$ there is a block set $\{H_i\}_{i<\theta}$ with $H_i \subseteq \kappa_i - \sup_{j<i}\kappa_j$ and $|H_i| = \kappa_i$ which is homogeneous for $\mathcal{P}$.

If the sequence is finite we write for example

$$(\kappa_0, \kappa_1, \kappa_2) \longrightarrow (\kappa_0, \kappa_1, \kappa_2)^{\lambda_0, \lambda_1, \lambda_2}$$

General Problem: Determine the exact partition properties for all cardinals and polarized properties for sequences of cardinals assuming AD.

Down low, say below $\aleph_{\omega_1}$, we can use the detailed inductive analysis of the cardinals. This relies on a detailed theory of the projective ordinals (essentially the first few Suslin cardinals)

Up higher we must use general methods relying on the general theory of Suslin cardinals and pointclasses under AD.

Conjecture: Every regular Suslin cardinal has the strong partition property.

Recall the following fundamental definition of descriptive set theory.

Definition

$A \subseteq \omega^\omega$ is κ-Suslin if there is a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ with

$$A = p[T] = \{x \colon \exists f \in \kappa^\omega \ \forall n \ (x \restriction n, f \restriction n) \in T\}.$$

Let $S(\kappa)$ be the κ-Suslin sets.
κ is a Suslin cardinal if $S(\kappa) - \bigcup_{\lambda<\kappa} S(\lambda) \neq \emptyset$.

Under AD the first ω Suslin cardinals are:

$$\omega, \quad \omega_1 = \delta^1_1, \quad \omega_\omega, \quad \omega_{\omega+1} = \delta^1_3, \quad \omega_{\omega^{\omega^\omega}}, \quad \omega_{\omega^{\omega^\omega}+1} = \delta^1_5, \ldots.$$

The corresponding Suslin classes enumerate the first ω many Levy classes closed under $\exists^{\mathbb{R}}$.

So, $S(\omega) = \boldsymbol{\Sigma}^1_1$, $S(\delta^1_1) = \boldsymbol{\Sigma}^1_2$, $S((\delta^1_3)^-) = \boldsymbol{\Sigma}^1_3$, $S(\delta^1_3) = \boldsymbol{\Sigma}^1_4$, etc.

In general, $S((\delta^1_{2n+1})^-) = \boldsymbol{\Sigma}^1_{2n+1}$ and $S(\delta^1_{2n+1}) = \boldsymbol{\Sigma}^1_{2n+2}$.

Definition

δ^1_n is the supremum of the lengths of the $\boldsymbol{\Delta}^1_n$ prewellorderings of the reals.

Fact

(Kechris) $\quad cof((\delta^1_{2n+1})^-) = \omega$ *and*
(Kunen-Martin) $\quad \delta^1_{2n+2} = (\delta^1_{2n+1})^+$.

Theorem (J)

All the δ^1_{2n+1} have the strong partition property. These are the only cardinals below the projective ordinals with the strong partition property.

Theorem (J)

Between δ^1_{2n+1} and δ^1_{2n+3} there are exactly $2^{n+1}-1$ regular cardinals κ. They all satisfy $\kappa \longrightarrow (\kappa)^{\delta^1_{2n+1}}$ but no higher.

Theorem (Kechris-Kleinberg-Moschovakis-Woodin)

Every regular limit Suslin cardinal has the strong partition property.

Proof shows many non-Suslin cardinals have the strong partition property.

Theorem

For every regular limit (inaccessible) Suslin cardinal κ we have

$$(\kappa, \kappa^{+}, \kappa^{++}) \longrightarrow (\kappa, \kappa^{+}, \kappa^{++})^{\kappa,\kappa,\kappa}.$$

This was part of joint work with A. Apter and B. Löwe.

Conjecture: For every inaccessible Suslin Cardinal κ we have:

$$(\kappa, \kappa^{+}, \kappa^{++}) \longrightarrow (\kappa, \kappa^{+}, \kappa^{++})^{\kappa,\kappa^{+},\kappa^{+}}.$$

Remark

We cannot increase the last exponent to κ^{++}.

