More Forcing Constructions

We use forcing to establish the consistency of some combinatorial principles which are of use in many constructions. First we introduce Jensen's diamond principle.

Definition 0.1. $\diamondsuit$ is the statement: there are $A_\alpha \subseteq \alpha$ for $\alpha < \omega_1$ such that for all $A \subseteq \omega_1$, $\{\alpha < \omega_1 \colon A \cap \alpha = A_\alpha\}$ is stationary.

Clearly $\diamondsuit$ implies CH as every subset of ω must appear among the A_α. This already shows that $\diamondsuit$ is not provable in ZFC. In fact it is known that GCH does not imply $\diamondsuit$ (Jensen). $\diamondsuit$ is thus a powerful strengthening of CH, and is useful in places where CH alone is not sufficient. First we show that we may force to make $\diamondsuit$ true.

Lemma 0.2. *Let M be a transitive model of ZFC. Then there is a generic extension $M[G]$ in which $\diamondsuit$ holds.*

Proof. Let $\mathbb{P}$ consist of all countable sequences $p = \{A_\beta\}_{\beta<\alpha}$ where $\alpha < \omega_1$ and $A_\beta \subseteq \beta$. We order $\mathbb{P}$ by extension, so $p \leqslant q$ iff $\mathrm{dom}(p) \geqslant \mathrm{dom}(q)$ and $p \restriction \mathrm{dom}(q) = q$. Let G be M generic for $\mathbb{P}$. Clearly $\mathbb{P}$ is countably closed, so $\omega_1^{M[G]} = \omega_1^M$. The generic can be identified with a sequence $\{A_\beta\}_{\beta<\omega_1}$. We show that this sequence witnesses $\diamondsuit$ in $M[G]$. Fix $A \subseteq \omega_1$, $A \in M[G]$, and a $C \subseteq \omega_1$, $C \in M[G]$ with $(C$ is c.u.b.$)^{M[G]}$. Let $A = \tau_G$, $C = \sigma_G$. Let $p_0 \in G$, $p_0 \Vdash (\sigma$ is c.u.b.$)$. Let $\dot{G}$ be the canonical name for the generic. We claim that $p_0 \Vdash \exists\alpha((\alpha \in \sigma) \wedge (\tau \cap \alpha = \dot{G}(\alpha)))$. Suppose not, and let $p_1 \leqslant p_0$, $p_1 \in G$ with $p_1 \Vdash \forall\alpha\ \neg((\alpha \in \sigma) \wedge (\tau \cap \alpha = \dot{G}(\alpha)))$. Working in M, we construct a sequence of conditions $q_0 = p_1 \geqslant q_1 \geqslant \dots$ as follows. Assume q_n has been defined. Let $q_{n+1} \leqslant q_n$ be such that

1. $\mathrm{dom}(q_{n+1}) > \mathrm{dom}(q_n)$ and there is a $\beta_n \in (\mathrm{dom}(q_n), \mathrm{dom}(q_{n+1}))$ such that $q_{n+1} \Vdash \check{\beta}_n \in \sigma$.
2. For every $\alpha < \mathrm{dom}(q_n)$, either $q_{n+1} \Vdash \check{\alpha} \in \tau$ or $q_{n+1} \Vdash \check{\alpha} \notin \tau$.

There is no problem getting q_{n+1} as $p_0 \Vdash (\sigma$ is unbounded $)$ and $\mathbb{P}$ is countably closed. Let $q = \bigcup_n q_n$. Let $\alpha = \mathrm{dom}(q) = \bigcup_n \mathrm{dom}(q_n)$. From (1) and the fact that $p_0 \Vdash (\sigma$ is closed$)$ it follows that $q \Vdash \check{\alpha} \in \sigma$. From (2), there is a $B \subseteq \alpha$, $B \in M$, with $q \Vdash \tau \cap \alpha = B$. Let $q' = q \cup \{\langle \alpha, B\rangle\}$. Then $q' \Vdash ((\check{\alpha} \in \sigma) \wedge (\tau \cap \check{\alpha} = \dot{G}(\check{\alpha}))$. This contradicts the definition of p_1. □

The forcing used in lemma 0.2 is isomorphic to a dense subset of $\mathrm{FN}(\omega_1, 2, \omega_1)$ ($\mathrm{FN}(\omega_1, 2, \omega_1)$ is isomorphic to $\mathbb{P} = \mathrm{FN}(A, 2, \omega_1)$, where $A = \{(\beta, \alpha) \colon \beta < \alpha < \omega_1\}$. The forcing used in the lemma is a dense subset of $\mathbb{P}$.) Thus we get:

Corollary 0.3. *Let M be a transitive model of ZFC and G be M-generic for $FN(\omega_1, 2, \omega_1)^M$. Then $\diamondsuit$ holds in $M[G]$.*

Exercise 1. Show that if M is a transitive model of ZFC and G is M-generic for $\mathrm{coll}(\omega_1, \kappa)$, for some $\kappa \geqslant \omega_1$, then $\diamondsuit$ holds in $M[G]$. [hint: First identify $\mathrm{coll}(\omega_1, \kappa) = \mathrm{FN}(\omega_1, \kappa, \omega_1)$ with $\mathbb{P} = \mathrm{FN}(A, \kappa, \omega_1)$ where $A = \{(\alpha, \beta) \colon \alpha < \beta < \omega_1\}$. A generic G for $\mathbb{P}$ induces a $G' \colon \omega_1^2 \to 2$ by $G'(\alpha, \beta) = 0$ iff $G(\alpha, \beta)$ is even. Show that G' gives a $\diamondsuit$ sequence in $M[G]$.]

We can generalize $\diamondsuit$ to higher cardinals.

Definition 0.4. Let κ be a regular cardinal. Then $\diamondsuit_\kappa$ is the statement that there is a sequence $A_\alpha \subseteq \alpha$, $\alpha < \kappa$, such that for all $A \subseteq \kappa$, $\{\alpha < \kappa \colon A \cap \alpha = A_\alpha\}$ is stationary.

The proof of lemma 0.2 generalizes to give:

Lemma 0.5. *Let M be a transitive model of ZFC and κ a regular cardinal of M. Then there is a generic extension $M[G]$ which preserves all cardinals and cofinalities $\leqslant \kappa$ in which $\diamondsuit_\kappa$ holds.*

Proof. Let $\mathbb{P}$ consist of all functions p with $\mathrm{dom}(p)$ an ordinal less than κ, and with $p(\alpha) \subseteq \alpha$ for all $\alpha \in \mathrm{dom}(p)$. Clearly $\mathbb{P}$ is $< \kappa$ closed (since κ is regular), and so $\mathbb{P}$ preserves all cardinalities and cofinalities $\leqslant \kappa$. Let G be M generic for $\mathbb{P}$. The proof that $\diamondsuit_\kappa$ holds in $M[G]$ is essentially identical to the proof of lemma 0.2. Fix $A = \tau_G \subseteq \kappa$, and $C = \sigma_G \subseteq \omega_1$ with $(C \text{ is c.u.b.})^{M[G]}$. Let $p_0 \in G$, $p_0 \Vdash (\sigma \text{ is c.u.b.})$. Suppose $p_1 \leqslant p_0$ with $p_1 \Vdash \forall \alpha\ \neg((\alpha \in \sigma) \wedge (\tau \cap \alpha = \dot{G}(\alpha)))$. We construct the sequence $q_0 = p_1 \geqslant q_1 \geqslant q_2 \geqslant \dots$ as before. We use now $\mathbb{P}$ being $< \kappa$ closed to get (2). As before, let $q = \bigcup_n q_n$ and let $q' = q \cup \{\langle B, \alpha \rangle\}$ where $\alpha = \mathrm{dom}(q) = \bigcup_n \mathrm{dom}(q_n)$ and $B \in M$, $q \Vdash \tau \cap \check{\alpha} = \check{B}$. This is a contradiction exactly as above. $\square$

The forcing of lemma 0.5 is again equivalent to $\mathrm{FN}(\kappa, 2, \kappa)$, thus we have:

Corollary 0.6. *Let M be a transitive model of ZFC and κ a regular cardinal of M. Let G be M-generic for $FN(\kappa, 2, \kappa)^M$. Then $\diamondsuit_\kappa$ holds in $M[G]$.*

There are two natural weakenings of $\diamondsuit$ which turn out to be equivalent to $\diamondsuit$ by a theorem of Kunen. One weakening is to replace "stationary" in the definition of $\diamondsuit$ with "non-empty." The second is to allow countably many sets at each stage $\alpha < \omega_1$ to do the guessing. the following theorem gives the equivalence.

