Applications of Descriptive Set Theory to Functional Analysis and Dynamics

S. Jackson

Department of Mathematics
University of North Texas

August, 2013
SUMIRFAS, College Station

Overview

We discuss to applications of descriptive set theory to classifying complexity, one in functional analysis, and one in dynamics.

The first result concerns the complexity of the uniform homeomorphism relation between separable Banach spaces. It is joint with Su Gao and Bunyamin Sari.

The second result concerns the complexity of a natural ideal associated to attractors from iterated function systems.

Basic objects of study in analysis are the Polish spaces X, the complete separable metric spaces.

Recall $(X, \mathcal{B})$ is a standard Borel space if $\mathcal{B}$ is a σ-algebra on X such that $(X, \mathcal{B})$ is isomorphic to the algebra of Borel sets in some Polish space.

Any Borel subset of a Polish space is a standard Borel space.

Examples: $\mathbb{R}^n_{\text{std}}$, ω^ω, 2^ω, probability measures on a Polish space, separable Banach spaces, $F(X)$.

Any two uncountable Polish spaces are Borel isomorphic, so among the standard Borel spaces the only invariants are the cardinalities $1, 2, \ldots, \omega, c$.

In Polish spaces, one classifies the complexities of sets through the notion of a reduction.

Let X, Y be Polish spaces and $A \subseteq X$, $B \subseteq Y$. We say A is Wadge reducible to B, $A \leq_w B$ if there is a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$, that is,

$$x \in A \Leftrightarrow f(x) \in B$$

[This definition is reasonable if X, Y are 0-dimensional, otherwise can use coarser notion of Borel reduction].

For sets in ω^ω complexity classes fall into a wellorderded hierarchy via Wadge's lemma and the Martin-Monk theorems.

Wadge's Lemma: Assume sufficient determinacy. Then for $A, B \subseteq \omega^\omega$ either $A \leq_w B$ or $B \leq_w A$.

So, if we identify the degree of A with A^c, then the degrees are linearly orderded.

Martin-Monk: Assume sufficient determinacy. Then the Wadge degrees are wellorderded.

So, the Wadge hierarcy gives a complete classification of the complexity degrees for sets in Polish space.

We wish a notion of complexity to classify objects more general than subsets of a Polish space.

Let E be an equivalence relation on a Polish space X. The corresponding quotient space is X/E.

Objects of the form X/E represent all objects for which there is a surjection of $\mathbb{R}$ onto the object. We usually restrict to Borel equivalence relations E (though the results usually apply to arbitrary E given determinacy).

In general, there is no reasonable (definable) way to regard X/E as a subset of a standard Borel space.

We generalize the notion of reduction to introduce notion of complexity for equivalence relations on standard Borel spaces.

Let E, F be Borel equivalence relations on standard Borel spaces X, Y.

Definition

- We say E is reducible to F, $E \leq F$, if there is a Borel function $f : X \to Y$ such that $xE\, y \Leftrightarrow f(x)F\, f(y)$.
- We say E is bi-reducible with F, $E \sim F$, if $E \leq F$ and $F \leq E$.
- We say E is emdeddable into F, $E \sqsubseteq F$, if in addition f is one-to-one.

Note that a reduction gives a definable injection from X/E to Y/F so reduction can be viewed as a notion of definable cardinality for these quotient spaces.

We say E is a countable (Borel) equivalence relation if all classes of E are countable.

If G is a Polish group and G acts on X, then the orbit equivalence relation E_G is defined by

$$xE_G\, y \Leftrightarrow \exists g \in G\ (g \cdot x = y).$$

The Feldman-Moore theorem says that every countable Borel equivalence relation is given by the Borel action of a countable group G. The case $G = \mathbb{Z}$ is the classical case of discrete-time dynamics.

So, we can study the equivalence relations E_G group by group.

The simplest equivalence relations are the smooth or tame ones.

Definition

E is smooth if there is a Borel reduction of E to equality relation on a Polish space.

So, for a smooth E, X/E can be regarded as a subset of a standard Borel space.

For countable Borel E, smooth is the same as saying there is a Borel selector for E.