We consider the possible cofinalities/measurability of $\aleph_1, \aleph_2, \aleph_3$.

- There are 3 possibilities of $\aleph_1$ (cof $= \omega$, *regular*, measurable).
- There are 4 possibilities for $\aleph_2$, and 5 possibilities for $\aleph_3$.
- 13 of the 60 total possibilities are "trivially inconsistent." For example, $\aleph_1$ regular, $\text{cof}(\aleph_2) = \aleph_1$, and $\text{cof}(\aleph_3) = \aleph_2$.

Theorem

Assuming suitable large cardinals, all of the remaining 47 *patterns are consistent with* ZF*.*

The following definition is used frequently.

Definition

We say $f\colon \alpha \to \mathrm{On}$ has uniform cofinality ω if there is an $f'\colon \alpha \times \omega \to \mathrm{On}$, inceasing in the second argument, such that

$$\forall \beta < \alpha \; f(\beta) = \sup_m f'(\beta, m)$$

We say f is of the correct type if f is increasing, everywhere discontinuous, and of uniform cofinality ω.

We can also define f having uniform cofinality ρ, or more generally f having uniform cofinality g for $g\colon \alpha \to \mathrm{On}$.

For example, $f(\alpha)$ has uniform cofinality α.

When proving partition properties from AD we always use an alternate form of the definition.

Definition

$\kappa \xrightarrow{\text{c.u.b.}} (\kappa)^{\lambda}$ if for every partition $\mathcal{P}$ of the functions $f \in (\kappa)^{\lambda}$ of the correct type, there is a c.u.b. $C \subseteq \kappa$ homogeneous for $\mathcal{P}$,

Fact:

$$\kappa \xrightarrow{\text{c.u.b.}} (\kappa)^{\lambda} \Rightarrow \kappa \longrightarrow (\kappa)^{\lambda}$$

$$\kappa \longrightarrow (\kappa)^{\omega \cdot \lambda} \Rightarrow \kappa \xrightarrow{\text{c.u.b.}} (\kappa)^{\lambda}$$

In particular, for any κ the notions of weak or strong partition property are unambiguous. We officially adopt the c.u.b. versions of these properties henceforth.

Theorem (Steel-Woodin)

The Suslin cardinals are closed below their supremum. Assuming AD^+*, they are closed.*

So, on the c.u.b. set of limit Suslin cardinals, κ has the strong partition property iff κ is regular.

Off this set things are more difficult.

All known proofs of partition properties use a general method of Martin.

Martin's theorem says roughly that to show $\kappa \longrightarrow (\kappa)^\lambda$ it suffices to code functions $f : \lambda \to \kappa$ within some pointclass $\boldsymbol{\Gamma}$ having some reasonable boundedness properties.

For example, Martin showed $\omega_1 \longrightarrow (\omega_1)^{\omega_1}$ using this method and the pointclass $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}^1_1$.

To show $\delta^1_3 \longrightarrow (\delta^1_3)^{\delta^1_3}$ one uses the pointclass $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}^1_3$.

Let κ be an inaccessible Suslin cardinal less than the supremum of the Suslin cardinals.

Since κ has the weak partition property, for each regular $\rho < \kappa$ there is a ρ-cofinal normal measure μ_ρ on κ.

In particular, $\mu = \mu_\omega$ is a normal measure on κ.

Remark

There are many more normal measures on κ than these. For every *thin* stationary set $S \subseteq \kappa$ there is a normal measure μ_S which concentrates on S. Such stationary sets are well-ordered. The order-type is much larger than κ.

General pointclass arguments show the following.

Fact

There is a pointclass $\boldsymbol{\Gamma}$ *closed under* $\forall^{\mathbb{R}}$, $\wedge$, $\vee$ *and with scale*$(\boldsymbol{\Gamma})$. *The norms* φ_i *of a* $\boldsymbol{\Gamma}$*-scale on a* $\boldsymbol{\Gamma}$*-complete set* P *map onto* κ.

Since κ is a limit of Suslin cardinals, $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ is closed under real quantification.

