V_κ, H_κ, and HOD

Recall that working in ZF (in fact, ZF$-$Power) we have previously defined V_α for all $\alpha \in$ ON. Recall that V_α consists of all sets of rank $< \alpha$ (remember V_α is a set). The next lemma shows what axioms of set theory hold in these sets.

Lemma 0.1. *Assume ZF. Then for any limit ordinal $\alpha > \omega$, $V_\alpha \models ZF-Replacement$. Assuming ZFC, for any limit ordinal $\alpha > \omega$, $V_\alpha \models ZFC-Replacement$.*

Proof. The empty set axiom holds in V_α as $\varnothing \in V_\alpha$ and the statement $\phi(x) = \neg \exists y \in x$ is Δ_0 so absolute. If $x, y \in V_\alpha$ then $\{x, y\}$ in V_α as α is a limit. Since the statement $\phi(x, y, z) = [x \in z \land y \in z \land \forall w \in z \ (w \approx x \lor w \approx y)]$ is Δ_0, hence absolute, it follows that $V_\alpha \models$ pairing. The union axiom is similar. By downwards absoluteness, foundation holds in any class (since we assuming foundation holds in V), so it holds in V_α. Since $\alpha > \omega$, $\omega \in V_\alpha$. As the statement $\phi(x) =$ "$x = \omega$" is absolute, $V_\alpha \models$ infinity. Extensionality holds in V_α as V_α is transitive. If $x \in V_\alpha$, then $\mathcal{P}(x) \in V_\alpha$ as α is a limit. The statement $\phi(x, y) = [y = \mathcal{P}(x)]$, written out, is Π_1 and hence downwards absolute. Since $v \models \phi(x, \mathcal{P}(x))$, it follows that $V_\alpha \models \phi(x, \mathcal{P}(x))$. Hence $V_\alpha \models$ Power set. To see comprehension, let $\phi(x_1, \ldots, x_n, y, z)$ be a formula and let $a_1, \ldots, a_n, b \in V_\alpha$. By comprehension in V applied to the formula $\phi^{V_\alpha}(\vec{x}, y, z)$ (which has the extra parameter V_α in it) there is set w such that $\forall z \ (z \in w \leftrightarrow z \in b \land \phi^{V_\alpha}(\vec{a}, b, z))$. Since $b \in V_\alpha$, we also have $w \in V_\alpha$. Hence $V_\alpha \models \exists w \ \forall z \ (z \in w \leftrightarrow z \in b \land \phi(\vec{a}, b, z))$, and so $V_\alpha \models$ Comprehension. Finally, suppose AC holds in V. Let $a \in V_\alpha$. By AC in V, let $\prec$ be a wellordering of a. Since $\prec \in \mathcal{P}\mathcal{P}\mathcal{P}(a)$ and α is limit, $\prec \in V_\alpha$. The statement $\phi(x, y) = [y$ is a wellordering of$x]$ is Π_1, and since it holds in V it holds in V_α. Thus, $V_\alpha \models$ AC. $\qquad\square$

Thus, in ZF we can prove that there are set models for all of ZF except replacement. More precisely, for any formulas ϕ of ZF$-$Replacement, ZF $\vdash \forall \alpha(\alpha > \omega \land \alpha$ limit$\rightarrow \phi^{V_\alpha})$.

Corollary 0.2. *ZF$-$Replacement $\nvdash$ ZF.*

Proof. Replacement clearly does not hold in $V_{\omega \cdot 2}$. Let ϕ be an instance of replacement so that ZF $\vdash (\neg \phi)^{V_{\omega \cdot 2}}$. If ZF$-$Replacement $\vdash$ ZF, then there would be finitely many axioms $\{\psi_1, \ldots, \psi_n\}$ of ZF$-$Replacement such that $\{\psi_1, \ldots, \psi_n\} \vdash \phi$. But, ZF $\vdash \psi_1^{V_{\omega \cdot 2}} \land \cdots \land \psi_n^{V_{\omega \cdot 2}}$, and so ZF $\vdash \phi^{V_{\omega \cdot 2}}$, a contradiction. $\qquad\square$

In the proof of corollary 0.2 we have implicitly assumed the consistency of ZF, though we could get by just assuming the consistency of ZF$-$Replacement.