Theorem 0.7. *(Kunen) The following are equivalent in ZFC.*

1. $\diamondsuit$
2. *There is a sequence $\{\mathcal{A}_\alpha\}_{\alpha < \omega_1}$, where each $\mathcal{A}_\alpha$ is a countable collection of subsets of α, such that for every $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 \colon A \cap \alpha \in \mathcal{A}_\alpha\}$ is stationary.*
3. *There is a sequence $\{\mathcal{A}_\alpha\}_{\alpha < \omega_1}$, where each $\mathcal{A}_\alpha$ is a countable collection of subsets of α, such that for every $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 \colon A \cap \alpha \in \mathcal{A}_\alpha\}$ is non-empty.*

Proof. First we show that (2) implies (1). Let $\pi \colon \omega \times \omega_1 \to \omega_1$ be a bijection. Let $D \subseteq \omega_1$ be c.u.b. such that for $\alpha \in D$, $\pi \restriction \omega \times \alpha$ is a bijection between $\omega \times \alpha$ and α. Let $\{\mathcal{A}_\alpha\}_{\alpha < \omega_1} = \{A^n_\alpha\}_{\alpha < \omega_1, n < \omega}$ witness (2). Let $B^{n,m}_\alpha = \{\beta < \alpha \colon \pi(m, \beta) \in A^n_\alpha\}$. We show that for some $n \in \omega$ that $\{B^{n,n}_\alpha\}_{\alpha < \omega_1}$ is a $\diamondsuit$ sequence. If not, then for each n let $E_n \subseteq \omega_1$, and C_n be c.u.b. witnessing the failure of $B^{n,n}_\alpha$ to be a $\diamondsuit$ sequence. Define E to code the E_n by: $\alpha \in E$ iff $\pi^{-1}(\alpha) = (k, \beta)$ and $\beta \in E_k$. By (2), let $\alpha \in D \cap \bigcap_n C_n$ such that $E \cap \alpha \in \mathcal{A}_\alpha$. Say $E \cap \alpha = A^n_\alpha$. But then $B^{n,n}_\alpha = \{\beta < \alpha \colon \pi(n, \beta) \in A^n_\alpha\} = \{\beta < \alpha \colon \pi(n, \beta) \in E \cap \alpha\} = E_n \cap \alpha$, a contradiction to the definition of E_n and $\alpha \in C_n$.

We next show (3) implies (2). Let $\{\mathcal{A}_\alpha\}_{\alpha < \omega_1}$ be as in (3). Without loss of generality we may assume the $\mathcal{A}_\alpha$ are increasing. We define a new sequence $\{\mathcal{A}'_\alpha\}_{\alpha < \omega_1}$ as follows. We view subsets of ω as coding bounded subsets of ω_1 in some manner (e.g., take a bijection between 2^ω and $\omega_1^{<\omega}$). For successor ordinals let $\mathcal{A}'_\alpha = \mathcal{A}_\alpha$. For α limit let $\mathcal{A}'_\alpha \supseteq \mathcal{A}_\alpha$ be such that:

(i) For all $n \in \omega$ and all $A \in \mathcal{A}_{\alpha+n}$, $\{\beta < \alpha \colon 2 \cdot \beta \in A\} \in \mathcal{A}'_\alpha$.

(ii) Suppose $A \in \mathcal{A}_\alpha$, $\beta < \alpha$ is a limit, and $x = \{n \in \omega \colon \beta + 2n + 1 \in A\}$ codes a subset A_β of α. Then $A_\beta \in \mathcal{A}'_\alpha$.

To see this works, fix $A \subseteq \omega_1$ and a c.u.b. $C \subseteq \omega_1$. We may assume C consists of limit ordinals. We must show that for some $\alpha \in C$ that $A \cap \alpha \in \mathcal{A}'_\alpha$. Define $B \subseteq \omega_1$ as follows. For each limit β let $x_\beta \subseteq \omega$ code $A \cap N_C(\beta)$, where $N_C(\beta)$ is the least element of C greater than β. Let $B = \{2 \cdot \gamma \colon \gamma \in A\} \cup \{\beta + 2n + 1 \colon \beta \text{ is limit } \wedge n \in x_\beta\}$.

By (3), fix now $\alpha < \omega_1$ such that $B \cap \alpha \in \mathcal{A}_\alpha$. If $\alpha \in C$ then $A \cap \alpha \in \mathcal{A}'_\alpha$ by (i) (with $n = 0$). Let γ be the largest element of C less than α. If $\alpha < \gamma + \omega$, then by (i) we still have $A \cap \gamma \in \mathcal{A}'_\gamma$. If $\alpha \geqslant \gamma + \omega$, then from (ii) we have that $A \cap \delta \in \mathcal{A}'_\delta$, where $\delta = N_C(\gamma)$.

Thus, in all cases we have an $\alpha \in C$ with $A \cap \alpha \in \mathcal{A}'_\alpha$. □

Another generalization of $\diamondsuit$ is the following.

Definition 0.8. Let κ be a regular cardinal and $S \subseteq \kappa$ stationary. Then $\diamondsuit_\kappa^S$ is the statement that there is a sequence $\{A_\alpha\}_{\alpha \in S}$ with $A_\alpha \subseteq \alpha$, such that for all $A \subseteq \kappa$, the set $\{\alpha \in S \colon A \cap \alpha = S_\alpha\}$ is stationary.

Lemma 0.9. *Let M be a transitive model of ZFC, and κ a regular cardinal of M. Let $\lambda > (2^{<\kappa})^M$ be a cardinal of M. Let G be M-generic for $FN(\lambda, 2, \kappa)$. Then in $M[G]$ we have that for every stationary $S \subseteq \kappa$, $\diamondsuit_\kappa^S$ holds.*

Proof. The forcing $\mathbb{P} = \mathrm{FN}(\lambda, 2, \kappa)$ is $< \kappa$ closed in M, so all cardinalities and cofinalities $\leqslant \kappa$ are preserved. Also, $\mathbb{P}$ is $(2^{<\kappa})^+$ c.c. in M.

First we show that for every $S \in M$ which is stationary in $M[G]$ that $\diamondsuit_\kappa^S$ holds in $M[G]$. Fix such an $S \in M$. Since $\lambda \cong \kappa^2 \oplus \lambda$, $\mathrm{FN}(\lambda, 2, \kappa) \cong \mathrm{FN}(\kappa^2, 2, \kappa) \times \mathrm{FN}(\lambda, 2, \kappa)$. Let $\mathbb{P}$ be the forcing of lemma 0.5, that is, conditions p are functions with $\mathrm{dom}(p) < \kappa$ and for all $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \subseteq \alpha$. Since $\mathbb{P} \times \mathrm{FN}(\lambda, 2, \kappa)$ is dense in $\mathrm{FN}(\kappa^2, 2, \kappa) \times \mathrm{FN}(\lambda, 2, \kappa)$, we may view G as a product $G = G_1 \times G_2$ where $G_1 \subseteq \mathbb{P}$, $G_2 \subseteq \mathrm{FN}(\lambda, 2, \kappa)$ (more precisely, $M[G] = M[G_1][G_2]$ where (G_1, G_2) is generic for the product).