Definition

E_0 is the equivalence relation on 2^ω given by

$$xE_0\, y \Leftrightarrow \exists n\, \forall m \geq n\ (x(m) = y(m)).$$

The Harrington-Kechris-Louveau theorem says that if E is a Borel equivalence relation then either E is smooth or $E_0 \sqsubseteq E$.

So, there is no complexity class of equivalence relation strictly between the smooth relation $E_=$ and E_0.

If G is a Polish group, G acts of $F(G)$ by the shift action

$$g \cdot F = \{gf : f \in F\}$$

We can view this action as being on 2^G by

$$g \cdot x(h) = x(g^{-1}h)$$

We call this the Bernoulli (left) shift action of G on 2^G. When G is countable, 2^G is a compact Polish space in the natural product topology.

The Bernoulli shift is more or less universal for

Fact

Let E the the orbit equivalence relation for a Borel action of the countable group G on a Polish space X. Then

$$E \leq E((2^\omega)^G) \leq E(2^{G\times\mathbb{Z}}).$$

A theorem of Becker-Kechris says that for general Polish G the ω-product of the shift action of G on $F(G)$ is universal for Borel G-actions.

There is also a universal Polish group (e.g., $\mathrm{ISO}(\mathbb{U})$), so that there is a universal Polish group action.

Countable Equivalence Relations

Theorem (Dougherty-J-Kechris)

The shift action of F_2 on 2^{F_2} is a universal countable Borel equivalence relation, that is, $E \leq E_{F_2}$ for any countable Borel E.

Definition

A countable Borel equivalence relation E is hyperfinite if E is the increasing union of relations E_n with finite classes.

Theorem (Slaman-Steel)

The following are equivalent:

- *E is hyperfinite.*
- *$E = E_G$ where $G = \mathbb{Z}$.*
- *The classes of E can be uniformly Borel ordered in type $\mathbb{Z}$ (or are finite).*

Theorem (Weiss)

Every Borel action by $\mathbb{Z}^n$ is hyperfinite.

Theorem (Gao-J)

Every Borel action by a countable abelian group is hyperfinite.

Hyperfiniteness problem (Kechris): Is every equivalence relation generated by a Borel action of an amenable group hyperfinite?

Adams-Kechris showed that the structure of the countable Borel equivalence relations above E_0 is complicated.

Some Non-countable Relations

- E_0^ω: Does not embed into any $\mathbf{\Sigma}^0_3$ equivalence relation.
- ℓ_p, $1 \leq p \leq \infty$. Given by action of Polish group. By Dougherty-Hjorth $\ell_p \leq \ell_q$ for $p \leq q$.
- ℓ_∞ by Rosendal is universal for K_σ equivalence relations.
- E_1: Defined as E_0 except on $\mathbb{R}$.
- $=^+$: Equality on countable sets of reals. Embeds any countable equivalence relations, given by S_∞ action.

The collection $\mathcal{B}$ of separable Banach spaces can be viewed as a Polish space by identifying separable Banach spaces with closed subspaces of $C[0,1]$. The subset of $F(C[0,1])$ corresponding to linear subspaces is a G_δ.

The collection $\mathcal{B}_b$ of separable Banach spaces with basis (X, B) can also be viewed as a Polish space.

- One way is to use a universal basis, so we identify (X, B) with an $x \in 2^\omega$.
- Or can identify $(X, (b_i))$ with the sequence $\| \sum_{i=1}^{k} a_i^n b_i \|_X$, where $(a_1^n, \dots, a_k^n)$ enumerates $\mathbb{Q}^{<\omega}$. This identifies $\mathcal{B}_b$ with a closed subset of $\mathbb{R}^\omega$.

There are Borel functions taking codes for members of $\mathcal{B}_b$ in one coding to codes in the other coding.