Exercise 1. Just assuming the consistency of ZF$-$Replacement, show that ZF$-$Replacement $\nvdash$ ZF. (hint: let ϕ be the instance of replacement which gives the existence of $\omega \cdot 2$. If ZF$-$Replacement $\nvdash \phi$, we're done, so assume ZF$-$Replacement $\vdash \phi$. Now follow the above prove for this ϕ.)

Exercise 2. Let $\psi_1, \ldots, \psi_n\}$ be axioms of ZF. Show that ZF$-$Replacement $\cup \{\psi_1, \ldots, \psi_n\} \nvdash$ ZF. (hint: Working in ZF, show that there is a least limit ordinal $\alpha > \omega$ such that $V_\alpha \models \psi_1 \land \cdots \land \psi_n$. Argue then that there must be a smaller limit ordinal $\beta < \alpha$ such that $V_\beta \models \psi_1 \land \cdots \land \psi_n$.)

We now define another family of sets called the H_κ. If P is any property, we say set x hereditarily has property P if every element of the transitive closure of x has property P.

Definition 0.3. (ZFC) Let κ be an infinite regular cardinal. H_κ is the collection of sets which hereditarily have size less than κ, that is, every element of the transitive closure of x has cardinality $< \kappa$.

We first give a simple reformulation of the definition.

Lemma 0.4. *For all sets x, $x \in H_\kappa$ iff $|\operatorname{tr cl}(x)| < \kappa$.*

Proof. Clearly if $|\operatorname{tr cl}(x)| < \kappa$ then $x \in H_\kappa$. Suppose now $x \in H_\kappa$. Recall $\operatorname{cl}_n(x)$ is defined by: $\operatorname{cl}_0(x) = x$, $\operatorname{cl}_{n+1}(x) = \operatorname{cl}_n(x) \cup (\cup \operatorname{cl}_n(x))$. If we assume inductively that $|\operatorname{cl}_n(x)| < \kappa$, then $\operatorname{cl}_{n+1}(x)$ is a $< \kappa$ union of sets each of which has size $< \kappa$ by definition of H_κ. Since κ is regular, it follows that $|\operatorname{cl}_{n+1}(x)| < \kappa$. Finally, $\operatorname{tr cl}(x) = \bigcup_n \operatorname{cl}_n(x)$ and using the regularity of κ again we have $|\operatorname{tr cl}(x)| < \kappa$. $\square$

The definition of H_κ shows it is a class, but we now show that is in fact a set.

Lemma 0.5. *For all infinite regular cardinals κ, $H_\kappa \subseteq V_\kappa$.*

Proof. Suppose $x \notin V_\kappa$, so $|x| \geqslant \kappa$. We know from lemma **??** that for every $\alpha < |x|$, there is a $y \in \operatorname{tr cl}(x)$ such that $|y| = \alpha$. This clearly forces $|\operatorname{tr cl}(x)| \geqslant \kappa$, so $x \notin H_\kappa$. $\square$

One could give another proof of lemma 0.5 as follows. Let $x \in H_\kappa$. Using AC, let $\lambda < \kappa$ be the cardinality of $\operatorname{tr cl}(x)$, and let $f\colon \lambda \to \{x\} \cup \operatorname{tr cl}(x)$ be a bijection. Define the relation E on λ by $\alpha E \beta$ iff $f(\alpha) \in f(\beta)$. Thus, (λ, E) "codes" the set $\{x\} \cup \operatorname{tr cl}(x)$. Clearly $\lambda, E \in V_\kappa$. Let π be the collapse map for (λ, E). We show by induction on E that for all $\alpha \in \lambda$ that $\pi(\alpha) \in V_\kappa$. We have $\pi(\alpha) = \{\pi(\beta)\colon \beta \in \lambda \wedge \beta E \alpha\}$. By induction, each $\pi(\beta)$ in the above expression is in V_κ, so $\pi(\alpha) \subseteq V_\kappa$. Since κ is regular and $\lambda < \kappa$, $\pi(\alpha) \subseteq V_\gamma$ for some $\gamma < \kappa$, and hence $\pi(\alpha) \in V_{\gamma+1} \subseteq V_\kappa$. In particular, $x \in V_\kappa$.

Note that if $x \subseteq H_\kappa$, then to show $x \in H_\kappa$ it is enough to show that $|x| < \kappa$. Also, since each ordinal is transitive, clearly $H_\kappa \cap \operatorname{ON} = \kappa$.

The next lemma shows that ZFC$-$Power holds in the H_κ.