There is a c.u.b. $C \subseteq \kappa$ of limit Suslin cardinals and every $\alpha \in C$ is *strongly reliable*. That is, if $\beta < \alpha$ then $\sup\{\varphi_n(x) \colon \varphi_0(x) \leq \beta\} < \alpha$.

Let $C_\omega = \{\alpha \in C \colon \mathrm{cof}(\alpha) = \omega\}$.

Some local pointclasses

For $\alpha \in C_\omega$, let $\boldsymbol{\Sigma}_0^\alpha$ be the pointclass of countable unions in $S(<\alpha)$,

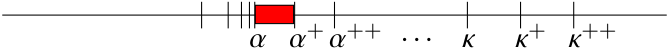
Then: scale($\boldsymbol{\Sigma}_0^\alpha$), scale($\boldsymbol{\Pi}_1^\alpha$), scale($\boldsymbol{\Sigma}_2^\alpha$), etc.

$\boldsymbol{\Pi}_1^\alpha$ admits scales with norms onto α^+.

α^+ is the supremum of the lengths of the $S(\alpha)$ wellfounded relations.

$S(\alpha) = \boldsymbol{\Sigma}_1^\alpha$, $S(\alpha^+) = \boldsymbol{\Sigma}_2^\alpha$.

We have the following picture.



$$\boldsymbol{\Sigma}^{\alpha}_{0} = \bigcup_{\lambda<\alpha} S(\lambda)$$

$$S(\alpha) = \boldsymbol{\Sigma}^{\alpha}_{1}, \qquad S(\alpha^+) = \boldsymbol{\Sigma}^{\alpha}_{2}$$

$$\text{scale}(\boldsymbol{\Sigma}^{\alpha}_{0},) \qquad \text{scale}(\boldsymbol{\Pi}^{\alpha}_{1})$$

Lemma

We have the following representations for κ^+, κ^{++}.

$$[\alpha \mapsto \alpha^+]_\mu = \kappa^+, \qquad [\alpha \mapsto \alpha^{++}]_\mu = \kappa^{++}$$

Let R be the binary relation corresponding to φ_0.

$$R(x,y) \leftrightarrow x, y \in P \wedge \varphi_0(x) \leq \varphi_0(y).$$

For $\alpha \in C$ let:

$$R^\alpha(x,y) \leftrightarrow x, y \in P \wedge \varphi_0(x) \leq \varphi_0(y) < \alpha.$$

Uniform Sets and Scales

Using the R^α we uniformly define sets

$$A^\alpha \in \mathbf{\Sigma}^\alpha_0, \qquad B^\alpha \in \mathbf{\Sigma}^\alpha_1, \qquad Q^\alpha \in \mathbf{\Pi}^\alpha_1$$

$$A_\alpha(\langle \tau, z\rangle) \leftrightarrow \exists n\, (\tau(z))_n \in R^\alpha.$$

$$B^\alpha(\langle \tau, z\rangle) \leftrightarrow \exists w\, \forall n\, R^\alpha(\tau(z, w, n)).$$

$$Q^\alpha(w) \leftrightarrow \forall z\, A^\alpha(\langle w, z\rangle).$$

We can uniformly assign scales and Suslin representations to the A^α, B^α, and Q^α.

- The scale φ_n on P gives a scale σ_n on R and by restriction scales on the R^α.
- From the scale σ_n on R we easily get a Suslin representation $A^\alpha = p[U^\alpha]$ which then gives a $\mathbf{\Sigma}^\alpha_0$-scale on A^α.
- Similarly we get a Suslin representation $B^\alpha = p[V^\alpha]$, where V^α is a tree on $\omega \times \alpha$.
- From periodicity and the scale on A^α we get a $\mathbf{\Pi}^\alpha_1$-scale on Q^α.

We code ordinals in C_ω as follows.

Let $W = \{x \colon \forall n\, (x)_n \in P\}$.

For $x \in W$ let $\psi(x) = |x| = \sup_n \varphi_n(x)$.
$W_\alpha = \{x \in W \colon \psi(x) \leq \alpha)\}$.

