Group Colorings and Shift Equivalence Relations

S. Gao, S. Jackson*, B. Seward

Department of Mathematics
University of North Texas

AMS-ASL Special Session on Logic and Dynamical Systems
January, 2009
Washington, DC

Overview

Recall the definition of a 2-coloring of a countable group G.

Definition

$c\colon G \to \{0,1\}$ is a 2-*coloring* if

$$\forall s \in G \; \exists T \in G^{<\omega} \; \forall g \in G \; \exists t \in T \; (c(gt) \neq c(gst)).$$

This definition was formulated independently by Pestov (c.f. paper of Glasner and Uspenski).

Significance of the definition.

Let E be the shift equivalence relation on $X = 2^G$, given by the action of G:

$$g \cdot x(h) = x(g^{-1}h).$$

Let F denote the *free part* of this space, that is,

$$x \in F \text{ iff } \forall g \neq 1 \ (g \cdot x \neq x).$$

1. Coloring property gives a *marker compactness* property. (MCP) Let $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$ be relatively closed complete sections of F. Then $\bigcap_n S_n \neq \emptyset$.
2. Coloring property is equivalent to a free orbit closure.

Main Result

Theorem

Every countable group G has the 2-coloring property.

- First proof works for abelian, solvable groups.
- Second proof works for general groups.

Main Result

Theorem

Every countable group G has the 2-coloring property.

- First proof works for abelian, solvable groups.
- Second proof works for general groups.

Other Combinatorial Reformulations

Other natural descriptive properties have combinatorial reformulations in terms of the group G.

Definition

Colorings c_1, c_2 of G are *orthogonal* ($c_1 \perp c_2$) if

$$\exists T \in G^{<\omega}\ \forall g_1, g_2 \in G\ \exists t \in T\ (c_1(g_1 t) \neq c_2(g_2 t)).$$

Fact

If $x, y \in F$, *then* $x \perp y$ *iff* $\overline{[x]} \cap \overline{[y]} = \emptyset$.

Definition

A coloring c is *minimal* if

$$\forall S \in G^{<\omega}\ \exists T \in G^{<\omega}\ \forall g \in G\ \exists t \in T\ \forall s \in S\ (c(s) = c(gts)).$$

Fact

$x \in F$ is minimal iff $\overline{[x]}$ is minimal
(i.e., for every $y \in \overline{[y]}$ we have $\overline{[x]} = \overline{[y]}$).

Extension of Result

Theorem

For every countable group G there is a perfect set of pairwise orthogonal, minimal orbits in F.

In fact, we get more.....

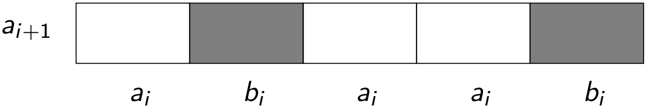
First consider the simplest case of $G = \mathbb{Z}$.

The following is not the argument that works in general, but has applications.

We define two sequences a_i, b_i from $2^{<\omega}$. We will have $\mathrm{lh}(a_i) = \mathrm{lh}(b_i)$. Can take $b_i = 1 - a_i$.

Each a_{i+1}, (and b_{i+1}) is a concatenation of a_i's and b_i's. (May assume $\mathrm{lh}(a_i) > i + 1$).

Let $a_{i+1} = a_i b_i a_i a_i b_i, \qquad b_{i+1} = b_i a_i b_i b_i a_i$.



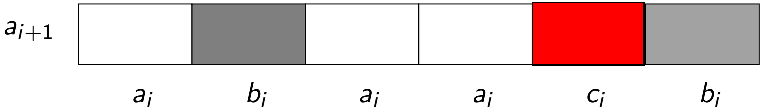
Let x be any concatenations of a_{i+1}'s and b_{i+1}'s. Then for $s = i + 1$, can take $T = \{0, 1, \ldots, 2\text{lh}(a_{i+1})\}$. to verify the 2-coloring property for this s.