Lemma 0.6. *(ZFC) For every axiom ϕ of ZFC$-$Power, ZFC$\vdash \forall \kappa \, [\kappa$ regular $\to \phi^{H_\kappa}]$.*

Proof. We work in ZFC. Let κ be an infinite regular cardinal. We show that all the axioms of ZFC except power hold in H_κ. Note for the following arguments that by definition H_κ is transitive. Foundation holds in H_κ as it holds in any set or class. Clearly, $\varnothing \in H_\kappa$, and by absoluteness then H_κ satisfies the emptyset axiom. If $x, y \in H_\kappa$, then $|\{x, y\}| < \omega < \kappa$, so $\{x, y\} \in H_\kappa$. By absoluteness, H_κ satisfies the pairing axiom. If $x \in H_\kappa$ then $\cup x \subseteq \operatorname{tr cl}(x)$, so $\operatorname{tr cl}(\cup x) \subseteq \operatorname{tr cl}(x)$, ansd so $|\operatorname{tr cl}(\cup(x))| \leqslant |\operatorname{tr cl}(x)| < \kappa$. Thus, $\cup(x) \in H_\kappa$. By absoluteness, H_κ satisfies the union axiom. Since $\kappa > \omega$, $\omega \in H_\kappa$ as we noted above. By absoluteness, H_κ satisfies the infinity axiom. Since H_κ is transitive, it satisfies extensionality (if $x \neq y$ are in H_κ, then there is a $z \in x$, say, with $z \notin y$. By transitivity, $z \in H_\kappa$).

To show comprehension, let $\phi(x_1, \ldots, x_n, y, z)$ be a formula, and let $a_1, \ldots, a_n, w \in H_\kappa$. We must show that $\exists v \in H_\kappa \; \forall y \in H_\kappa \; (y \in v \leftrightarrow (y \in w \wedge \phi^{H_\kappa}(\vec{a}, y, w)))$. By comprehension in V applied to the formula $\phi^{H_\kappa}(\vec{x}, y, z)$, there is a $v \in V$ such that $\forall y \; (y \in v \leftrightarrow (y \in w \wedge \phi^{H_\kappa}(\vec{a}, y, w)))$. In particular, $\forall y \in H_\kappa \; (y \in v \leftrightarrow (y \in w \wedge \phi^{H_\kappa}(\vec{a}, y, w)))$. Thus, it suffices to observe that $v \in H_\kappa$, which follows as $v \subseteq w \in H_\kappa$ (any subset of an element of H_κ is in H_κ).

To show replacement, let $\phi(x_1, \ldots, x_n, y, z, w)$ be a formula, $a_1, \ldots, a_n, A \in H_\kappa$, and assume $(\forall y \in A \; \exists z \; \phi(\vec{a}, y, z, A))^{H_\kappa}$. Thus, $\forall y \in A \; \exists z \in H_\kappa \; \phi^{H_\kappa}(\vec{a}, y, z, A)$. Applying replacement in V to the formula $[\phi^{H_\kappa} \wedge z \in H_\kappa]$, there is a set B such that $\forall y \in A \; \exists z \in B \cap \in H_\kappa \; \phi^{H_\kappa}(\vec{a}, y, z, A)$. By AC, there is a $B' \subseteq B$ with $|B'| \leqslant |A|$ such that $\forall y \in A \; \exists z \in B' \cap \in H_\kappa \; \phi^{H_\kappa}(\vec{a}, y, z, A)$. Since $B' \subseteq H_\kappa$ and $|B'| < \kappa$, we have $B' \in H_\kappa$. Thus, $(\exists B \; \forall y \in A \; \exists z \in B \; \phi(\vec{a}, y, z, A))^{H_\kappa}$.

Finally, to see AC holds in H_κ, let $a \in H_\kappa$. By AC in V, let $<$ be a wellordering of a. Since $< \; \subseteq a \times a$, $|<| < \kappa$. Easily $< \; \subseteq H_\kappa$, as every ordered pair $\langle x, y \rangle$ is in H_κ if $x, y \in H_\kappa$. So, $< \; \subseteq H_\kappa$, and hence $< \; \in H_\kappa$. By downwarda absoluteness, $(<$ is a wellordering$)^{H_\kappa}$. $\qquad\square$

Corollary 0.7. *ZFC$-$Power $\nvdash$ Power.*

Proof. Similar to the proof of corollary 0.2, using now the fact that ZFC $\vdash (\neg \; \text{Power})^{H_{\omega_1}}$. $\qquad\square$