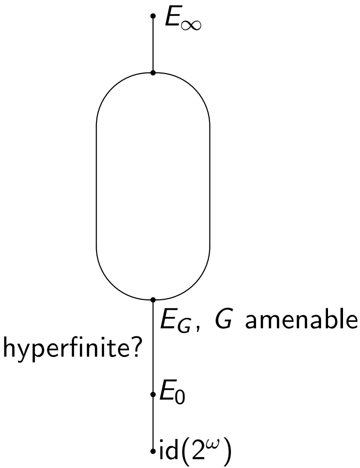
Some known classifications

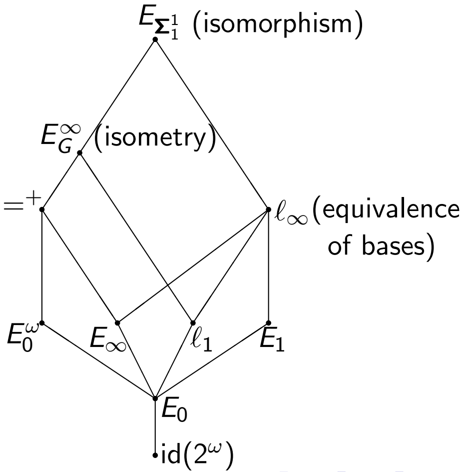
(Ferenczi, Louveau, Rosendal) Isomorphism on separable Banach spaces is bireducible with $\mathbf{\Sigma}^1_1$-complete equivalence relation.

(Rosendal) Equivalence of bases is complete K_σ equivalence relation (ℓ_∞).

(Melleray) Isometry between separable Banach spaces is universal orbit equivalence relation.

(Gao-J-Sari) Uniform homeomorphism between separable Banach spaces is at least ℓ_∞.


E_∞
E_G, G amenable
hyperfinite?
E_0
id(2^ω)


$E_{\boldsymbol{\Sigma}^1_1}$ (isomorphism)
E_G^∞ (isometry)
$=^+$
ℓ_∞ (equivalence of bases)
E_0^ω
E_∞
ℓ_1
E_1
E_0
id(2^ω)

Recall that for linearly isomorphic Banach spaces X, Y, the Banch-Mazur distance is defined by

$$d(X, Y) = \inf\{\|T\|\|T^{-1}\| \colon T \colon X \to Y \text{ is an isomorphism }\}$$

Definition

X, Y are locally equivalent if there is a $C > 0$ such that for every finite dimensional subspace E of X there is a subspace F of Y with $d(E, F) \leq C$ and vice-versa. We write $X \equiv_L Y$.

Theorem (Ribe)

If X, Y are uniformly homeomorphic then $X \equiv_L Y$.

Fact

Local equivalence is a $\boldsymbol{\Sigma}^3_0$ equivalence relation on $\mathcal{B}$ or on $\mathcal{B}_b$.

Corollary

E_0^ω, and hence $=^+$ is not reducible to $\equiv_L$ on either $\mathcal{B}$ or $\mathcal{B}_b$.

In fact we have:

Theorem (Gao-J-Sari)

$\equiv_L$ (on either $\mathcal{B}$ or $\mathcal{B}_b$) is bi-reducible with ℓ_∞.

On the other hand, local equivalence differs from uniform homeomorphism:

Theorem (GJS)

There is a single Borel class $\mathcal{C}$ of separable Banach spaces under uniform homeomorphism such that $=^+$ is reducible to the isomorphism relation on $\mathcal{C}$.

Sketch of proof of reducibility

Let $I_i = [l_i, r_i] \subseteq (1,2)$ with $|I_i| = \frac{1}{2^{i+1}}$, and let $n_i \in \mathbb{Z}^+$.

Let $\vec{p} = (p_i)$ with $p_i \in I_i$.

Let

$$S_{\vec{p}} = \left(\sum_{i=1}^{\infty} \ell_{p_i}^{n_i} \right)_2$$

Fix a sequence $b_i \in \mathbb{Z}^+$ with $\limsup b_i = \infty$.

Let $\sigma_i \colon [0, b_i] \to I_i$ be an affine bijection.

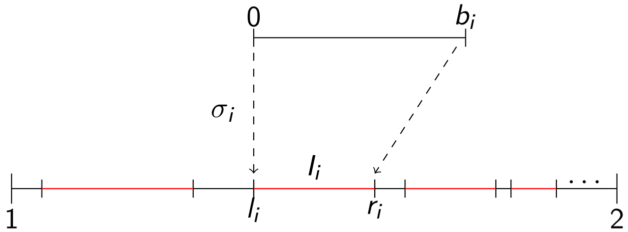


Figure: Basic setup for the embedding

- $I_i = [l_i, r_i]$, $|I_i| = 2^{-(i+1)}$
- $\sigma_i \colon [0, b_i] \to I_i$

We choose the n_i carefully, they must grow fast, but not too fast.