The scale φ_n on P easily give a scale on W, let $W = p[T_W]$ be the Suslin representation.

We define two trees T^+ and T^{++} on $\omega \times \kappa$ to analyze the ultrapowers $[\alpha \mapsto \alpha^+]_\mu$ and $[\alpha \mapsto \alpha^{++}]_\mu$.

Will have:

Lemma

For any $f\colon \kappa \to \kappa$ with $f(\alpha) < \alpha^+$ there is a $x \in \omega^\omega$ with T^+_x wellfounded and such that $\forall^*_\mu \alpha\ f(\alpha) < |T^+_x \restriction \alpha|$.

This immediately shows that $[\alpha \mapsto \alpha^+]_\mu \leq \kappa^+$, and the lower bound follows easily from the normality of μ.

For the tree T^{++} we will have:

Lemma

For any $f\colon \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$ there is a $x \in \omega^\omega$ with T_x^{++} wellfounded such that $\forall^*_\mu \alpha\ (f(\alpha) < [\beta \mapsto |T_x^{++} \restriction \beta|]_{\mu_\alpha})$.

The lemma and the normality of μ_α show that $[\alpha \mapsto \alpha^{++}]_\mu \leq \kappa^{++}$.

Note: We don't immediately get the lower bound.

These lemmas will also give a coding of the functions into κ^+ or κ^{++} we use to get the partition relations.

Construction of T^+

Recall B^α is universal for $\mathbf{\Sigma}^\alpha_1$. From the Kunen-Martin theorem and the coding lemma we have that α^+ is the supremum of the lengths of the $\mathbf{\Sigma}^\alpha_1$ wellfounded relations. We code these using B^α.

We define a tree U^2 on $\omega^2 \times (\omega^2 \times \kappa^2)$ as follows.

A branch $(x, y, z, w, \vec{\alpha}, \vec{\beta})$ through U^2 should witness (for $x \in W$): that $z_0, z_1, \dots$ is a decreasing chain in the $\mathbf{\Sigma}^{\psi(x)}_1$ relation coded by y. This relation is the set of (c, d) such that

$$\exists w \; \forall n \; \langle y(\langle c, d\rangle, w\rangle_n = (e, f) \in R^{\psi(x)}$$

The $\vec{\alpha}, \vec{\beta}$ witness this as follows:

- The ordinals $\vec{\alpha}$ witness that all the (e, f) are in R.
- The $\vec{\beta}$ witness that for each (e, f) there is an n such that $(f, x_n) \in R$.

Key Point: For $x \in W_\alpha$ (possibly $\psi(x) < \alpha$) and y coding a $\boldsymbol{\Sigma}_1^{\psi(x)}$ wellfounded relation, the tree $U^2_{x,y} \restriction \alpha$ is wellfounded (not just $U^2_{x,y} \restriction \psi(x)$).

Suppose $f : \kappa \to \kappa$ and $f(\alpha) < \alpha^+$ for $\alpha \in C$.

Consider the game G_f:

I	r
II	x, y

II wins the run iff

$$(r \in W) \to [x \in W \wedge \psi(x) \geq \psi(r) \wedge |B_y^{\psi(x)}| > f(|x|)].$$

A boundedness argument shows that II has a winning strategy.

This suggests the following definition of the tree T^+ on $(\omega)^2 \times \kappa \times (\omega)^2 \times \kappa$:

$(\sigma, r, \vec{\alpha}, x, y, \vec{\beta}) \in [T^+]$ iff:

1. $(r, \vec{\alpha}) \in [T_W]$.
2. $\sigma(r) = (x, y)$
3. $(x, y, \vec{\beta}) \in [U^2]$.

Then T^+_σ is wellfounded and $|T^+_\sigma \upharpoonright \alpha| > f(\alpha)$ for μ almost all α.