Lemmas 0.1, 0.6 are stated and proved in the metatheory, however me may formalize their statements and proofs within ZFC. Lemma 0.6, for example, then becomes the statement ZF $\vdash \forall\alpha\forall n \; ((\alpha > \omega \; \text{limit} \wedge \theta(n) \to \rho(n, V_\alpha)))$ where θ is the formula which represents the relation $R(n) \leftrightarrow n$ codes an axiom of ZF$-$Replacement. [The only thing slightly problematic in the formalizations is where we consider the relativizations $\phi^{V_\kappa}, \phi^{H_\kappa}$. But lemma **??** gives us a formula ρ which expresses the formalizations of these notions, e.g., the formalization of $(\theta_n)^{V_\kappa}$ is the statement $\rho(n, V_\kappa)$.]

Here ρ is the set satisfaction formula of lemma **??**. Likewise, lemma 0.6 when formailzed becomes the statement ZFC $\vdash \forall\kappa\forall n \; ((\kappa \; \text{an uncountable regular cardinal}) \wedge \theta'(n) \to \rho(n, H_\kappa)))$, where now θ' represents $R'(n) \leftrightarrow n$ codes an axiom of ZF$-$Power.

The completeness theorem of first order logic may also be formailzed within ZFC (or ZF for countable theories, which we consider here). It then becomes the statement that ZF $\vdash \forall T \subseteq \omega \; (\text{CON}(T) \leftrightarrow \exists M \; \forall n \in T \; \rho(n, M))$. Using this, we may reformulate the formalzed lemmas 0.1, 0.6 as consistency results as follows.

Corollary 0.8. *ZFC$\vdash$ CON(ZFC$-$Replacement).*

Corollary 0.9. *ZFC$\vdash$ CON(ZFC$-$Power).*

From Gödel's theorem, we know that ZFC $\nvdash$ CON(ZFC), but these corollaries show that if we weaken ZFC a bit, we can prove its formal consistency within ZFC.

We showed that $H_\kappa \subseteq V_\kappa$ for all infinite regular κ. When does equality hold? Now $V_\kappa \subseteq H_\kappa$ iff $\forall\alpha < \kappa \; (V_\alpha \in H_\kappa)$ (since if $V_\alpha \in H_\kappa$ then any subset of V_α is in H_κ). Since each V_α is transitive, this is then equivalent to $\forall\alpha < \kappa \; (|V_\alpha| < \kappa)$. By definition of the $\beth$ function, $|V_{\omega+\alpha}| = \beth(\alpha)$. So, $V_\kappa = H_\kappa$ iff $\forall\alpha < \kappa \; \beth(\alpha) < \kappa$. Now $\beth(\alpha) \geqslant \alpha$, so $\beth(\alpha+1) \geqslant 2^{|\alpha|}$ for all α. Thus, for V_κ to equal H_κ we must have $\forall\lambda < \kappa \; 2^\lambda < \kappa$. Since κ is assumed regular, we must then have that κ is strongly inaccessible. Conversely, if κ is strongly inaccessible, the following exercise shows this condition is satisfied.

Exercise 3. Show that if κ is strongly inaccessible then $\forall\alpha < \kappa \; (\beth(\alpha) < \kappa)$.

We thus have:

Fact 0.10. For all infinite regular cardinals κ, $H_\kappa = V_\kappa$ iff κ is strongly inaccessible.

Corollary 0.11. *ZFC$\nvdash \exists\kappa \; (\kappa \; \text{is strongly inaccessible}).*

Proof. If ZFC $\vdash$ $\exists\kappa$ (κ is strongly inaccessible), then from the formalized lemmas 0.6, 0.1 we would have ZFC $\vdash$ $\exists\kappa$ $\forall n$ $(\theta(n) \to \rho(n, H_\kappa))$ where θ represents $R(n) \leftrightarrow \theta_n \in$ ZFC, and ρ again is the set satisfaction formula. By the formalized completeness theorem, this would give ZFC $\vdash$ CON(ZFC), a contradiction to Gödel's theorem. $\qquad\square$

Note that the proof of corollary 0.11 actually showed the following.

Corollary 0.12. *ZFC* $+$ $\exists\kappa$ *(κ is strongly inaccessible)* $\vdash$ *CON(ZFC).*

We sometimes state corollary 0.12 by saying that ZFC$+\exists\kappa$ (κ is strongly inaccessible) has a greater consistency strength that ZFC.