Let

$$n_i = 2^{b_i 2^i}$$

We can easily reduce $\ell_\infty(\mathbb{R})$ to $\ell_\infty([0, b_i])$, so we use the latter.

Given $x \in \prod[0, b_i]$, we define $\pi(x)$ to be the separable Banach space $\pi(x) = S_{\vec{p},\vec{n}}$ where $\vec{p} \in \prod I_i$ is given by

$$p_i = \sigma_i(x(i))$$

We check that π is a reduction from $\ell_\infty([0, b_i])$ to the uniform homeomorphism relation on $\mathcal{B}$.

Note that (we assume the b_i are increasing)

$$n_{i+1}^{1/r_{i+1}} \geq n_{i+1}^{1/2} = 2^{b_{i+1}2^i} \geq n_i \geq n_i^{1/l_i}$$

Lemma

$S_{\vec{p}}$, $S_{\vec{q}}$ are uniformly homeomorphic iff $\sup_i n_i^{|\frac{1}{p_i}-\frac{1}{q_i}|} < \infty$.

Proof. If $n_i^{|\frac{1}{p_i}-\frac{1}{q_i}|} < C$ for all i, then $d(\ell_{p_i}^{n_i}, \ell_{q_i}^{n_i}) \leq C$, just take the identity map.

Let $p = p_{i_0}$, $q = q_{i_0}$, $n = n_{i_0}$, and assume $p < q$.

Let $T\colon \ell_p^n \to S_{\vec{q}}$ be a linear embedding. We show that $\|T\|\|T^{-1}\| \geq C(n^{1/p-1/q})$ for some absolute constant C.

To do this we consider the type $2, n$ constants of ℓ_p^n and $S_{\vec{q}}$.

Recall the type p, n constant $T_{p,n}(X)$ of a Banach space X is the smallest constant such that for all n vectors $x_1, \dots, x_n \in X$ we have:

$$\left(\operatorname*{Ave}_{\epsilon_i=\pm 1} \|\sum_{i=1}^{n} \epsilon_i x_i\|^2\right)^{1/2} \leq T_{p,n}(X) \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}$$

X is of type p if $T_p(X) = \sup_n T_{p,n}(X) < \infty$.

If $T\colon X \to Y$ is a linear embedding then

$$T_{p,n}(X) \leq \|T\|\|T^{-1}\| T_{p,n}(Y).$$

We have (c.f. Tomczak-Jaegermann):

- $T_p(\ell_q^n) \geq n^{\max(0,1/q-1/p)}$
- $T_p(\ell_q^n) \leq cq^{1/2} n^{\max(0,1/q-1/p)} \quad (1 \leq q < \infty)$
- For $n \leq k$, $T_{p,n}(\ell_q^k) \leq cq^{1/2} n^{\max(0,1/q-1/p)}$

Fix $T\colon \ell_p^n \to S_{\vec{q}}$. We estimate $T_{2,n}(\ell_p^n)$.

On the one hand it is at least $n^{1/p-1/2}$.

On the other hand, it is $\leq \|T\|\|T^{-1}\|T_{2,n}(S_{\vec{q}})$.

A computation using the previous facts and $n_i^{2/q_i} < n^{2/q}$ for $i < i_0$ gives that

$T_{2,n}(S_{\vec{q}}) \leq \sqrt{2}cn^{1/q-1/p}$

and so $\|T\|\|T^{-1}\| \geq \frac{1}{\sqrt{2}c}n^{1/p-1/q}$.

This proves the lemma.

Suppose $\vec{x}, \vec{y} \in \prod [0, b_i]$.