Construction of T^{++}

We first construct a tree V^2 on $(\omega)^2 \times \kappa$ with the following properties:

1. $(x, y) \in [V^2]$ iff $x \in W$ and for all n, $(y)_n$ codes a $\mathbf{\Sigma}_1^{|x|}$ wellfounded relation $B_{(y)_n}^{\psi(x)}$.
2. If $x \in W$, $\psi(x) \in C$ then there is a c.u.b. $D \subseteq \alpha^+$ such that if $\gamma \in D$, $y \in \omega^\omega$ and for all n $(y)_n$ codes a $\mathbf{\Sigma}_1^{\psi(x)}$ wellfounded relation of rank $< \gamma$, then $V_{x,y}^2 \restriction \gamma$ is illfounded.

main point: We can translate the $\mathbf{\Pi}_1^{\psi(x)}$ statement asserting the wellfoudedness of the $B_{(y)_n}^{\psi(x)}$ into $\mathbf{\Pi}_1^{\beta}$ statements for any $\beta \geq \psi(x)$ (use the $(x)_i$ as in the definition of U).

Suppose $x \in W$, $\psi(x) = \alpha \in C$, and $g \colon \alpha^+ \to \alpha^+$. Play the game

G_g:

I	z
II	w

II wins the run iff:

$$(\forall n\, B^\alpha_{(z)_n} \text{ is wellfounded }) \to (B^\alpha_w \text{ is wellfounded } \wedge |B^\alpha_w| > g(\sup_n |B^\alpha_{(y)_n}|))$$

By boundedness, II has a winning strategy τ for any G_g.

Suppose now $f\colon \kappa \to \kappa$ with $f(\alpha) < \alpha^{++}$.

Play the game G_f:

$$\begin{array}{ll} \text{I} & r \\ \text{II} & x, \tau \end{array}$$

r, x will be in W and τ will be strategy for a game G_g where $[g]_{\mu_\alpha} > f(\alpha)$, where $\alpha = \psi(x)$.

More precisely, II wins the run iff:

$$
\begin{aligned}
r \in W \rightarrow (x \in W \wedge \psi(x) = \alpha \geq \psi(r)) \\
\wedge \forall y \, [\forall n \, B^{\alpha}_{(y)_n} \text{ is wellfounded } \rightarrow \\
B^{\alpha}_{\tau(y)} \text{ is wellfounded } \wedge |B^{\alpha}_{\tau(y)}| \geq g(\sup_n |B^{\alpha}_{(y)_n}|)]
\end{aligned}
$$

for some $g \colon \alpha^+ \to \alpha^+$ with $[g]_{\mu_\alpha} \geq f(\alpha)$.

II has a winning strategy σ for any f, and this suggests the definition of T^{++}:

$(\sigma, r, \vec{\alpha}, x, \tau, y, z, \vec{\beta}, \vec{\gamma}) \in T^{++}$ iff:

1. $(r, \vec{\alpha}) \in [T_W]$.
2. $\sigma(r) = (x, \tau)$.
3. $(x, y, \vec{\beta}) \in [V^2]$
4. $\tau(y) = z$.
5. $(x, z, \vec{\gamma}) \in [U^2]$

The properties of U^2 and V^2 show that T^{++} has the desired property.

The countable exponent θ case.

Fix a bijection $\pi\colon \omega \cdot \theta \to \omega$.

We code cofinally in κ^+, κ^{++} many ordinals using sections of our trees: T_x^+, T_x^{++}.

Suppose $\mathcal{P}$ is a partition of the block functions from $3 \times \theta$ to $(\kappa, \kappa^+, \kappa^{++})$.

Consider the game $G_{\mathcal{P}}$:

$$\begin{array}{ll} \text{I} & x, y, z \\ \text{II} & x', y', z' \end{array}$$

(1) If there is an $j < \omega \cdot \theta$ such that $(x)_{\pi(j)} \notin P$ or $(x')_{\pi(j)} \notin P$, then player I wins iff for the least such j, $(x)_{\pi(j)} \in P$.

(2) Suppose next that there is an $\alpha < \kappa$ such that one of the following holds.

(a) There is a $j < \omega \cdot \theta$ such that either $T^{+}_{(y)_{\pi(j)}} \restriction \alpha$ or $T^{+}_{(y')_{\pi(j)}} \restriction \alpha$ is illfounded.