Exercise 4. Show corollary 0.11 without using Gödel's theorem. [Hint: Suppose ZFC $\vdash$ $\exists\kappa$ (κ is strongly inaccessible), and let $\psi_1, \ldots, \psi_n$ be a finite fragment of ZFC such that $\{\psi_1, \ldots, \psi_n\} \vdash \exists\kappa$ (κ is strongly inaccessible). Working in ZFC, let κ be the least strongly inaccessible cardinal. Use lemmas 0.6, 0.1 and an absoluteness argument to show that there is a strongly inaccessible $\lambda < \kappa$.]

V_κ and H_κ are sets which we showed satisfy certain fragments of ZFC. We now consider a certain class, HOD, which we show (assuming ZFC) satisfies all of ZFC. In fact, HOD will be a transitive class containing all of the ordinals, a so-called *inner model* of ZFC.

We would like to say a set x is *ordinal-definable* if there is a formula $\phi(x_1, \ldots, x_n, y)$ and ordinals $\alpha_1, \ldots, \alpha_n$ such that x is the unique set x such that $\phi(\alpha_1, \ldots, \alpha_n, y)$. We would then define HOD to be the sets which are hereditarily ordinal definable. The problem with this is that this definition, at least the way it is stated, is not a legitimate formula of set theory, i.e., doesn't really seem to define a class. The problem, as we mentioned before, is that we cannot define formual satisfaction over classes, only over sets. However, the reflection theorem provides a way to circumvent this logical problem. For any formula ϕ and ordinals $\alpha_1, \ldots, \alpha_n$, the reflection theorem says that there is a $\beta > \max\{\alpha_1, \ldots, \alpha_n\}$ such that $\forall y \in V_\beta$ $(\phi(\vec{\alpha}, y)^{V_\beta} \leftrightarrow \phi(\vec{\alpha}, y))$. Thus, there should be no loss of generality in interpreting our formulas in the V_β. This suggests the following definition.

Definition 0.13. OD is the class defined by:

$$x \in \text{OD} \leftrightarrow \exists\beta \in \text{ON} \; \exists\alpha_1, \ldots, \alpha_n < \beta \; \exists n \in \omega \; \forall y \in V_\beta \; [(y \approx x) \leftrightarrow \rho(n, \langle\vec{\alpha}, y\rangle, V_\beta)]$$

where ρ is the formula for set satisfaction from lemma **??**. Also,

$$x \in \text{HOD} \leftrightarrow \forall y \in \text{tr cl}(x) \; (y \in \text{OD}).$$

Note that the quantification over $\alpha_1, \ldots, \alpha_n$ should really be expressed as a quantification over sets $z = \langle\alpha_1, \ldots, \alpha_n\rangle$ coding the sequence.

Our remarks above concerning the reflection theorem now easily give the following.

Lemma 0.14. *Let $\phi(x_1, \ldots, x_n, y)$ be a formula of set theory. Then*

$$ZF \vdash \forall\alpha_1, \ldots, \alpha_n \in ON \; \forall x \; [\forall y \; (y \approx x \leftrightarrow \phi(\vec{\alpha}, y)) \to y \in OD].$$

By definition HOD is transitive, and it trivially contains all of the ordinals. The next theorem shows that working in ZF we can show that HOD satisfies ZFC. We will need th following ordring on the finite sequences of ordinals.

Definition 0.15. The Gödel ordering $<_G$ on $\mathrm{ON}^{<\omega}$ is defined by:

$$\langle \alpha_1, \ldots, \alpha_n \rangle <_G \langle \beta_1, \ldots, \beta_m \rangle \leftrightarrow \max\{\alpha_1, \ldots, \alpha_n\} < \max\{\beta_1, \ldots, \beta_m\}$$
$$\vee \, (\max\{\alpha_1, \ldots, \alpha_n\} = \max\{\beta_1, \ldots, \beta_m\} \wedge n < m)$$
$$\vee \, (\max\{\alpha_1, \ldots, \alpha_n\} = \max\{\beta_1, \ldots, \beta_m\} \wedge n = m \wedge \vec{\alpha} <_{\mathrm{lex}} \vec{\beta}),$$

where $<_{\mathrm{lex}}$ denotes lexicographic order.

The following lemma is clear from the definition.