To get a coloring, take any x such that for each i, x is a concatenation of a_i's and b_i's.

Easily modify to get a perfect set of pairwise orthogonal 2-colorings.

For example, for $w \in 2^\omega$ define $x(w)$ as above but using $a_{i+1} = a_i b_i a_i a_i c_i b_i$ where

$$c_i = \begin{cases} a_i & \text{if } x(i) = 0 \\ b_i & \text{if } x(i) = 1 \end{cases}$$



Each $x(w)$ has the following *marker identification property*:

(MIP) There is a finite $A \subseteq \mathbb{Z}$ such that for any $k \in \mathbb{Z}$, whether $k \cdot x$ is the start of an a_i or b_i is determined by $k \cdot x \restriction A$.

In fact A depends only on i, not on w.

If i is least such that $w_1(i) \neq w_2(i)$, then $x(w_1) \perp x(w_2)$ follows from the marker identification property, using roughly $A_i + |a_i|$.

Extending To Other G

We can extend this method to show the following.

Theorem

Suppose $\mathbb{Z} \trianglelefteq G$. Then G has the 2-coloring property.

proof sketch.

Let $x_1\mathbb{Z}, x_2\mathbb{Z}, \dots$ be the cosets of $\mathbb{Z}$ in $G = \{g_1, g_2, \dots\}$. If $g \notin \mathbb{Z}$, then g induces a fixed-point-free permutation π_g on the cosets. We use the algorithm above to color each coset $x_i\mathbb{Z}$ with a 2-coloring c_i. At step i, if $g_i \in \mathbb{Z}$ then we define the a_i, b_i for each coset as above. If $g_i \notin \mathbb{Z}$ then consider $\pi_i = \pi_{g_i}$. On each orbit of π_i, if $\pi_i(x\mathbb{Z}) = y\mathbb{Z}$, then define the a_i, b_i for $x\mathbb{Z}$ and for $y\mathbb{Z}$ such that the colorings will be orthogonal, and by a fixed set A_i (not depending on x and y).

To see this works, for $s \in G$ take cases as to whether $s \in \mathbb{Z}$. If $s \in \mathbb{Z}$, the 2-coloring property is satisfied by the argument that each c_n is a 2-coloring. If $s = g_i \notin \mathbb{Z}$, then for $g \in x_j\mathbb{Z}$, $gs \in x_k\mathbb{Z}$ for some $j \neq k$, and the set A_i witnesses the 2-coloring property for g and gs (by the orthogonality of c_j and c_k). □

These methods give:

Corollary

Every abelian, and in fact, every solvable group has the 2-coloring property.

This method can be used to show more, for example:

Theorem

Let $\mathbb{Z} \trianglelefteq G$. Then the set of 2-colorings of G is $\mathbf{\Pi}^0_3$-complete.

In summary, these methods show:

- Every abelian or solvable group has the 2-coloring property.
- If $\mathbb{Z} \trianglelefteq G$ or $S \trianglelefteq G$ where S is infinite solvable, then G has the 2-coloring property.
- Show directly the free group F_n has the 2-coloring property.

These methods give:

Corollary

Every abelian, and in fact, every solvable group has the 2-coloring property.

This method can be used to show more, for example:

Theorem

Let $\mathbb{Z} \trianglelefteq G$. Then the set of 2-colorings of G is $\mathbf{\Pi}^0_3$-complete.

In summary, these methods show:

- Every abelian or solvable group has the 2-coloring property.
- If $\mathbb{Z} \trianglelefteq G$ or $S \trianglelefteq G$ where S is infinite solvable, then G has the 2-coloring property.
- Show directly the free group F_n has the 2-coloring property.

These methods give:

Corollary

Every abelian, and in fact, every solvable group has the 2-coloring property.

This method can be used to show more, for example:

Theorem

Let $\mathbb{Z} \trianglelefteq G$. Then the set of 2-colorings of G is $\mathbf{\Pi}^0_3$-complete.

