Colorings and Supercolorings on Countable Groups

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We consider the shift action of a countable group G on the compact Polish space $X = 2^G$.

$$g \cdot x(h) = x(g^{-1}h)$$

This is a continuous action of G on the space 2^G. Let $[x]$ denote the orbit of x under this action.

We let $F(2^G)$ denote the free part of the space:

$$x \in F(2^G) \Leftrightarrow \forall g \neq 1_G \ (g \cdot x \neq x)$$

$F(2^G)$ is an invariant G_δ in X.

A subflow means a closed invariant subset of 2^G.

Every orbit closure $\overline{[x]}$ is a subflow.

We say a subflow $A \subseteq 2^G$ is a free subflow if $A \subseteq F(2^G)$.

We say A is a minimal subflow if A does not properly contain any subflows. So, $A = \overline{[x]}$ for all $x \in A$.

2-colorings

Definition

A 2-coloring of a group G is an $x\colon G \to \{0,1\}$ satisfying the following: for every $s \neq 1_G$, there is a finite $T = T(s) \subseteq G$ such that:

$$\forall g \in G \; \exists t \in T \; (x(gt) \neq x(gst)).$$

Definition

A super 2-coloring of G is a 2-coloring in which $|T(s)|$ is bounded independently of s.

We say a group is 2-colorable (super 2-colorable) if it has a 2-coloring (super 2-coloring).

Remark

We can't require that $T(s)$ be independent of s.

Remark

The notion of 2-coloring is due to GJS and independently to Glassner-Uspensky.

The significance of 2-colorings is contained in the following.

Theorem

G is 2-colorable iff 2^G has a free subflow.

Proof.

Suppose $x \in 2^G$ is a 2-coloring. We show that $\overline{[x]} \subseteq F(2^G)$.

Suppose $y = \lim g_n^{-1} \cdot x \in \overline{[x]}$, and let $s \leq 1_G$. Let $T = T(s)$ be as in a 2-coloring. Let n be large enough so that $y \restriction T \cup sT = g_n^{-1} \cdot x \restriction T \cup sT$.

Let $t \in T$ be such that $x(g_n s) \neq x(g_n st)$. Then $y(t) = x(g_n t) \neq x(g_n st) = y(st) = s^{-1} \cdot y(t)$.

So, $y \in F(2^G)$.

$\square$

Suppose next that $\overline{[x]} \subseteq F(2^G)$.

Let $s \neq 1_G$. For $y \in \overline{[x]}$ let $t(y) \in G$ be least such that $y(t) \neq s^{-1} \cdot y(t) = y(st)$. The map $y \mapsto t(y)$ is continuous from $\overline{[x]}$ to G.

By compactness of $\overline{[x]}$ there is a finite $T \subseteq G$ containing the range of this map on $\overline{[x]}$.

In particular, this T works for all $y = g^{-1} \cdot x \in [x]$, and this shows that x is a 2-coloring: for all $g \in G$ there is a $t \in T$ such that

$$x(gt) = g^{-1} \cdot x(t) \neq s^{-1} \cdot g^{-1} \cdot x(t) = x(gst)$$

Recall in a super 2-coloring we bound the size of $|T(s)|$ independently of s.

Definition

For $A \subseteq 2^G$ we define the *n-closure* of A by:

$$y \in \overline{A}^n \Leftrightarrow \forall F \in G^n \; \exists x \in A \; (y \restriction F = x \restriction F).$$

So, $\overline{A}^1 \supseteq \overline{A}^2 \supseteq \cdots \overline{A}^n \supseteq \cdots \supseteq \overline{A}^\infty = \overline{A}$.

Each $\overline{A}^n$ is a closed set.

Each $\overline{[x]}^n$ is a closed invariant set in 2^G (i.e., is a subflow).

Note that $\overline{[x]}^1 = 2^G$ unless x is the constant 0 or constant 1 element.

Fact

If G has super 2-coloring of degree n (i.e., $|T(s)| = n$), then there is an orbit $[x]$ with $\overline{[x]}^{2n} \subseteq F(2^G)$.

So, the existence of a super 2-coloring on G implies the existence of a n-free orbit.

Proof.

Let $x \in 2^G$ be a super 2-coloring of degree n. Let $y \in \overline{[x]}^{2n}$. To show $y \in F(2^G)$, fix $s \neq 1_G$. Let $T = T(s)$ of size n be as in super 2-coloring.

Let $F = T \cup sT$. Let $g \in G$ be such that $y \upharpoonright F \; g^{-1} \cdot x \upharpoonright F$. Let $t \in T$ be such that $x(gt) \neq x(gst)$. Then
$y(t) = x(gt) \neq x(gst) = y(st)$, so $s^{-1} \cdot y \neq y$.

$\square$

So,

There is a super 2-coloring of degree n of G
$\Rightarrow$ There exists a $2n$-free orbit
$\Rightarrow$ There is a free orbit
$\Leftrightarrow$ There is a 2-coloring of G.

Question

Which groups admit 2-colorings or super 2-colorings. Which shift actions admit n-free orbits?

Theorem (GJS)

Every countable group G has a 2-coloring.