Lemma 0.16. *The Gödel ordering is a class wellordering of $\mathrm{ON}^{<\omega}$, all of whose initial segments are sets.*

Theorem 0.17. *For all axioms ϕ of ZFC, $ZF \vdash \phi^{HOD}$.*

Proof. Foundation holds in HOD, as it holds in any class with the ϵ relation. Extensionality holds as HOD is transitive. Consider pairing. Let $x, y \in \mathrm{HOD}$. Fix $n \in \omega$, $\alpha_1, \ldots, \alpha_n \in \mathrm{ON}$, and $\beta_1 \in \mathrm{ON}$ such that $\forall z \in V_\beta \; [(z \approx x) \leftrightarrow \rho(n, \langle \vec{\alpha}, y \rangle, V_\beta)$, and similarly let $m, \gamma_1, \ldots, \gamma_k, \beta_2$ witness $y \in \mathrm{HOD}$. Then,

$$z \in \{x, y\} \leftrightarrow \exists w \; [w = V_{\beta_1} \wedge \rho(n, \langle \vec{\alpha}, z \rangle, w)] \vee \exists w \; [w = V_{\beta_2} \wedge \rho(m, \langle \vec{\gamma}, z \rangle, w)]$$

The righthand is a formula in set theory with ordinal parameters n, m, $\vec{\alpha}$, $\vec{\gamma}$, β_1, β_2. Thus, $\{x, y\} \in \mathrm{OD}$. Since $\{x, y\} \subseteq \mathrm{HOD}$, we have $\{x, y\} \in \mathrm{HOD}$.

Infinity holds in HOD by absoluteness, since $\omega \in \mathrm{HOD}$ (HOD contains all the ordinals).

Consider the union axiom. Let $x \in \mathrm{HOD}$, and fix witnesses n, $\vec{\alpha}$, β. Then

$$z \in \cup x \leftrightarrow \exists w \; \exists y \; [w = V_\beta \wedge y \in w \wedge \rho(n, \langle \vec{\alpha}, y \rangle, w) \wedge \exists u \in y(z \in u)].$$

Again, the righthand side is a formula with ordinal parameters n, $\vec{\alpha}$, β, and so $\cup x \in \mathrm{OD}$. Since $\cup x \subseteq \mathrm{tr\,cl}(x) \subseteq \mathrm{HOD}$, this shows $\cup x \in \mathrm{HOD}$. By absoluteness, HOD satisfies the union axiom.

To show comprehension, let $\phi(x_1, \ldots, x_n, y, z)$ be a formula, and $a_1, \ldots, a_n, A \in \mathrm{HOD}$. We must show that $B \in \mathrm{HOD}$, where

$$B = \{z \in A : \phi^{\mathrm{HOD}}(a_1, \ldots, a_n, A, z)\}.$$

Note that B exists in V by comprehension in V. Then,

$$x = B \leftrightarrow \forall z \; [z \in B \leftrightarrow \exists u_1, \ldots u_n \; \exists v \; (u_1 = a_1 \wedge \cdots \wedge u_n = a_n \wedge v = A$$
$$\wedge \, \phi^{\mathrm{HOD}}(u_1, \ldots, u_n, v, z))],$$

where in place of "$u_1 = a_1$" we use a formula with only ordinal parameters (the witnesses for $a_1 \in \mathrm{HOD}$) and likewise for the other a_i and A. This shows $B \in \mathrm{OD}$, and since $B \subseteq \mathrm{HOD}$ we have $B \in \mathrm{HOD}$. Clearly, $(\forall z \; (z \in B \leftrightarrow z \in A \wedge \phi(a_1, \ldots, a_n, A, z)))^{\mathrm{HOD}}$, and so HOD satisfies comprehension.

To show power set, let $x \in \mathrm{HOD}$. Let $\alpha \in \mathrm{ON}$ be large enough so that $\mathcal{P}(x) \in V_\alpha$. Note that $V_\alpha \in \mathrm{OD}$ (it is definable from α). Since HOD is a class, $y = V_\alpha \cap \mathrm{HOD} \in \mathrm{OD}$ as well, and so $y \in \mathrm{HOD}$. Clearly $(\forall z \; (z \subseteq x \rightarrow z \in y)^{\mathrm{HOD}}$, so HOD satisfies power set.

