Examples of Ordinal Arithmetic

We'll use frequently the exercise that $\omega^\alpha + \omega^\beta = \omega^\beta$ if $\alpha < \beta$.

Lemma 0.1. *$(\omega^{\alpha_0} \cdot k_0 + \omega^{\alpha_1} \cdot k_1 + \cdots \omega^{\alpha_n} \cdot k_n) \cdot \beta = (\omega^{\alpha_0} \cdot k_0 \cdot \beta)$, if β is a limit and $= (\omega^{\alpha_0} \cdot k_0 \cdot \beta) + (\omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_n} \cdot k_n)$ if β is a successor (T stands for "tail.")*

Proof. By induction on β. For $\beta = 0$ it is trivial (provided we consider 0 a limit ordinal!; it is also trivial for $\beta = 1$ if you prefer to start there). Let α denote $\omega^{\alpha_0} \cdot k_0 + \omega^{\alpha_1} \cdot k_1 + \cdots \omega^{\alpha_n} \cdot k_n$. Let T (for "tail") denote $\omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_n} \cdot k_n$.

Case I.) β is a successor, say $\beta = \gamma + 1$. Then $\alpha \cdot \beta = \alpha \cdot (\gamma + 1) = \alpha \cdot \gamma + \alpha = (\omega^{\alpha_0} \cdot k_0 \cdot \gamma) + ?T + (\omega^{\alpha_0} \cdot k_0 + \omega^{\alpha_1} \cdot k_1 + \cdots \omega^{\alpha_n} \cdot k_n)$ where $?T$ denotes T if γ is a successor and denotes 0 if γ is a limit. From the exercise $T + \omega^{\alpha_0} \cdot k_0 = \omega^{\alpha_0} \cdot k_0$, so in either case this becomes $= \omega^{\alpha_0} \cdot k_0 \cdot \gamma + \omega^{\alpha_0} \cdot k_0 + T = \omega^{\alpha_0} \cdot k_0 \cdot (\gamma + 1) + T = \omega^{\alpha_0 + 1} \cdot k_0 \cdot \beta + T$ which is what is claimed.

Case II.) β is a limit. Then

$$\alpha \cdot \beta = \alpha \cdot (\sup_{\beta' < \beta} \beta) = \sup_{\beta' < \beta} \alpha \cdot \beta' = \sup_{\beta' < \beta} (\omega^{\alpha_0} \cdot k_0 \cdot \beta' + T),$$

where we have assumed without loss of generality that β' is a successor ordinal (every limit is a limit of successors). Now $\omega^{\alpha_0} \cdot k_0 \cdot \beta' + T < \omega^{\alpha_0} \cdot k_0 \cdot \beta' + T + \omega^{\alpha_0} \cdot k_0 = \omega^{\alpha_0} \cdot k_0 \cdot \beta' + \omega^{\alpha_0} \cdot k_0 = \omega^{\alpha_0} \cdot k_0 \cdot (\beta' + 1)$. Since $\beta' + 1 < \beta$ whenever $\beta' < \beta$ (as β is a limit) the supremum is bounded by $\sup_{\beta' < \beta} (\omega^{\alpha_0} \cdot k_0 \cdot (\beta' + 1)) = \omega^{\alpha_0} \cdot k_0 \cdot \beta$. Clearly this is also a lower bound for the supremum, and we are done. □

Using this lemma, ordinal multiplication is now straightforward. For example, here is the example we did in class:

$$\begin{aligned}
&(\omega^{\omega^2 + \omega} \cdot 3 + \omega^2 \cdot 3 + 11) \cdot (\omega^{\omega^2} \cdot 2 + \omega^\omega \cdot 2 + 17) \\
&\quad = \omega^{\omega^2 + \omega} \cdot 3 \cdot (\omega^{\omega^2} \cdot 2 + \omega^\omega \cdot 2 + 17) + \omega^2 \cdot 3 + 11 \\
&\quad = \omega^{\omega^2 + \omega} \cdot (\omega^{\omega^2} \cdot 2 + \omega^\omega \cdot 2 + 51) + \omega^2 \cdot 3 + 11 \\
&\quad = \omega^{\omega^2 \cdot 2} \cdot 2 + \omega^{\omega^2 + \omega \cdot 2} \cdot 2 + \omega^{\omega^2 + \omega} \cdot 51 + \omega^2 \cdot 3 + 11
\end{aligned}$$

which is now in Cantor normal form.