Corollary

Every shift action of a countable group G has a free subflow.

On the other hand, we don't know the answer to the following:

Question

Does $\mathbb{Z}$ have a super 2-coloring?

We do know that there are groups G which admit super 2-colorings.

However, it is easy to see:

Fact

$2^{\mathbb{Z}}$ *has a 3-free orbit.*

Proof.

Define a sequence $s_n \in 2^{[\ell_n, r_n]}$, for some ℓ_n, r_n, as follows.

Let $s_1 = 0101$, where $\ell_1 = -2$, $r_1 = 1$.

Each s_{n+1} will be a concatenation of copies of s_n and $\bar{s}_n$ (the digit flip of s_n).

First, concatenate copies of s_n to get s'_{n+1} which has length a multiple of $n+1$. Then let $s_{n+1} = \bar{s}'_{n+1} {}^\frown s'_{n+1} {}^\frown \bar{s}'_{n+1}$. □

Theorem

Let $G = \oplus_{i=1}^{\infty} G_i$ where each G_1 is non-trivial. Then G has a super 2-coloring of degree 5.

Fix an injection $\pi \colon G \to \omega$ with $\pi(g) > |g| = \max \operatorname{supp}(g)$. Let $\pi'(g) = \pi(g) + 1$. For $s \neq 1_G$, let $\sigma(s)$, $\sigma'(s)$ be a non-identity element of $G_{\pi(s)}$, $G_{\pi'(s)}$.

idea: $\sigma(s)$ will serve as a "switch" for the shift s, turning on or off a contribution pertaining to s in the coloring.

For $s \neq 1_G$, let $m(s) = \min \operatorname{supp}(s)$.

For each G_i and $h \neq 1_{G_i}$ in G_i we define a coloring $\chi_{i,h}$ of G_i by:

Fix coset representatives (containing 1_{G_i}) in G_i for $\langle h \rangle$.

Let $\chi_{i,h}(g) = 0$ if the least n such that $g = xh^n$ is even, where x is a coset representative, and $= 1$ if n is odd.

We set:

$$x(g) = \sum_{h \neq 1_G} \chi_{\pi(h),\sigma(h)}(g(\pi(h))) \cdot \chi_{m,h(m)}(g(m)) + \chi_{\pi'(h),\sigma'(h)}(g(\pi'(h))) \cdot \chi_{m,h^{-1}(m)}(g(m))$$

where $m = m(h)$.

Note that the above sum is a finite sum.

To show x is a super 2-coloring, fix $s \neq 1_G$.

Let $T(s) = \{1_G, \sigma(s), \sigma^2(s), \sigma'(s), {\sigma'}^2(s)\}$.

Let $g \in G$, and assume that $x(g) = x(gs)$. Select either $\pi(s)$ or $\pi'(s)$ depending on whether $\chi_{m,s(m)}(g(m)) \neq \chi_{m,s(m)}(gs(m))$ (where $m = m(s)$) or $\chi_{m,s(m)}(g(m)) \neq \chi_{m,s^{-1}(m)}(gs(m))$.

Say the first holds.

Consider either $\sigma(s)$ or $\sigma^2(s)$ depending on whether $\chi_{\pi(s),\sigma(s)}(g(\pi(s))) \neq \chi_{\pi(s),\sigma(s)}(g\sigma(s)(\pi(s)))$.

Say the first holds.

Then $x(g\sigma) \neq x(gs\sigma)$.

Theorem

A free product of groups G_i with $|G_0| \geq 5$ has a super 2-coloring of degree 6.

Corollary

Every free group on ≥ 2 generators has a super 2-coloring.

Proof. For simplicity, assume $G = G_0 \otimes G_1$.

Let a_0, a_1, a_2, a_3 be non-identity elements of G_0.

Let b_0 be a non-identity element of G_1.

A code word is a word of the form

$$(g \neq 1, a_0) b_0 a_0 b_0 \cdots a_0 b_0 (g \neq 1, a_0) b_0 a_0 b_0 \cdots a_0 b_0$$

Define the coloring x of G as follows.

If $g \in G$ does not end in a code word, set $x(g) = 0$.

Suppose

$$\begin{aligned} g =& (g_0) h_1 g_1 h_1 \cdots g_k h_k \cdot \\ & (g \neq 1, a_0) b_0 a_0 b_0 \cdots a_0 b_0 (g \neq 1, a_0) b_0 a_0 b_0 \cdots a_0 b_0. \end{aligned}$$

Say there are i, j occurrences of b_0 in the two segments of the code word.

If $j \notin [1, 10]$, then set $x(g) = 0$. Let

$$g' = (g_0)h_1 g_1 h_1 \cdots g_k h_k g a^{-1}_{j/2-2},$$

if j is even (where $a_{-1} = 1_{G_0}$) and for j odd use $a^{-1}_{(j+1)/2-2}$.

Let s be the i^{th} non-identity element of G.

Then if j is even let $x(g) = \chi_s(g')$, which as before is the coloring using coset representatives for $\langle s \rangle$ (if j is odd, use $\chi_{s^{-1}}(g')$).

To see this works, fix $s \neq 1_G$.