$\pi(\vec{x}) = S_{\sigma_i(x(i)),\vec{n}}$ is uniformly homeomorphic to $\pi(\vec{y}) = S_{\sigma_i(y(i)),\vec{n}}$

$\Leftrightarrow \sup_i \; n_i^{\left|\frac{1}{\sigma_i(x(i))} - \frac{1}{\sigma_i(y(i))}\right|} < \infty$

$\Leftrightarrow \sup_i \; \log(n_i)|\sigma_i(x(i)) - \sigma_i(y(i))| < \infty$

$\Leftrightarrow \sup_i \; (b_i 2^i)(\frac{|x(i)-y(i)|}{b_i 2^{i+1}}) = \frac{1}{2} \sup_i |x(i) - y(i)| < \infty$

$\Leftrightarrow \vec{x} \; \ell_\infty \; \vec{y}$.

This shows $\ell_\infty = \ell_\infty(\mathbb{R})$ reduces to uniform homeomorphism on separable Banach spaces.

We also get that ℓ_∞ is bi-reducible with $\equiv_L$.

For this we need the reverse reduction from $\equiv_L$ to ℓ_∞.

This follows from the following fact.

For any Polish (X, d) let $\vec{x} F_X \vec{y}$ iff the d-Hausdorff distance between $\{x(i)\}$ and $\{y(i)\}$ is finite.

Fact

For any Polish (X, d), $F_X \leq \ell_\infty$.

Dynamics

Let $f_i \colon \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \dots, k$, be contraction mappings, that is $\|f_i(x) - f_i(y)\| \leq c\|x - y\|$ where $c < 1$.

There is a unique compact set $K = K(f_1, \dots, f_n)$ such that $K = f_1(K) \cup \dots \cup f_k(K)$.

We say $\vec{f}$ satisfies the porosity condition if there is a $p > 0$ such that for all balls B of radius r there is a sub-ball $B' \subseteq B$ of radius pr such that $B' \cap K = \emptyset$.

If we fix a ball $B(r)$ and a $c < 1$, then the set of $\vec{f}$ which are contractions of $B(r)$ with constant c and is a Polish space $F(r, c)$, and the set of $\vec{f}$ satisfying the porosity condition is a closed subset of $F(r, c)$.

We let *IFS* denote the $K \in F(\mathbb{R}^n)$ of the form $K = K(\vec{f})$ for some iterated function system $\vec{f}$ satisfying the porosity condition.

We let I_{ifs} denote the σ-ideal generated by the sets in *IFS*. So, $I_{\text{ifs}} \subseteq X$ is a σ-ideal of closed subsets of $\mathbb{R}^n$.

We recall the following dichotomy theorem of Kechris and Woodin:

Theorem (Kechris,Woodin)

Let $\mathcal{I}$ be a $\mathbf{\Pi}^1_1$ σ-ideal of closed sets in a compact metric space. Then either $\mathcal{I}$ is $\mathbf{\Pi}^0_2$ or $\mathcal{I}$ is $\mathbf{\Pi}^1_1$-complete.

Every $F \in I_{\text{ifs}}$ is meager. However the meager ideal is $\mathbf{\Pi}^0_2$. A straightforward computation shows that I_{ifs} is $\mathbf{\Sigma}^1_2$.

Theorem

I_{ifs} is $\mathbf{\Pi}^1_1$-complete.

More generally, Let $\mathcal{P} = \{p_\alpha\}_{\alpha<\lambda}$ be a family of porosity functions, $p_\alpha \colon \omega \to \omega$.

Let $I^{\mathcal{P}}_{\text{ifs}}$ be the σ-ideal generated by the $K(\vec{f})$ which have a porosity function dominated by a $p_\alpha \in \mathcal{P}$.

Theorem

Let $\mathcal{P}$ be be given and assume there is a $g\colon \omega \to \omega$ such that

$$\forall p \in \mathcal{P}\ \exists^\infty n\ (\frac{p(n+1)}{p(n)} < g(n)).$$

Then $I_{ifs}^{\mathcal{P}}$ is $\boldsymbol{\Pi}^1_1$-hard.

Corollary

There is an uncountable Borel $\mathcal{P}$ such that $I_{\text{ifs}}^{\mathcal{P}}$ is $\boldsymbol{\Pi}^1_1$-hard.

When $\mathcal{P}$ is countable we get the upper bound: $I_{\text{ifs}}^{\mathcal{P}}$ is $\boldsymbol{\Pi}^1_1$.