Replacing M by $M[G_2]$, it is enough to show that if M is a transitive model of ZFC, κ is regular in M, G is M-generic for $\mathbb{P}$, and $S \in M$ is stationary in $M[G]$, then $M[G]$ satisfies $\diamondsuit_\kappa^S$. To see this, let $A = \tau_G \subseteq \kappa$ and $C = \sigma_G$ be c.u.b. in $M[G]$. Suppose $p \Vdash \neg \exists \alpha\ (\alpha \in \sigma \cap \check{S} \wedge \dot{G}(\alpha) = \tau \cap \alpha)$. In $M[G]$ define

$$
\begin{aligned}
D = &\{\alpha < \kappa \colon \forall \beta < \alpha\ \exists \gamma, \delta < \alpha\ (\gamma > \beta \wedge G{\restriction}\delta \Vdash (\check{\gamma} \in \sigma) \\
&\wedge (G{\restriction}\delta \Vdash \check{\beta} \in \tau \vee G{\restriction}\delta \Vdash \check{\beta} \notin \tau))\}.
\end{aligned}
$$

Easily $M[G]$ satisfies that D is a c.u.b. subset of κ (recall κ is regular in $M[G]$). Since S is stationary in $M[G]$, let $\alpha \in S \cap D$, with $\alpha > \mathrm{dom}(p)$. Let $q = G{\restriction}\alpha$. Thus, $q \leqslant p$. From $\alpha \in D$ we get that $q \Vdash (\check{\alpha} \in \sigma)$. Also, there is a $B \in M$ such that $q \Vdash \tau \cap \alpha = \check{B}$. Let $q' = q \cup \{\langle \alpha, B \rangle\}$. Then $q' \leqslant p$ and $q' \Vdash (\check{\alpha} \in (\check{S} \cap \sigma) \wedge \dot{G}(\check{\alpha}) = \tau \cap \check{\alpha})$, a contradiction.

Returning to the proof of the theorem, suppose now $S \in M[G]$ is stationary in $M[G]$, where G denotes our $\mathrm{FN}(\lambda, 2, \kappa)$ generic. Assume for convenience that λ is regular in M. For any $\rho < \lambda$, we may write $\mathrm{FN}(\lambda, 2, \kappa) \cong \mathrm{FN}(\rho, 2, \lambda) \times \mathrm{FN}(\lambda - \rho, 2, \kappa)$, and view G as a product $G_\rho^- \times G_\rho^+$ accordingly. It suffices to show that for some $\rho < \lambda$ that $S \in M[G_\rho^-]$. For then we may apply the previous paragraph (using $M[G_\rho^-]$ as our ground model) to conclude that $\diamondsuit_\kappa^S$ holds in $M[G]$. In fact, we show that any $S \subseteq \kappa$, $S \in M[G]$ lies in some $M[G_\rho^-]$. Since $\mathbb{P} = \mathrm{FN}(\lambda, 2, \kappa)$ is λ c.c., there is a nice name for S in M of size $< \lambda$ (using λ regular in M). Let τ be such a nice name, that is, $|\tau|^M < \lambda$. Clearly then, $S = \tau_G = \tau_{G_\rho^-}$ for large enough ρ (i.e., ρ greater than the domains of all conditions in $\cup^3(\tau)$). Thus, $S \in M[G_\rho^-]$.

If λ is not regular in M, the argument of the previous paragraph still goes through writing instead $\mathrm{FN}(\lambda, 2, \kappa) = \mathrm{FN}(T, 2, \kappa) \times \mathrm{FN}(\lambda - T, 2, \kappa)$ for some $T \subseteq \lambda$ of size $\leqslant 2^{<\kappa} < \lambda$ (but T may now be cofinal in λ). □

1. Suslin's Hypothesis

We say a linear order $(X, <)$ is *dense* if whenever $x < y$ then there is a $z \in X$ with $x < z < y$. We say $(x, <)$ is without endpoints if there is no least or maximal element of X. We say $(X, <)$ is separable if there is a countable set $D \subseteq X$ such that every non-empty interval (x, y) contains a point of D. We say $(X, <)$ is c.c.c. if there is no uncountable family of pairwise disjoint open sets (equivalently, intervals). The linear order is separable or c.c.c. iff X viewed as a topological space (with the order topology) is separable or c.c.c. We say $(X, <)$ is *complete* if every non-empty bounded set in X has a least upper bound and a greatest lower bound.

Recall that for topological spaces in general, 2^{nd} countable $\Rightarrow$ separable $\Rightarrow$ c.c.c. A countable product of 2^{nd} countable spaces is second countable, but an ω_1 product of $\mathbb{R}_{std}$ is not 2^{nd} countable (or even first countable). A $\leqslant 2^\omega$ length product of separable spaces is separable, but a $(2^\omega)^+$ product of $\mathbb{R}_{std}$ is not separable. Finally, an arbitrary product of second countable spaces is c.c.c. (we'll discuss products of c.c.c. more later).

The following lemma is an important but elementary fact from analysis. It is just asserting that $\mathbb{R}$ is the unique completion of $\mathbb{Q}$. The proof is left to the exercises.

Lemma 1.1. $(\mathbb{R}, <_{std})$ *is the unique, up to isomorphism of linear orders, linear order* $(X, <)$ *satisfying:*

(1) $(X, <)$ *is dense and without endpoints.*
(2) $(X, <)$ *is complete.*
(3) $(X, <)$ *is separable.*

Exercise 2. (Cantor) Show that any two countable dense linear orders without endpoints are isomorphic. [hint: Use a "back and forth" argument. Construct $f = \bigcup_n f_n$ in ω stages, where each f_n will be an isomorphism from a subset of the first order of size n to a subset of the second order. At even stages $n = 2m$, arrange that the m^{th} element of the first order is in the domain of f_n. At odd stages $n = 2m+1$, arrange that the m^{th} element of the second order is in the range of f_n.]

Exercise 3. Prove lemma 1.1. [hint: Start with exercise 2. Then extend the f of that exercise to the completions of the countable dense sets, using the fact that both are complete.]

Note that the requirement that $(X, <)$ not have endpoints is rather trivial: if it has them then we can simply remove them without effecting the other properties of lemma 1.1.

A natural question, raised by Suslin, is whether the characterization of the real line, lemma 1.1, continues to hold if we weaken the requirement of separability to that of being c.c.c. The statement that it does is called Suslin's hypothesis, SH.

Definition 1.2. Suslin's hypothesis, SH is the statement that $(\mathbb{R}, <_{std})$ is the unique, up to isomorphism of linear orders, linear order $(X, <)$ which is dense, without endpoints, complete, and c.c.c.

A counterexample to SH is called a *Suslin line.* That is, a Suslin line is a linear order $(X, <)$ which is dense, without endpoints, complete, and c.c.c., but not separable. Thus, SH is the statement that there are no Suslin lines.

The following lemma says that the existence of a Suslin line is equivalent to the existence of a linear order $(X, <)$ which is c.c.c. but not separable, as the other properties can be easily arranged.

Lemma 1.3. *If there is a linear order* $(X, <)$ *which is c.c.c. but not separable, then there is a Suslin line.*

Proof. Let $(X, <)$ be a c.c.c. but not separable linear order. Define an equivalence relation on X by $x \sim y$ iff (x, y) is separable. Each equivalence class $[x]$ is an interval of X. Distinct equivalence classes correspond to disjoint intervals. The classes then inherit an order from X, namely $[x] < [y]$ iff $x < y$ (equivalently, all the points of $[x]$ are less than any point of $[y]$). Let $\tilde{X}$ be the set of equivalence classes with this induced order. We claim that $\tilde{X}$ is dense in itself, and c.c.c. but not separable. Granting this, we can then finish by removing the endpoints of $\tilde{X}$, if any, then taking the completion. It is easy to check that the completion is still c.c.c. and not separable (see the following exercise). First note that every equivalence class $I = [x]$ is separable. To see this, let (x_n, y_n) be a maximal family of parirwise disjoint intervals contained in I (which must be countable as X is c.c.c.). Let D_n be dense in (x_n, y_n), and let $D = \bigcup_n D_n$. Then D together with the first and last elements of I (if any) is dense in I.

To see $\tilde{X}$ is dense in itself, suppose $[x] < [y]$. If $([x], [y]) = \varnothing$, then $(x, y) \subseteq ([x] \cup [y])$, and thus $D \cup E$ is dense in (x, y) where D is dense in $[x]$ and E is dense in $[Y]$. Thus, $x \sim y$, so $[x] = [y]$, a contradiction.