In summary, these methods show:

- Every abelian or solvable group has the 2-coloring property.
- If $\mathbb{Z} \trianglelefteq G$ or $S \trianglelefteq G$ where S is infinite solvable, then G has the 2-coloring property.
- Show directly the free group F_n has the 2-coloring property.

Two main ideas:

1. Get reasonable marker regions for general groups.
2. Exploit polynomial versus exponential growth.

Marker Regions

Question

What kind of marker regions can we get for general groups?

Say a group G has *regular markers* if there are $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$, each E_i an equivalence relation on G with finite classes each of which is a translate by a fixed set $A_i \subseteq G$, and such that $\bigcup_i E_i = G \times G$.

Question

Which groups admit regular markers?

Marker Regions

Question

What kind of marker regions can we get for general groups?

Say a group G has *regular markers* if there are $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$, each E_i an equivalence relation on G with finite classes each of which is a translate by a fixed set $A_i \subseteq G$, and such that $\bigcup_i E_i = G \times G$.

Question

Which groups admit regular markers?

Marker Regions

Question

What kind of marker regions can we get for general groups?

Say a group G has *regular markers* if there are $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$, each E_i an equivalence relation on G with finite classes each of which is a translate by a fixed set $A_i \subseteq G$, and such that $\bigcup_i E_i = G \times G$.

Question

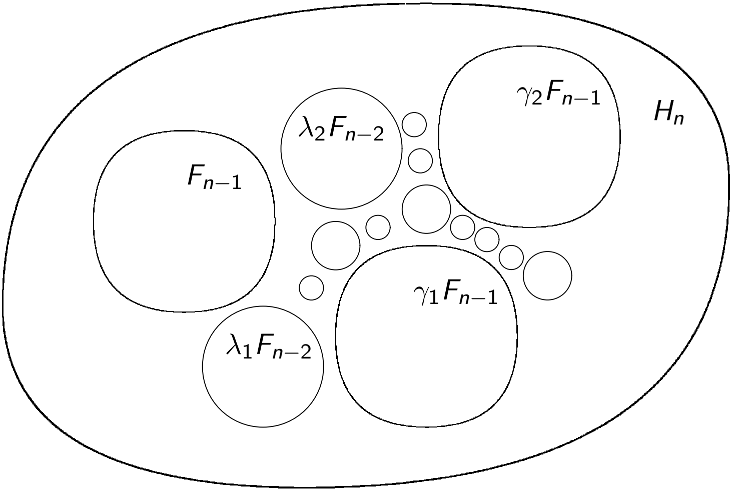
Which groups admit regular markers?

For general groups we get the following marker structure.

Will have marker sets $\mathbf{\Delta}_1 \supseteq \mathbf{\Delta}_2 \supseteq \mathbf{\Delta}_3 \supseteq \cdots$ (each $\mathbf{\Delta}_n \subseteq G$).

Will have $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ (each $F_n \subseteq G$ finite).

- The $\mathbf{\Delta}_n$ translates of F_n are maximally disjoint.
- Each F_n will be a disjoint union of copies of $F_1, \dots, F_{n-1}$.
- (homogeneity) Within any copy γF_n of F_n, the points in $\mathbf{\Delta}_k$ ($k \leq n$) are precisely the translates $\gamma(\mathbf{\Delta}_k \cap F_n)$ of the points in F_n.
- (fullness) If a copy δF_k intersects γF_n ($k \leq n$) then $\delta F_k \subseteq \gamma F_n$.