Let $T(s)$ be 1_G together with all elements of the form

$$\{a_{-1}, \dots, a_3\} b_0 a_0 b_0 \cdots a_0 b_0 \cdot a_1 b_0 a_0 b_0 \cdots a_0 b_0$$

where if i, j are as before, then the i^{th} non-identity element is s, and $j \in [1, 10]$. There at most 51 such elements.

Given $g \in G$, consider g, gs. At least one of $a_{-1}, \dots, a_3$ has the property ga_k and gsa_k both end with a term $g \in G_0$, $g \notin \{1_{G_0}, a_0\}$.

Let w be the codeword coding this i and where j codes this value of k and is even or odd depending on whether $\chi_s(g) \neq \chi_s(gs)$ or $\chi_{s^{-1}}(g) \neq \chi_{s^{-1}}(gs)$.

Let $t = a_k w \in T(s)$.

Then $(gt)' = g$ and $(gst)' = gs$, and so $x(gt) \neq x(gst)$.

Corollary

Every free product of infinite groups has an n-free orbit for some n.

more on n-free orbits

Definition

We say $x, y \in 2^G$ are orthogonal, $x \perp y$, if there is a finite $T \subseteq G$ such that for any $g_1, g_2 \in G$, there is a $t \in T$ with $x(g_1 t) \neq y(g_2 t)$.

We say a family x_α, $\alpha \in 2^\omega$ is strongly pairwise orthogonal if whenever $\alpha \neq \beta$ then $x_\alpha \perp x_\beta$, and the set $T_{\alpha,\beta}$ depends only on $\min\{n \colon \alpha(n) \neq \beta(n)\}$.

Theorem (GJS)

For any countable group G there is strongly pairwise orthogonal family of 2-colorings of G.

Corollary

For any G, there is a perfect set of pairwise disjoint free subflows of 2^G.

So, a natural question would be:

Question

For every group G, does there exists a perfect set of pairwise disjoint n-closed free subflows (for some n)?

This is equivalent to saying there is a perfect set x_α, $\alpha \in 2^\omega$, of n-free colorings such that for all $\alpha \neq \beta$ we have $x_\beta \notin \overline{[x_\alpha]}^n$.

We call this a perfect n-free set.

Theorem

There is a perfect 5-free set of colorings of $\mathbb{Z}$*.*

To show this, we modify the previous construction of a 3-free coloring of $\mathbb{Z}$.

Suppose at stage n we have defined s_u for $u \in 2^n$. For $u' \in 2^{n+1}$ extending $u \in 2^n$, we will have that $s_{u'}$ is a concatenation of copies of s_u and $\bar{s}_u$.

First concatenate copies of each s_u for form t_u whose lengths are all (the same) multiple of $(n+1)!$.

Now proceed in $\binom{2^n}{2}$ stages.

For each $u_1 \neq u_2$, concatenate the current t_{u_1} according to the pattern 010101 and concatenate the current t_{u_2} according to the pattern 001001. Concatenate the other t_u to have the same length as well.

At the end of the $\binom{2^n}{2}$ stages, the current t_u is the new s_u.

This works since any concatenation of 010101 and its flips cannot agree on 7 consecutive digits with a concatenation of 001001 and its flips.

Theorem

Assume $\mathbb{Z} \trianglelefteq G$. Then 2^G has a 7-free orbit.

Let $g_0\mathbb{Z}, g_1\mathbb{Z}, \dots$ enumerate the cosets of $\mathbb{Z}$ in G, with $g_0 = 1_G$.

We construct by induction on n sequences s_n^i for all i, which are partial colorings for $g_i\mathbb{Z}$. The construction is a modification of the previous theorem.

First extend each s_n^i to t_n^i by concatenating copies of s_n^i so that the length of t_n^i is a multiple of $s = n + 1$.

Let $\chi_s(g_i) \in \{0, 1, 2\}$ be defined as follows.

Let $\bar{g}_i$ be the coset representative in $G/\mathbb{Z}$ of g_i with respect to the subgroup $\langle s \rangle \mathbb{Z}$.

Set $\chi_s(g_i) = 0$ if $g_i\mathbb{Z} = \bar{g}_i s^k \mathbb{Z}$ where k is even and if s has finite odd order r in $G/\mathbb{Z}$ then $k \neq r - 1$. If k is odd, set $\chi_s(g_i) = 1$, and if k is even and $k = r - 1$ with r odd, then set $\chi_s(g_i) = 2$.

Extend t_n^i to s_{n+1}^i by concatenating copies of t_n^i and its flip according to the pattern 010101010101 if $\chi_s(g_i) = 0$, according to 001001001001 if $\chi = 1$, and according to 000100010001 if $\chi = 2$.

This works since no sub-words of length 7 from concatenations of these patterns and their flips can be equal for the three different patterns.

Remark

The above argument is similar to an argument that showed that if $\mathbb{Z} \trianglelefteq G$ then G has a 2-coloring. This proof, however, did not generalize to show that all G have 2-colorings.

Similar aguments might show:

Theorem

Every Abelian group has a 7-free orbit.

Conjecture

Every countable G has an n-free orbit for some n.