To see $\tilde{X}$ is not separable, if $\{[d_n]\}_{n \in \omega}$ were dense in $\tilde{X}$, then let for each n $D_n \subseteq [d_n]$ de dense in $[d_n]$. Then $D = \bigcup_n D_n$ is dense in X, a contradiction.

To see $\tilde{X}$ is c.c.c., suppose $([x_\alpha], [y_\alpha])$ is an antichain in $\tilde{X}$. Then (x_α, y_α) is an antichain in X, so must be countable. □

Exercise 4. Let $(X, <)$ be a linear order. The completion $\hat{X}$ of X can de defined by adding points $\hat{x}$ for all cuts (bounded above, downward closed sets) $S \subseteq X$ which do not have a least upper bound in X. The point $\hat{x}$ is greater than any element of S but less than any element of X which is greater than all the elements of X. Show that X is dense in $\hat{X}$. Show that if $\hat{x} \in \hat{X} - X$, then X has no largest element below $\hat{x}$, and X has no least element above $\hat{x}$. Show that X is separable iff $\hat{X}$ is separable. [if $\hat{D}$ is countable dense in $\hat{X}$, show that $D \cup E$ is dense in X where $D = \hat{D} \cap X$ and E is chosen so that for all nonempty intervals $(\hat{d}_1, \hat{d}_2)$ where $\hat{d}_1$, $\hat{d}_2 \in \hat{D}$ we have $E \cap (\hat{d}_1, \hat{d}_2) \neq \varnothing$.] Show that X is c.c.c. iff $\hat{X}$ is c.c.c. [if $(\hat{x}, \hat{y})$ is a non-empty interval in $\hat{X}$, show that there are $x, y \in X$ with $\hat{x} \leqslant x < y \leqslant \hat{y}$ and (x, y) non-empty in X.]

Suslin's hypothesis we formulated by Suslin around 1920. It was reformulated in terms of a certain kind of tree, called Suslin trees, by Kurepa in the 30's. As we will see, SH is independent of ZFC, by results from the late 60's. Jech and Tennenbaum showed the consistency of ZFC + $\neg$SH, and Solovay and Tennenbaum the consistency of ZFC + SH. Jensen showed that Suslin trees exist in L, that is, $\neg$SH holds in L (we discuss these points in more detail below).

2. Various Trees

As we mentioned, Kurepa introduced the notion of a Suslin tree, and showed $\neg$SH is equivalent to the existence of a Suslin tree. In other words, the existence of a Suslin line is equivalent to the existence of a Suslin tree. We prove this in this section, and introduce a few other types of trees of interest.

Definition 2.1. A *tree* is a partially ordered (i.e., transitive, irreflexive) set $(T, <_T)$ with the property that for every $x \in T$, $\{y \in T : y <_T x\}$ is well-ordered. for $x \in T$ we write $|x|_T$ (or just $|x|$ if T is understood) to denote the order-type of $\{y \in T : y <_T x\}$. We call this the rank or height of x in T. The height of T is defined by $|T| = \sup\{|x|_T + 1 : x \in T\}$. By the αth level of T we mean the set of $x \in T$ of height α. A *chain* of T is a subset which is linearly ordered by $<_T$. A *branch* b of T means a chain which is closed downwards (i.e., if $x \in b$ and $y <_T x$, then $y \in b$). An antichain A of T is a subset of T of pairwise incomparable elements.

We are interested in trees of size κ, for various cardinals κ, which satisfy a certain non-triviality condition:

Definition 2.2. Let $\kappa \in$ CARD. A κ-tree is a tree T of height κ such that all levels of the tree have size $< \kappa$. That is, $\forall \alpha < \kappa\ |\{x \in T : |x|_T = \alpha\}| < \kappa$.

We now introduce several particular trees of interest.

Definition 2.3. Let κ be a cardinal. A κ Aronszajn tree is a κ tree with no branch of size κ. An Aronszajn tree refers to an ω_1 Aronszajn tree.

Requiring more we get the notion of a Suslin tree.

Definition 2.4. Let κ be a cardinal. A κ Suslin tree is a κ tree with no chains of size κ and no antichains of size κ. A Suslin tree refers to an ω_1 Suslin tree.

In other words, a κ Suslin tree is a κ Aronszajn tree with no antichains of size κ.

Definition 2.5. Let κ be a cardinal. A κ Kurepa tree is a κ tree with $\geqslant \kappa^+$ many branches of length κ. A Kurepa tree refers to an ω_1 Kurepa tree.

It is not immediately clear if any of these kinds of trees exist. We will see that Aronszajn trees exist in ZFC, but the existence of Suslin and Kurepa trees is independent of ZFC.

If desired, a κ tree can, with perhaps a fairly trivial modification, be viewed as a subtree of $(\kappa^-)^\kappa$, where $\kappa^- = \sup\{\lambda \in \text{CARD} : \lambda < \kappa\}$. The modifications required are: we assume the tree has a single root, that is, a single element of height 0 (if not, we add one), and if $x \neq y \in T$ have limit height, then $\{z : z <_T x\} \neq \{z : z <_T y\}$ (this can be arranged by adding extra elements at limit levels, one for each branch of that height, that sit immediately below the old points of that limit height). Given these adjustments, it is now straightforward to define an isomorphism π between

T and a subtree of $(\kappa^-)^\kappa$. If $x \in T$ has height α, then $\pi(x)$ will be a sequence with domain α (define $\pi(x)$ by induction on $|x|_T$, at limit stages take unions of the $\pi(y)$ for $y <_T x$, and at successor steps use the fact that any $x \in T$ has $< \kappa$ many immediate extensions).

Definition 2.6. A tree T is *branching* if every $x \in T$ has at least two distinct immediate extensions. T is said to be pruned if it has a single root and for every $x \in T$ and every $\alpha < |x|_T$ (with $\alpha < |T|$) there is a $y \in T$ of height α with $x <_T y$.

In other words, a pruned tree has the property that every element of the tree has arbitrarily high extensions in the tree.

If T is a κ tree and κ is regular, then there is a canonical subtree T' of T which is a pruned κ tree. Let T' be those $x \in T$ which have κ many extension in T (which implies x has extensions of arbitrary height). It is easy to check that T' is downward closed subtree of T, and that it is pruned (except it may have more than one root; in that keep only the part of T' above a particular root). [To see it is pruned, take $x \in T'$. Let $\alpha > |x|_T$. Let $S \subseteq T$ be κ many extensions of x, all of which have height $> \alpha$. κ many elements of S must extend a single $y \in T$ of height α. Then $y \in T'$.]

Before investigating these trees, we first make the connection with Suslin's hypothesis.

3. Suslin Trees and Suslin's Hypothesis

The following lemma connects Suslin's hypothesis with Suslin trees.

Lemma 3.1. *(ZFC) There is a Suslin line iff there is a Suslin tree.*

Proof. Suppose first that $(X, <)$ is a Suslin line. We construct the tree T out of the intervals $I = (x, y)$ in X. Rather than take all intervals (which does not give a tree), we pick the intervals $I_\alpha = (x_\alpha, y_\alpha)$, for $\alpha < \omega_1$, inductively so that they do form a tree. If I_β for $\beta < \alpha$ have been defined, let $C = \bigcup_{\beta<\alpha}\{x_\beta, y_\beta\}$ be the set of endpoints so far constructed. This set is countable, so it is not dense in X (recall X is not separable). Let I_α be a non-empty interval with $I_\alpha \cap C = \emptyset$. Continue to define I_α for all $\alpha < \omega_1$. If $\alpha \neq \beta$, then I_α and I_β are either disjoint, or one is contained in the other. Let T be the set whose elements are the intervals I_α constructed, and define $I <_T J$ iff $J \subseteq I$. $(T, <_T)$ is easily a tree (note that if I_α and I_β both contain J, and $\alpha < \beta$ then $I_\alpha \supseteq I_\beta$. Thus the $<_T$ predecessors of J are ordered by their indices.). We show that T is a Suslin tree. An uncountable antichain of T would be an uncountable family of intervals of X which are pairwise disjoint, a contradiction since X is c.c.c. (if I, J in T are not disjoint, then one contains the other so they are comparable in T). Suppose there were an uncountable branch, say $J_0 <_T J_1 < \cdots < J_\alpha < \ldots$. If x_α denote the left endpoint of J_α, then the x_α form an ω_1 increasing sequence from X. Then the intervals $(x_{2\cdot\alpha}, x_{2\cdot(\alpha+1)})$ are non-empty, pairwise disjoint, a contradiction. Thus, T is a Suslin tree.