(b) There is a $\beta < \alpha^{+}$ and a $j < \omega \cdot \theta$ such that either $T^{++}_{(z)_{\pi(j)}} \restriction \beta$ or $T^{++}_{(z')_{\pi(j)}} \restriction \beta$ is illfounded.

Let $\alpha < \kappa$ be least such that (a) or (b) above holds. If (a) holds, let j be least such that (a) holds for α and this j. In this case, Player I wins provided $T^{+}_{(y)_{\pi(j)}}$ is wellfounded. If (a) does not hold at α, but (b) does, let (β, j) be lexicographically least such that (b) holds. Player I wins in this case provided $T^{++}_{(z)_{\pi(j)}} \restriction \beta$ is wellfounded.

Assume II has a winning strategy τ.

We define c.u.b. sets $C_0 \subseteq \kappa$, $C_1 \subseteq \kappa^+$, and $C_2 \subseteq \kappa^{++}$.

For example, to define C_2 we define for $\alpha \in C$, $\beta, \gamma < \alpha^+$ and $j < \omega \cdot \theta$:

$$\begin{aligned} A_{\alpha,\beta,\eta,j} = \{(x, y, z) :\ & \forall j\ ((x)_{\pi(j)} \in P_0 \wedge \varphi_0((x)_{\pi(j)}) < \alpha) \\ & \wedge \forall \alpha' < \alpha\ \forall \beta\ < (\alpha')^+\ \forall j\ (T^+_{(y)_{\pi(j)}} \restriction \alpha \text{ and } T^{++}_{(z)_{\pi(j)}} \restriction \beta \text{ are w.f.}) \\ & \wedge \forall j\ |T^+_{(y)_{\pi(j)}} \restriction \alpha| < \beta \wedge \forall (\beta', j') \leq_{\text{lex}} (\beta, j)\ (|T^{++}_{(z)_{\pi(j)}} \restriction \beta| \leq \eta)\}. \end{aligned}$$

We have: $A_{\alpha,\beta,\gamma,j} \in \mathbf{\Delta}^{\alpha}_1$.

Since τ is winning for Player II, for each $(x, y, z) \in A_{\alpha,\beta,\eta,j}$, if $\tau(x, y, x) = (x', y', z')$ then $\forall(\beta', j') \leq_{\text{lex}} (\beta, j)$ $T^{++}_{(z')_{\pi(j')}} \restriction \beta$ is wellfounded.

By boundedness,

$$\rho_2(\alpha, \beta, \eta, j) := \sup\{|T^{++}_{(z')_{\pi(j')}} \restriction \beta| : (x', y', z') \in \tau[A_{\alpha,\beta,\eta,j}] \wedge j' \leq j\} < \alpha^+.$$

Let $C^{\alpha}_2 \subseteq \alpha^+$ be c.u.b. closed under ρ_2. The C^{α}_2 lift to $C_2 \subseteq \kappa^{++}$.

We need the following modifications to the countable exponent case.

- We code functions $F: \kappa \to \kappa$ using the uniform coding lemma.
- We use the Kechris-Woodin theory of generic codes to quantify over codes for ordinals less than κ.
- We modify the payoff of the game as follows:

I plays $x, y, z, \quad$ II plays x', y', z'

We now consider the least $\alpha < \kappa$ which is bad for one of the players meaning we check (in this order):

- For some $\delta < \alpha$, y or y' is not good at α: for almost all codes a of δ, if $R^{\delta}_{y}(a, b)$ then $T^{+}_{b} \restriction \alpha$ is wellfounded.
- For some $\delta < \alpha$ and $\beta < \alpha^{+}$, z is not good at (δ, α, β): for almost all codes a of δ, if $R^{\delta}_{z}(a, b)$ then $T^{++}_{z} \restriction \beta$ is wellfounded.
- x or x' does not code an ordinal at α.

To decide who wins, take least α such that one of these cases holds. If first case holds, take least δ. If not, but second case holds take least (β, δ).

An Extension

We can also show:

$$\{\kappa^{+n}\}_{n\in\omega} \longrightarrow (\{\kappa^{+n}\}_{n\in\omega})^{\kappa}.$$