Now let's turn to exponentiation. To warm up, let's first compute α^m where $m \in \omega$.

Lemma 0.2. $(\omega^{\alpha_0} \cdot k_0 + \omega^{\alpha_1} \cdot k_1 + \cdots \omega^{\alpha_n} \cdot k_n)^m = \omega^{\alpha_0 \cdot m} \cdot k_0 + \omega^{\alpha_0 \cdot (m-1) + \alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_0 \cdot (m-1) + \alpha_{n-1}} \cdot k_{n-1} + \begin{cases} \omega^{\alpha_0 \cdot (m-1) + \alpha_n} \cdot k_n & \textit{if } \alpha_n > 0 \\ \alpha^{m-1} \cdot k_n & \textit{if } \alpha_n = 0 \end{cases}$

Note that if $\alpha_n = 0$ (i.e., α is a successor ordinal), then the last term requires us to recursively compute α^{m-1}.

Proof. First assume that $\alpha_n > 0$, so α is a limit. Since $\alpha^m = \alpha^{m-1} \cdot \alpha$, and α is a limit ordinal, applying the multiplication rule $m - 1$ times gives

$$\begin{aligned}
\alpha^m =& \alpha^{m-1} \cdot \alpha = (\omega^{\alpha_0} \cdot k_0)^{m-1} \cdot \alpha = (\omega^{\alpha_0} \cdot k_0)^{m-1} \cdot (\omega^{\alpha_0} \cdot k_0 + \cdots + \omega^{\alpha_n} \cdot k_n) = \\
&\omega^{\alpha_0 \cdot m} \cdot k_0 + \omega^{\alpha_0 \cdot (m-1) + \alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_0 \cdot (m-1) + \alpha_n} \cdot k_n
\end{aligned}$$

Next assume that $\alpha_0 = 0$, so that last term is k_n and hence α is a successor ordinal. Let $H = \omega^{\alpha_0} \cdot k_0 + \cdot + \omega^{\alpha_{n-1}} \cdot k_{n-1}$ (H stands for "head"). Then $\alpha^m = \alpha^{m-1} \cdot (H + k_n) = \alpha^{m-1} \cdot H + \alpha^{m-1} \cdot k_n$. Since H is a limit ordinal, the first term, using the multiplication lemma, becomes $(\omega^{\alpha_0} \cdot k_0))^{m-1} \cdot H = \omega^{\alpha_0 \cdot m} \cdot k_0 + \omega^{\alpha_0 \cdot (m-1)+\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_0 \cdot (m-1)+\alpha_{n-1}} \cdot k_{n-1}$ and the result follows. $\square$

Example. Let's compute $(\omega^{\omega^2+\omega} \cdot 3 + \omega^\omega \cdot 7 + 2)^3$. Let α denote $\omega^{\omega^2+\omega} \cdot 3 + \omega^\omega \cdot 7 + 2$. Using the previous lemma we have:

$$
\begin{aligned}
&(\omega^{\omega^2+\omega} \cdot 3 + \omega^\omega \cdot 7 + 2)^3 = \omega^{(\omega^2+\omega)\cdot 3} \cdot 3 + \omega^{(\omega^2+\omega)\cdot 2+\omega} \cdot 7 + \alpha^2 \cdot 2 \\
&\quad = \omega^{\omega^2 \cdot 3+\omega} \cdot 3 + \omega^{\omega^2 \cdot 2+\omega \cdot 2} \cdot 7 + \alpha^2 \cdot 2 \\
&\quad = \omega^{\omega^2 \cdot 3+\omega} \cdot 3 + \omega^{\omega^2 \cdot 2+\omega \cdot 2} \cdot 7 + \omega^{(\omega^2+\omega)\cdot 2} \cdot 3 \cdot 2 + \omega^{\omega^2+\omega+\omega} \cdot 7 \cdot 2 + \alpha \cdot 4 \\
&\quad = \omega^{\omega^2 \cdot 3+\omega} \cdot 3 + \omega^{\omega^2 \cdot 2+\omega \cdot 2} \cdot 7 + \omega^{\omega^2 \cdot 2+\omega} \cdot 6 + \omega^{\omega^2+\omega \cdot 2} \cdot 14 + \alpha \cdot 4 \\
&\quad = \omega^{\omega^2 \cdot 3+\omega} \cdot 3 + \omega^{\omega^2 \cdot 2+\omega \cdot 2} \cdot 7 + \omega^{\omega^2 \cdot 2+\omega} \cdot 6 + \omega^{\omega^2+\omega \cdot 2} \cdot 14 + \omega^{\omega^2+\omega} \cdot 12 + \omega^\omega \cdot 28 + 8
\end{aligned}
$$

Computing α^β when β is a limit ordinal is easier.

Lemma 0.3. *If β is a limit, then $(\omega^{\alpha_0} \cdot k_0 + \cdots + \omega^{\alpha_n} \cdot k_n)^\beta = \omega^{\alpha_0 \cdot \beta}$.*

Proof. We prove this by induction on β. For $\beta = \omega$, note that $(\omega^{\alpha_0} \cdot k_0 + \cdots + \omega^{\alpha_n} \cdot k_n)^m$ lies between $\omega^{\alpha_0 \cdot m}$ and $(\omega^{\alpha_0} \cdot (k_0+1))^m = \omega^{\alpha_0 \cdot m} \cdot (k_0 + 1) < \omega^{\alpha_0 \cdot m+1}$, and both of these sup up to $\omega^{\alpha_0 \cdot \omega}$.

If β is a limit of limit ordinals, then using induction we have

$$(\alpha)^\beta = \sup_{\substack{\beta'<\beta \\ \beta' \text{ limit}}} \omega^{\beta'} = \sup_{\substack{\beta'<\beta \\ \beta' \text{ limit}}} \omega^{\alpha_0 \cdot \beta'} = \omega^{\alpha_0 \cdot \beta}$$

If β is not a limit of limit ordinals, then $\beta = \beta' + \omega$ for some limit ordinal β'. Then

$$\alpha^\beta = \alpha^{\beta'+\omega} = \alpha^{\beta'} \cdot \alpha^\omega = \omega^{\alpha_0 \cdot \beta'} \cdot \omega^{\alpha_0 \cdot \omega} = \omega^{\alpha_0 \cdot \beta' + \alpha_) \cdot \omega} = \omega^{\alpha_0 \cdot (\beta'+\omega)} = \omega^{\alpha_0 \cdot \beta}.$$

$\square$

Since every ordinal β is of the form $\lambda + m$ for some limit ordinal λ, we now have complete rules for ordinal exponentiation.

Example. We compute α^β where $\alpha = \omega^{\omega^2+\omega} \cdot 3 + \omega^\omega \cdot 7 + 2$ (as in the above example) and $\beta = \omega^3 + 3$.

$$
\begin{aligned}
&\alpha^\beta = \alpha^{\omega^3} \cdot \alpha^3 = \omega^{(\omega^2+\omega)\cdot\omega^2} \cdot \alpha^3 = \omega^{\omega^4} \cdot \alpha^3 = \\
&\omega^{\omega^4+\omega^2 \cdot 3+\omega} \cdot 3 + \omega^{\omega^4+\omega^2 \cdot 2+\omega \cdot 2} \cdot 7 + \omega^{\omega^4+\omega^2 \cdot 2+\omega} \cdot 6 + \omega^{\omega^4+\omega^2+\omega \cdot 2} \cdot 14 \\
&\quad + \omega^{\omega^4+\omega^2+\omega} \cdot 12 + \omega^{\omega^4+\omega} \cdot 28 + \omega^4 \cdot 8
\end{aligned}
$$