Suppose next that T is a Suslin tree, and we construct a Suslin line. Let X be the set of maximal branches of T. We fix an order on T (say by identifying it with ω_1) and order the branches of T lexicographically. This defines the linear ordering $(X, <)$. To see it is c.c.c., suppose (x_α, y_α), $\alpha < \omega_1$ was an ω_1 sequence of pairwise disjoint non-empty intervals. Let $z_\alpha \in (x_\alpha, y_\alpha)$. Let β_0 be the least ordinal $< |z_\alpha|$ such that $x_\alpha(\beta_0) \neq z_\alpha(\beta_0)$, and let β_1 be the least ordinal $< |z_\alpha|$

such that $y_\alpha(\beta_1) \neq z_\alpha(\beta_1)$. Let $\beta = \max\{\beta_0, \beta_1\}$. Let $a_\alpha = z(\beta) \in T$. Then $\{a_\alpha\}$ is an uncountable antichain in T, a contradiction.

To see $(X, <)$ is not separable, let $A = \{b_b\}$ be a countable subset of X. Choose $\alpha < \omega_1$ of height greater than the supremum of the heights of the branches b_n. There is some $z \in T$ of height α which has three distinct extensions (in fact, ω_1 many) in T (as T is an ω_1 tree). This defines a non-empty interval in X which contains only branches of length $> \alpha$. Hence, this gives a non-empty interval missing A, so A is not dense.

This shows $(X, <)$ is c.c.c. but not separable. From lemma 1.3 this gives a Suslin line. Alternatively, we can modify the tree T directly so that $(X, <)$ as just constructed is dense in itself. To do this, first prune T so that, without loss of generality, every $x \in T$ has extensions to arbitrarily high levels (the pruned subtree is clearly still a Suslin tree). Then consider levels T_{α_0}, $T_{\alpha_1}, \dots$ of T such that all points of T of level α_η have ω many extensions at level $\alpha_{\eta+1}$. This defines a subtree T' of T (the union of the points at some level α_η) which is still an ω_1 tree (and thus still a Suslin tree). The tree T' is ω-splitting (i.e., every element has infinitely many immediate extensions). If we order the extensions of any point of T' in order type $\mathbb{Q}$, then X will clearly be dense in itself. We can then take the completion of $(X, <)$ to get a Suslin line. □

4. Aronszajn Trees

Recall an Aronszajn tree is an ω_1 with no ω_1 branch. The next lemma shows we can construct them in ZFC.

Lemma 4.1. *(ZFC) There is an ω_1 Aronszajn tree.*

We'll give two construction of an Aronszajn tree.

first proof. We construct the tree as a subtree of $\mathbb{Q}^{\omega_1}$. The α^{th} level of the tree will consist of increasing sequences $t \in \mathbb{Q}^\alpha$ with $\sup(t)$ finite (i.e., $\operatorname{ran}(t)$ is a bounded set of rationals.). We will have $t <_Y u$ iff u extends t. We construct the levels of the tree, $T\alpha$ inductively and will also satisfy $(*)$: for any $t \in T_\alpha$ and any $\beta > \alpha$ and $q > \sup(t)$, there is a $u \in T_\beta$ with $t <_T u$ and $\sup(u) < q$.

If T_α is defined , we let $T_{\alpha+1}$ consist of all $t^\frown q$ where $q \in Q$ and $q > \sup(t)$. This clearly maintains $(*)$.

Suppose now α is a limit and T_β has been defined for all $\beta < \alpha$. For each $t \in T_{<\alpha} = \bigcup_{\beta<\alpha} T_\beta$ and each $q > \sup(t)$, choose a sequence of ordinals $|t|_T < \alpha_0 < \alpha_1 < \dots$ with $\sup_n \alpha_n = \alpha$ and choose rationals $\sup(t) < r_0 < r_1 < \dots$ with $\sup_n r_n < q$. By $(*)$, choose then $t <_T t_0 <_T t_1 <_T t_2 <_T \dots$ where $t_i \in T_{\alpha_i}$ and $\sup(t_i) < r_i$. Put then $u = \cup t_n$ in T_α. Clearly we have maintained $(*)$, and T_α is countable. Thus, T is an ω_1 tree. It clearly has no ω_1 branch, since that would give an ω_1 sequence of distinct rationals. □

second proof. We now give a second construction due to Kunen. We will construct a sequence $\{s_\alpha\}_{\alpha<\omega_1}$ satisfying:

(1) S_α is a one-to-one function from α to ω.
(2) If $\alpha < \beta$ then $s_\beta \restriction \alpha$ agrees with s_α except on a finite set.
(3) $\omega - \operatorname{ran}(s_\alpha)$ is infinite.

Granting this, we let T be the set of all $s_\alpha \restriction \beta$ where $\beta \leqslant \alpha$, that is, all initial segments of all of the s_α. We order T again by extension. Clearly T is a tree. T is an ω_1 tree from property (2), since there are countably many $s \in \alpha^\omega$ which agree with s_α up to a finite set. From (1) there are clearly no ω_1 branches through T.

It remains to construct the s_α, which we do by induction. For successor steps, let $s_{\alpha+1} = s_\alpha{}^\frown n$ where $n \notin \operatorname{ran}(s_\alpha)$. Suppose α is a limit, and let $\{\alpha_n\}$ be an increasing sequence with supremum α. Begin with s_{α_0}. Get $t_{\alpha_1} = s_{\alpha_0} \cup u$ where u is the result of changing s_{α_1} on a finite subset of $\alpha_1 - \alpha_0$ so that t_{α_1} is one-to-one. In general, get $t_{\alpha_{n+1}} = t_{\alpha_n} \cup u$ where u is the result of changing $s_{\alpha_{n+1}}$ on a finite subset of $\alpha_{n+1} - \alpha_n$ so that $t_{\alpha_{n+1}}$ is one-to-one. We can do this from properties (2) and (3). Let $t = \bigcup_n t_{\alpha_n}$, then t satisfies properties (1) and (2). We can then modify t to get s_α by, for example, changing the values at the α_n, say by $s_\alpha(\alpha_n) = t(\alpha_{2n})$. s_α now satisfies (1)-(3). □

The second proof modifies to get κ-Aronszajn trees for κ a successor of a regular, assuming GCH.

Lemma 4.2. *Let $\kappa = \lambda^+$ where λ is regular and $2^{<\lambda} = \lambda$. Then there is a κ Aronszajn tree.*

Proof. We construct s_α for $\alpha < \kappa$ satisfying:

(1) S_α is a one-to-one function from α to λ.
(2) If $\alpha < \beta$ then $s_\beta \restriction \alpha$ agrees with s_α except on a set of size $< \lambda$.
(3) $\operatorname{ran}(s_\alpha)$ is non-stationary in λ.

Granting this, we again let T be the tree os initial segments of the s_α. This is a κ tree since for each $\alpha < \kappa$ there are at most $\lambda^{<\lambda} = 2^{<\lambda} = \lambda$ many $s \in \lambda^\alpha$ which agree with s_α except on a set of size $< \lambda$.