$\gamma_2 F_{n-1}$
H_n
$\lambda_2 F_{n-2}$
F_{n-1}
$\gamma_1 F_{n-1}$
$\lambda_1 F_{n-2}$

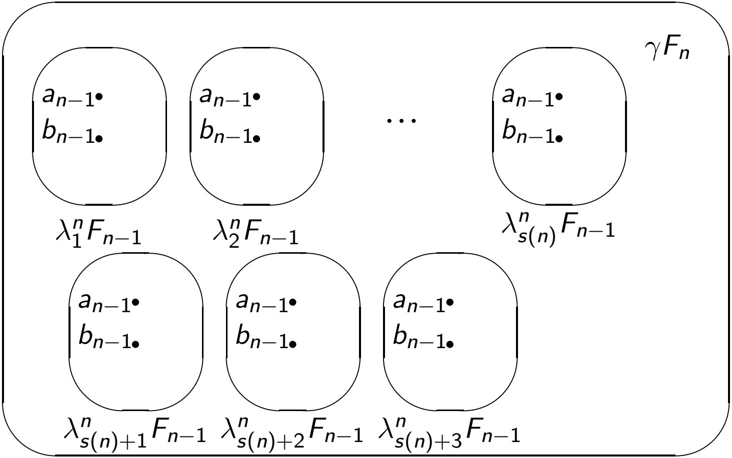


Figure: The labeling of the F_{n-1} copies inside an F_n copy

We define a coloring $c = \bigcup c_n$, which will then be extended to the 2-coloring c'.

c will color all points except those in

$$D = \bigcup_n \mathbf{\Delta}_n \{\lambda_1^n, \ldots, \lambda_{s(n)}^n\} b_{n-1}.$$

In extending c_{n-1} to c_n we color the above points except for those in $\mathbf{\Delta}_n \lambda_1^n, \ldots, \mathbf{\Delta}_b \lambda_{s(n)}^n$, and $\mathbf{\Delta}_n\{a_n, b_n\}$ where:

$$a_n \doteq \lambda_{s(n)+2}^n \, a_{n-1}$$
$$b_n \doteq \lambda_{s(n)+3}^n \, b_{n-1}.$$

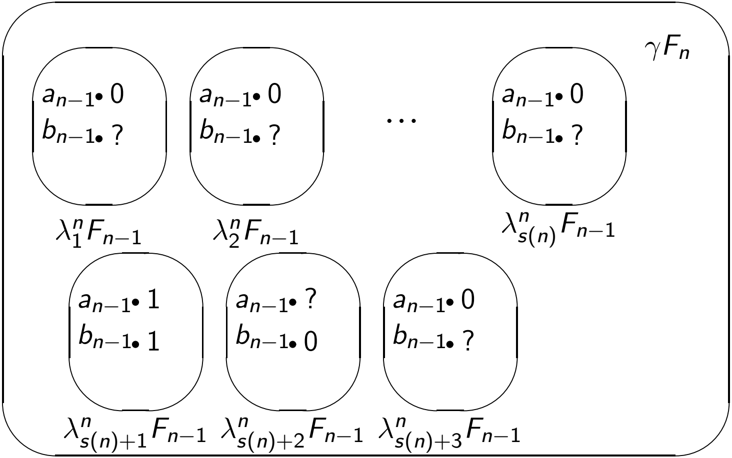


Figure: Extending c_{n-1} to c_n.

We extend c to c' by coloring the points of D so as to get a 2-coloring. Exploit polynomial versus exponential growth.

At stage n we extend c to points of $\mathbf{\Delta}_n\{\lambda_1^n, \dots, \lambda_{s(n)}^n\}b_{n-1}$ to take care of coloring property for $s = g_n \in H_n$.

Let $g \in G$ and consider the pair g, gs. By maximal disjointness of F_n copies, $gf \in \mathbf{\Delta}_n$ for some $f \in F_n F_n^{-1}$. Done unless $gsf \in \mathbf{\Delta}_n$. In this case

$$gsf = gf(f^{-1}sf) \in (gf)F_n F_n^{-1} H_n F_n F_n^{-1}.$$

So there are about $|H_n|^5$ many points to consider, and there $2^{s(n)}$ many "colors" available, where $s(n)$ is linear in $|H_n|$.