To show replacement, let $\phi(x_1, \ldots, x_n, y, z, w)$ be a formula, $a_1, \ldots, a_n, A \in \mathrm{HOD}$, and assume $(\forall y \in A \; \exists z \; \phi(\vec{a}, y, z, A))^{\mathrm{HOD}}$. Let $\alpha \in \mathrm{ON}$ be such that $\forall y \in A \; \exists z \in V_\alpha \; \phi(\vec{a}, y, z, A))^{\mathrm{HOD}}$. As $V_\alpha \in \mathrm{OD}$, $B = V_\alpha \cap \mathrm{HOD} \in \mathrm{HOD}$. Then $(\forall y \in A \; \exists z \in B \; \phi(\vec{a}, y, z, A))^{\mathrm{HOD}}$.

Finally, to show AC holds in HOD, let $x \in$ HOD. Every $y \in x$ is in HOD and has a witness sequence n, $\vec{\alpha}$, β. To compare two elements y, z of x, we take the $<_G$ least witnesses for y and z, and compare them in the $<_G$ ordering. In more detail, define a wellordering of x by:

$$y \prec z \leftrightarrow y \in x \wedge z \in x \wedge \exists n \in \omega \; \exists \langle \vec{\alpha} \rangle \in \mathrm{ON}^{<\omega} \; \exists \beta_1 \in \mathrm{ON}$$

$$\exists m \in \omega \; \exists \langle \vec{\gamma} \rangle \in \mathrm{ON}^{<\omega} \; \exists \beta_2 \in \mathrm{ON}$$

$$[\forall u \; (u \approx y \leftrightarrow \exists w \; (w \approx V_{\beta_1} \wedge \rho(n, \langle \vec{\alpha}, u \rangle, w)))$$

$$\wedge \; \forall \langle n', \vec{\alpha}', \beta_1' \rangle <_G \langle n, \vec{\alpha}, \beta_1 \rangle \; \neg \forall u \; (u \approx y \leftrightarrow \exists w \; (w \approx V_{\beta_1'} \wedge \rho(n', \langle \vec{\alpha}', u \rangle, w)))]$$

$$\wedge \; [\forall u \; (u \approx z \leftrightarrow \exists w \; (w \approx V_{\beta_2} \wedge \rho(m, \langle \vec{\gamma}, u \rangle, w)))$$

$$\wedge \; \forall \langle m', \vec{\gamma}', \beta_2' \rangle <_G \langle m, \vec{\gamma}, \beta_2 \rangle \; \neg \forall u \; (u \approx z \leftrightarrow \exists w \; (w \approx V_{\beta_2'} \wedge \rho(m', \langle \vec{\gamma}', u \rangle, w)))]$$

$$\wedge \; \langle n, \vec{\alpha}, \beta_1 \rangle <_G \langle m, \vec{\gamma}, \beta_2 \rangle$$

Here for "$y \in x$" we substitute an appropriate formula with only ordinal parameters, and likewise for $z \in x$.

Easily $\prec$ is a wellordering, and the above formula shows $\prec \in$ OD. Since $\prec \subseteq$ HOD, we have $\prec \in$ HOD, and by downwards absoluteness ($\prec$ is a wellordering of x)$^{\mathrm{HOD}}$. $\qquad \square$

Corollary 0.18. *If ZF is consistent, then so is ZFC.*

Proof. Suppose ZF is consistent, but ZFC is not. Then ZF $\vdash \neg$AC. Let $\psi_1, \ldots, \psi_n$ be finitely many axioms of ZF such that $\{\psi_1, \ldots, \psi_n\} \vdash \neg$AC. From theorem 0.17, ZF $\vdash \psi_1^{\mathrm{HOD}} \wedge \cdots \wedge \psi_n^{\mathrm{HOD}}$. Hence, ZF $\vdash (\neg$AC$)^{\mathrm{HOD}}$. But, ZF $\vdash$ AC$^{\mathrm{HOD}}$ from theorem 0.17, a contraction to the consistency of ZF. $\qquad \square$

Thus, the axiom of choice cannot introduce a contradiction into mathematics, unless one was already present. This is a significant statement in view of the many "pathological" sets that may be constructed with AC. For eaxample, with AC a standard construction gives a non-measurable set of reals, and it can be shown that AC is necessary for the construction (more on this later).

Finally, we note that although corollary 0.18 was proved in the meta-theory, the argument is readily formalized, which gives the following version of the corollary.

Corollary 0.19. $CON(ZF) \vdash CON(ZFC)$.

Thus, the theories ZF and ZFC have the same consistency strength.