We construct the s_α by induction as before. Successor steps are trivial. Suppose α is a limit ordinal. We assume $\operatorname{cof}(\alpha) = \lambda$, as the other case is easier. Fix $\{\alpha_i\}_{i<\lambda}$ increasing, continuous, and cofinal in α. We construct sequence $t_{\alpha_i} \in \lambda^{\alpha_i}$ by induction on i as before. For $i < \lambda$ a limit we take unions. Properties (1) and (2) are immediate, and (3) follows since a $< \lambda$ intersection of sets c.u.b. in λ is c.u.b. in λ. We let $t_{\alpha_{i+1}} = t_{\alpha_i} \cup u$, where $u = s_{\alpha_{i+1}} \restriction (\alpha_{i+1} - \alpha_i)$, except we change the values on $< \lambda$ many points of $(\alpha_{i+1} - \alpha_i)$ to get $t_{\alpha_{i+1}}$ one-to-one. Using (2) and (3) and the trivial fact that every c.u.b. subset of λ has size λ, there is no problem defining $t_{\alpha_{i+1}}$ (we redefine $s_{\alpha_{i+1}} \restriction (\alpha_{i+1} - \alpha_i)$ on a set of size $< \lambda$ to have values in a c.u.b. set which $\operatorname{ran}(t_{\alpha_i})$ misses). We still clearly have that each $\operatorname{ran}(t_{\alpha_i})$ misses a c.u.b. subset of λ, say C_i.

Let $t = \bigcup_{i<\lambda} t_{\alpha_{i+1}}$. t then satisfies (1) and (2), and we must adjust it to get s_α also satisfying (3). Let $\{\beta_i\}_{i<\lambda}$ be an increasing, continuous sequence with $\beta_i \in \bigcap_{j<i} C_j$. Define s_α to be t, except t takes value β_i we define s_α to take value β_{i+1}. Clearly s_α then satisfies (3) (since $\{\beta_i\}_{i \in \text{Limit}}$ is c.u.b.). s_α still satisfies (2) since the modification of t to s_α changes $< \lambda$ many values of $t \restriction \alpha_i$ for any $i < \lambda$ (since $\operatorname{ran}(t \restriction \alpha_i)$ doesn't contain any β_j for $j > i$). □

If we don't assume CH, then there may or may not be ω_2 Aronszajn trees (Mitchell). For κ strongly inaccessible, there is a κ Aronszajn tree iff κ is weakly compact, a mild large cardinal axiom (the existence of weakly compact cardinals is consistent with $V = L$). On the other hand, Jensen showed that in L, there is a κ Suslin, hence a κ Aronszajn, tree for all regular κ which are not weakly compact.

5. Suslin Trees

Unlike Aronszajn trees, we cannot construct a Suslin tree in ZFC. We first show that $\diamondsuit$ implies the existence of a Suslin tree.

Theorem 5.1. *$\diamondsuit$ implies that there is a Suslin tree.*

Proof. Fix a $\diamondsuit$ sequence $\{A_\alpha\}_{\alpha<\omega_1}$. We construct the levels of the tree $T_\alpha = \{x \in t \colon |x|_T \leqslant \alpha\}$ by induction. We will have that $T \subseteq \omega_1$. We will also maintain that every $x \in T$ has extensions to all higher levels. Let T_0 consist of just the ordinal 0. Given T_α, let $T_{\alpha+1}$ be defined by extending every $x \in T_\alpha$ to ω many immediate extensions in $T_{\alpha+1}$. Suppose now α is limit. Let $T_{<\alpha} = \bigcup_{\beta<\alpha} T_\beta$ be the part of the tree so far constructed. If A_α is not a maximal antichain in $T_{<\alpha}$, then define T_α by picking for every $x \in T_{<\alpha}$ a branch b_x of $T_{<\alpha}$ of length α containing x, and extending this branch to a point in T_α. Suppose now that A_α is a maximal antichain of $T_{<\alpha}$. We define T_α to "seal-off" this antichain, that it, prevent it from growing further. For each $x \in T_{<\alpha}$, let b_x be a branch of $T_{<\alpha}$ of length α which contains x and some element of A_α. We can do this since every x is comparable with an element of A_α. T_α is defined by extending each such b_x to a point of T_α.

Let $T = \bigcup_{\alpha<\omega_1} T_\alpha$. Clearly T is a pruned ω_1 tree. Suppose $A \subset \omega_1$ is a maximal antichain of T. Let $C \subseteq \omega_1$ be c.u.b. such that for $\alpha \in C$, $T_{<\alpha} \subseteq \alpha$ and $A \cap \alpha$ is a maximal antichain of $T_{<\alpha}$. From $\diamondsuit$, let $\alpha \in C$ be such that $A \cap \alpha = A_\alpha$. Then at stage α in the construction we defined T_α so that all elements of height α extend an element of A_α. This shows that $A \cap \alpha$ is a maximal antichain of T, so $A = A \cap \alpha$, and thus A is countable. □

Corollary 5.2. *It is consistent with ZFC that there is a Suslin line.*

Constructing κ-Suslin trees for higher regular (non weakly compact) cardinals requires more that $\diamondsuit_\kappa$. However, it is easier to force directly the existence of these trees. In fact, this was the original argument of Jech and Tennenbaum. To get a Suslin tree we can force with either countable trees (Jech) or finite trees (Tennenbaum). We sketch both proofs. The first proof works for all regular cardinals as well.

For the first proof (κ now a regular cardinal of M), let the partial order $\mathbb{P}$ consist of pruned trees T, $|T|, \kappa$, of height $\alpha+1$ for some $\alpha < \kappa$ (i.e., for any $x \in T$ has an extension to the highest level α of T). For convenience we also require T to be splitting, and we assume also $T \subseteq \kappa$, that is, the elements of T are ordinals less than κ. We define $T_1 \leqslant T_2$ iff $|T_1| \geqslant |T_2|$ and $T_1 \upharpoonright |T_2| = T_2$ (here $T \upharpoonright \beta$ denotes the elements of T of height $< \beta$ in T). Thus, T_1 must extend T_2 "vertically."

$\mathbb{P}$ is countably closed, but not in general $< \kappa$ closed (the problem is with the pruned condition; an increasing union of conditions of length ω_1 may fail to be extendible to a condition as there may be no branches cofinal in the union). However, $\mathbb{P}$ is $< \kappa$ distributive, which is enough to get that $\mathbb{P}$ preserves all cofinalities and cardinalities $\leqslant \kappa$. To see this, let $\{D_\eta\}_{\eta<\rho}$, $\rho < \kappa$, be a $< \kappa$ collection of dense sets in $\mathbb{P}$. We define conditions p_η of height $\alpha_\eta + 1$ inductively. We will have $p_0 \geqslant p_1 \geqslant \dots$. As we define p_η we will also define for each $x \in p_\eta$ a function f_x which gives a brach of p_η containing x of height $\alpha_\eta + 1$. If $x \in p_{\eta_1} \geqslant p_{\eta_2}$, then we will have that the f_x functions are compatible. At successor steps, if p_η is defined we let $p_{\eta+1} \in D_{\eta+1}$ be any extension of p_η of some successor height $\alpha_{\eta+1} + 1 > \alpha_\eta + 1$, and extend all of the f_x function for $x \in p_\eta$ as well as define the f_x functions for $x \in p_{\eta+1} - p_\eta$.

There is no problem doing this as $p_{\eta+1}$ extends p_η and is pruned. For η limit, let $\beta = \sup\{\alpha_i : i < \eta\}$. Let $T = \bigcup_{i<\eta} p_i$. Our branch functions define for each $x \in T$ a branch f_x of T of height β. Let $T' \leqslant T$ have height $\beta+1$ and obtained by extending each branch f_x to level $\beta+1$ of T'. T' is now a condition in $\mathbb{P}$. Let $p_\eta \in D_\eta$ extend T', and extend the branch fuctions of T' appropriately. Continuing, we define a condition p_ρ which extends conditons in all of the D_η. Thus, $\bigcap_{\eta<\rho} D_\eta$ is dense, so $\mathbb{P}$ is $< \kappa$ distributive.

Let G be M generic for $\mathbb{P}$, where M is a transitive model of ZFC. We may identify G with a pruned κ tree (G has height κ since any condition T can be extended to a condition T' of height $\alpha+1$ for any $|T| \leqslant \alpha+1 < \kappa$).