So we have:

- If $|\mathcal{P}| < \mathfrak{b}$, then $I^{\mathcal{P}}_{\text{ifs}}$ is $\mathbf{\Pi}^1_1$ hard.
- If $\mathcal{P}$ is countable then $I^{\mathcal{P}}_{\text{ifs}}$ is $\mathbf{\Pi}^1_1$-complete.
- If $\mathcal{P} = \omega^\omega$ then $I^{\mathcal{P}}_{\text{ifs}}$ is $\mathbf{\Pi}^0_2$.
- There are uncountable Borel $\mathcal{P}$ for which $I^{\mathcal{P}}_{\text{ifs}}$ is $\mathbf{\Pi}^1_1$ hard.

Corollary

The ideal I^{C}_{ifs} of all $K(\vec{f})$ which have a computable porosity function is $\mathbf{\Pi}^1_1$-complete.

Some questions

Question

Are there any Δ^1_1 $\mathcal{P}$ for which $I^{\mathcal{P}}_{\text{ifs}}$ is $\mathbf{\Sigma}^1_2$-complete?

Question

For which Δ^1_1 $\mathcal{P}$ is $I^{\mathcal{P}}_{\text{ifs}}$ $\mathbf{\Pi}^0_2$, $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$-complete?

Enough to consider $n = 1$, in fact work in $[0, 1]$.

Fix a reasonable homeomorphism h between ω^ω and $\mathrm{Irr} \subseteq [0, 1]$.

Let Tr be the Polish space of trees on ω and $W \subseteq Tr$ the set of wellfounded trees. So, W is $\mathbf{\Pi}^1_1$-complete.

We define a continuous map $\pi\colon T \to K([0, 1])$ reducing W to I_{ifs}.

If $T \in W$ then $\pi(T)$ will be countable, and hence in I_{ifs}. If $T \notin W$ then we will have $\pi(T) \subseteq \pi_0(T) \cup \mathbb{Q}$ where $\pi_0(T) \subseteq \mathrm{Irr} \cap [0, 1]$ is a closed subset of Irr.

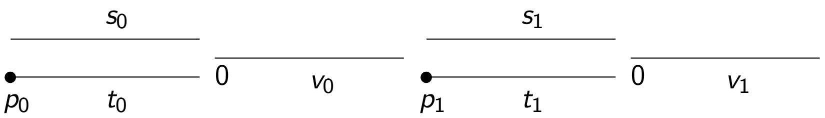
We actually define $\pi_0 \colon Tr \to Tr$.

Fix a function $b \colon \mathbb{Q} \times \omega^{<\omega} \to \omega$ such that for rational porosity number p and $s \in \omega^{<\omega}$, we have that for any iterated function system $\vec{f}$ with porosity $\geq p$ there is a sequence $t \in \omega^{<\omega}$ of length $\leq b(p, s)$ such that $h(s^\frown t) \cap K(\vec{f}) = \emptyset$.

This uses the fact that h is reasonable. Actually, $b(p, s)$ depends only on p and $|s|$.

For each n fix a continuous surjection $h_{1/n}$ from ω^ω to the Polish space of iterated function systems on $[0, 1]$ with porosity $p \geq \frac{1}{n}$ and $c \leq 1 - \frac{1}{n}$.

Sketch of proof of lower bound



- s_i build a branch through T.
- t_i build an IFS. Let $\bar{t} = \bar{t}_0{}^\frown \bar{t}_1{}^\frown \cdots$ (drop first digits) and let $t_i' = (\bar{t})_i = \bar{t}_{\langle i,0\rangle}{}^\frown \bar{t}_{\langle i,1\rangle} \cdots$.
- v_i get out of $K(\vec{f})$.
- $h_{1/t_i(0)}(t_i')$ is consistent with being an iterated function system $\vec{f}$ such that $K(\vec{f}) \cap N_{u_0{}^\frown v_0{}^\frown \cdots {}^\frown v_i} = \emptyset$.
- $|v_{i+1}| = b(p, u_0{}^\frown v_0{}^\frown \cdots {}^\frown u_{i+1})$ where $p = \frac{1}{t_{i+1}(0)}$.