We claim that G is a κ-Suslin tree. Suppose $\tau \in M^{\mathbb{P}}$ and $T_0 \Vdash$ (τ is a maximal antichain of $\dot{G}$). Let $D \subseteq P$ be those conditions T such that for some $A \subseteq T$ we have

(1) A is a maximal antichain of T.
(2) $\sup\{|a| : a \in A\} < |T|$ (i.e., there is a level of T such that all elements of A are below that level).
(3) $T \Vdash (\check{A} \subseteq \tau)$.

We claim that D is dense below T_0. For let $T \leqslant T_0$. As $\mathbb{P}$ is $< \kappa$ distributive, we may get $T_1 \leqslant T$ such that for all $x \in T_0$, there is a $a \in T_1$ such that x is compatible with a and $T_1 \Vdash (a \in \tau)$. In general, define $T_{n+1} \leqslant T_n$ so that for all $x \in T_n$, there is a $a \in T_{n+1}$ such that x is compatible with a and $T_{n+1} \Vdash (a \in \tau)$. Let $T = \bigcup_n T_n$. Then there is a maximal antichain A of such that $T \Vdash (A \subseteq \tau)$. Extend T to T' of height $|T|+1$ as follows. For each $x \in T$, let b_x be a branch of T containing x and an element of A, with b_x of height $|T|$. For each $x \in T$, put a point in T' which extends all the elements of b_x (i.e., extend the branch b_x). This defines T', and we have $T' \in D$. Thus, D is dense in $\mathbb{P}$. Let $T \in G \cap D$. Let $A \subseteq T$ witness $T \in D$. We must have $A = \tau_G$, since if $T' \leqslant T$ and $x \in T' - T$, then x is above some element of A. Thus, $\tau_G = A$ has size $< \kappa$ in $M[G]$. This shows G is a Suslin tree.

Corollary 5.3. *Let M be a transitive model of ZFC and κ a regular cardinal of M. Then there is a κ-distributive forcing (hence preserves all cofinalities and cardinalites $\leqslant \kappa$) such that in $M[G]$ there is a κ-Suslin tree.*

The second proof uses finite trees. $\mathbb{P}$ now consists of finite trees $T \subseteq \omega_1$ satisfying: if $\alpha <_T \beta$ then $\alpha < \beta$. We define $T_1 \leqslant T_2$ iff $<_{T_1} \restriction (T_2 \times T_2) = <_{T_2}$. We again identify a generic G with a tree G on ω_1 (to see it is a tree, note that if $\alpha \in G$ then the $<_G$ predecessors of α are $<_G$ ordered in their usual order as ordinals, hence the $<_G$ predecessors are well-ordered).

First we show that $\mathbb{P}$ is c.c.c. If $\{T_\alpha\}_{\alpha<\omega_1}$ were an antichain, then by the Δ system lemma we may assume that each T_α consists of a root R and a set $A_\alpha = \{a_\alpha^1, \dots, a_\alpha^n\}$ (of some fixed size n), where the A_α are pairwise disjoint. We may further assume that the T_α orderings on the root R are all the same. Further, we may assume that for $r \in R$, $r <_{T_\alpha} a_\alpha^k$ iff $r <_{T_\beta} a_\beta^k$ for all α, β. We may also assume that $a_\alpha^k <_{T_\alpha} a_\alpha^l$ iff $a_\beta^k <_{T_\beta} a_\beta^l$ for all $k, l \leqslant n$ and all α, β. Thus, T_α and T_β look the same except for the values of the ordinals in A_α, A_β. However it is now easy to get a common extension of any two of the T_α (e.g., the union of T_α and T_β is now a condition).

We show that G has no uncountable antichain (from which it also follows that G is an ω_1 tree (for another argument, see the exercise below). Suppose $T \Vdash$ (τ is an uncountable antichain of $\dot{G}$). Get an ω_1 sequence T_α of conditions extending

T and ordinals $\eta_\alpha \in T_\alpha$ with $T_\alpha \Vdash (\check{\eta}_\alpha \in \tau)$, and the η_α are distinct. Thin the T_α to a Δ system as above, $T_\alpha = R \cup A_\alpha$. It is easy to see that for any $\alpha \neq \beta$ we can get a common extension of T_α and T_β in which η_α is comparable with η_β, a contradiction.

Corollary 5.4. *The existence of a Suslin tree is consistent with ZFC + ¬CH. In particular, the existence of a Suslin tree does not imply* $\diamondsuit$.

6. Kurepa Trees

Recall a Kurepa tree is an ω_1 tree with at least ω_2 branches of length ω_1. We can give an easy reformulation of this which does not mention trees.

Lemma 6.1. *There is a Kurepa tree iff there is a family $\mathcal{F}$ of subsets of ω_1 satisfying:*

(1) $|\mathcal{F}| \geqslant \omega_2$.
(2) $\forall \alpha < \omega_1 \; |\{A \cap \alpha \colon A \in \mathcal{F}\}| \leqslant \omega$.

We call a family $\mathcal{F}$ as in lemma 6.1 a *Kurepa* family.

Proof. Assume first that T is a Kurepa tree. Without loss of generality we may assume $T \subseteq \omega_1$ and if $\alpha <_T \beta$ then $\alpha < \beta$. Then $\mathcal{F}$ = the set of ω_1 branches through T is a Kurepa family (note that ib b is a branch through T, then $b \cap \alpha$ is determined by the α^{th} level of b).

Conversely, assume $\mathcal{F}$ is a Kurepa family. Let T be the subtree of $2^{<\omega_1}$ consisting of all initial segments of characteristic functions of $A \in \mathcal{F}$. Easily T is a Kurepa tree (note that the element of T of height α correspond to the elements $A \cap \alpha$ for $A \in \mathcal{F}$). □

Just as $\diamondsuit$ implies the existence of a Suslin tree, there is a combinatorial principle $\diamondsuit^+$ which implies the existence of a Kurepa tree. $\diamondsuit^+$ implies $\diamondsuit$, however the existence of a Kurepa tree does not imply the existence of a Suslin tree. Although we show here the consistency of the existence of Kurepa tree directly by forcing, we state the principle $\diamondsuit^+$. We note also that $\diamondsuit^+$, like $\diamondsuit$ holds in L.

Definition 6.2. $\diamondsuit^+$ is the statement that there is a sequence $\{\mathcal{A}_\alpha\}_{\alpha<\omega_1}$, where each $\mathcal{A}_\alpha$ is a countable family of subsets of α, such that for all $A \subseteq \omega_1$, there is a c.u.b. $C \subseteq \omega_1$ such that for all $\alpha \in C$ we have $A \cap \alpha \in \mathcal{A}_\alpha$ and $C \cap \alpha \in \mathcal{A}_\alpha$.

In view of theorem 0.7, it is clear that $\diamondsuit^+$ implies $\diamondsuit$.

Theorem 6.3. *Let M be a transitive model of ZFC+ CH. Then there is a countably closed $\mathbb{P} \in M$ such that if G is M-generic for $\mathbb{P}$ then $M[G]$ satisfies that there is Kurepa tree.*

Proof. $\mathbb{P}$ consists of pairs (T, f) where T is a pruned countable subtree of $2^{<\omega_1}$ of height $\alpha < \omega_1$, and f is a function with domain a countable subset of ω_2 such that for $\alpha \in \text{dom}(f)$, $f(\alpha)$ is a branch through T. We say $(T_1, f_1) \leqslant (T_2, f_2)$ if $T_1 \restriction |T_2| = T_2$ (i.e., T_1 extends T_2 vertically), $\text{dom}(f_2) \subseteq \text{dom}(f_1)$, and for all $\alpha \in \text{dom}(f_2)$, $f_1(\alpha)$ extends $f_2(\alpha)$ (i.e., f_1 extends all of the branches of f_2, and may give new ones as well).

Clearly $\mathbb{P}$ is countably closed. Let T be the tree produced by the generic, and F the function produced (in the obvious manner). Thus, F is a map from ω_2^M to the length ω_1 branches of T (note: $\omega_1^M = \omega_1^{M[G]}$). It is easy to see that for a generic

G, the resulting function F will also be one-to-one (we may extend any (T, f) with $\alpha, \beta \in \mathrm{dom}(f)$ to a (T', f') where $f'(\alpha) \neq f'(\beta)$). Thus, in $M[G]$ the ω_1 tree T has $\geqslant \omega_2^M$ branches of length ω_1. We show finally that $\mathbb{P}$ is ω_2-c.c. in M, which shows that $\omega_2^M = \omega_2^{M[G]}$ and completes the proof.

Suppose $\{(T_\alpha, f_\alpha)\}_{\alpha<\omega_2}$ were an antichain of $\mathbb{P}$. We may assume that all of the trees have the same height $\beta < \omega_1$. By CH in M, there are $2^\beta = \omega_1$ many trees of height β, so we may assume that all of the T_α are equal to a fixed tree T_0. Again using CH, we may thin the antichain so the $\mathrm{dom}(f_\alpha)$ form a Δ system on ω_2, say $\mathrm{dom}(f_\alpha) = R \cup A_\alpha$ where the A_α are pairwise disjoint. We may assume that that the f_α all agree on the root R, as there are only $\omega_1^\omega = 2^\omega = \omega_1$ many choices for $f \restriction R$. At this point, any two members of the antichain are compatible, a contradiction. □

7. A Model in which there are no Kurepa Trees

Starting with a model M of ZFC+ there is an inaccessible cardinal, we produce a generic extension $M[G]$ in which there are no Kurepa trees. The inaccessible cardinal is necessary since if M satisfies ZFC+ there are no Kurepa trees then ω_2^M is inaccessible in L [Work in M. If ω_2 is not inaccessible in L, then $\omega_2 = (\omega_2)^{L[A]}$ for some $A \subseteq \omega_1$. However $\diamondsuit^+$ holds in any $L[A]$ for $A \subseteq \omega_1$. Thus there is a Kurepa tree in $L[A]$, and this remains a Kurepa tree in $L[A]$ as $\omega_2 = (\omega_2)^{L[A]}$.]

To motivate the forcing, note that if T is a Kurepa tree, then T will remain a Kurepa tree in any larger model unless ω_2 is collapsed in the larger model. Conversely, collapsing ω_2 by a countably closed forcing will kill the Kurepa trees of the ground model (by lemma 7.3), but may introduce new Kurepa trees, so it will be necessary to collapse the new ω_2, etc. This suggests we collape to ω_1 all the ordinals below some large cardinal.

Definition 7.1. Let κ be a regular cardinal. The Silver collapse of κ to ω_2 is the forcing $\mathbb{P}$ consisting of functions with domain a countable subset of $\{(\alpha, \beta) \colon \alpha < \kappa \wedge \beta < \omega_1\}$ and $f(\alpha, \beta) < \alpha$ for all $(\alpha, \beta) \in \mathrm{dom}(f)$.

It is clear that if g is generic for $\mathbb{P}$, then $(|\kappa| \leqslant \omega_2)^{M[G]}$, as in $M[G]$ all ordinals $\alpha < \kappa$ are onto images of ω_1. It is also clear that $\mathbb{P}$ is countably closed, so ω_1 is preserved in forcing with $\mathbb{P}$.

Lemma 7.2. *Let κ be a strongly inaccessible cardinal of M. Then $\mathbb{P}$ is κ-c.c. Thus, κ is a cardinal of $M[G]$, and hence $\kappa = \omega_2^{M[G]}$.*

Proof. Suppose $\{p_\alpha\}_{\alpha<\kappa}$ were an antichain of size κ. Let $d_\alpha = \mathrm{dom}(p_\alpha)$. We can view d_α as a countable subset of κ. We use the Δ-system argument. We may assume all the d_α has order-type $\tau < \omega_1$. Let $\eta(\beta) = \sup\{d_\alpha(\beta) \colon \alpha < \kappa\}$. There must be a least $\beta_0 < \tau$ such that $\eta(\beta_0) = \kappa$ [otherwise there is an $\eta < \kappa$ such that $d_\alpha \subseteq \eta$ for all $\alpha < \kappa$. Using κ strongly inaccessible this gives $< \kappa$ many possibilities for the p_α.] We may then thin the $\{p_\alpha\}$ sequence so the d_α form a Δ-system with root $R \in \kappa^\tau$. There are $< \kappa$ many possibilities for $p_\alpha \restriction R$, using again the inaccessibilty of κ. So we may assume $p_\alpha \restriction R$ is constant, and this gives a contradiction to the p_α being an antichain. □

Lemma 7.3. *Let $\mathbb{P}$ be a countably closed forcing in M. If $T \in M$ is an ω_1 tree of M, then any branch of T in $M[G]$ lies in M.*

Proof. Suppose $\tau \in M^{\mathbb{P}}$ and $p \Vdash (\tau \text{ is a branch of } \check{T} \wedge \tau \notin M)$. We define conditions p_s for $s \in 2^{<\omega}$ and also $x_s \in T$. Let $p_\varnothing = p$ and x_0 be the root of T. Given p_s and x_s, let $p_{s^\frown 0}$ and $p_{s^\frown 1}$ extend p_s and $x_{s^\frown 0}$, $x_{s^\frown 1}$ be two extensions of x_s in T which are incompatible in T, and with $p_{s^\frown i} \Vdash x_{s^\frown i} \in \tau$. We can do this as otherwise p_s determines the branch τ_G. Let $\alpha < \omega_1$ be greater than the heights of all the x_s in T. For each $r \in 2^\omega$, let p_r extend all of the p_s for s an initial segment of r, and (extending p_r if necessary), let $x_r \in T$ have height α with $p_r \Vdash \check{x}_r \in \check{T}$ (we can do this as p forces that τ has elements of all heights $\alpha < \omega_1$, as otherwise some $q \leqslant p$ would determine τ_G.) We clearly have that the $\{x_r\}$ are distinct elements of T of height α, contradicting T being an ω_1 tree. $\square$

Theorem 7.4. *Let M be a transitive model of ZFC and κ an inaccessible cardinal of M. Let $\mathbb{P} \in M$ be the Silver collapse of κ to ω_2 as defined in M. Then if G is M-generic for $\mathbb{P}$, $M[G]$ satisfies that there are no Kurepa trees.*

Proof. Suppose $T = \tau_G \in M[G]$ and $(T \text{ is a Kurepa tree})^{M[G]}$. We may assume $T \subseteq \omega_1^{M[G]} = \omega_1^M$. Let σ be a nice name for T, that is $\tau_G = \sigma_G$ and σ is of the form $\bigcup_{\alpha<\omega_1} \{\check{\alpha}\} \times A_\alpha$ where A_α is an antichain of $\mathbb{P}$. From lemma 7.2 it follows that $|\sigma| < \kappa$. For any $\lambda < \kappa$ we may write $\mathbb{P} = \mathbb{P}^{\leqslant\lambda} \times \mathbb{P}^{>\lambda}$ where $\mathbb{P}^{\leqslant\lambda}$ consists of those $p \in \mathbb{P}$ whose domain consists only of pairs (α, β) where $\alpha \leqslant \lambda$, and $\mathbb{P}^{>\lambda}$ those p whose domain consists only of pairs (α, β) with $\alpha > \lambda$. Let $\lambda < \kappa$ be large enough so that $\operatorname{tr}\operatorname{cl}(\sigma) \cap \mathrm{ON} \subseteq \lambda$. Write $G = G^{\leqslant\lambda} \times G^{>\lambda}$. Then $T = \sigma_G = \sigma_{G^{\leqslant\lambda}} \in M[G^{\leqslant\lambda}]$. From lemma 7.3 there are at most $(2^{\omega_1})^{M[G^{\leqslant\lambda}]}$ many branches of T in $M[G]$. Since κ is inaccessible, a simple name counting argument shows that $\rho = (2^{\omega_1})^{M[G^{\leqslant\lambda}]} < \kappa$. However, ρ has cardinality ω_1 in $M[G]$, as forcing with $\mathbb{P}^{>\lambda}$ clearly collapses ρ to cardinality ω_1. thus T has $\leqslant \omega_1$ many branches in $M[G]$. $\square$